Chapter 4, Sequences, Series, and Spirals

Unifying Mysticism and Mathematics

To Reveal Love, Peace, Wholeness, and the Truth

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The image on the front cover is a symbol of Indra's Net of Jewels or Pearls in Huayan Buddhism, visualized as a dewy spider's web in which every dewdrop contains the reflection of the light emanating from all the other dewdrops, like nodes in a mathematical graph, and the relationships between them. Indra's Net is thus a symbol of the interconnectedness of all beings in the Universe, as Consciousness, illustrating that none of us is ever separate from any other being for an instant, despite appearances to the contrary, embodied in conflict-ridden monetary economic systems.

To present the relationships between beings in particular circumstances in as comprehensible a way as possible, this chapter contains 116 tables and 266 figures, many of which are grouped together in further tables. As this chapter is rather like one in a mathematics textbook, there are, of course, many formulae, for those who feel comfortable with such abstractions. Although these reveal much beauty, for me, they are of secondary importance in creating a coherent picture of the psychodynamics of society, explaining who we are and our ultimate destiny as a species.

In this edition, I have also changed the subtitle of this book, which was previously *To Realize Love, Peace, Wholeness, and the Truth*. For Love, Peace, Wholeness, and the Truth *are* Reality, as particular denotations of the Ineffable Absolute. We cannot realize the Absolute—make it real. By unifying mysticism and mathematics, the conceptualizing mind becomes translucent, revealing the Eternal Reality that is ever Present, etymologically meaning 'before being' or 'prior to existence'.

Such a revelation is apocalyptic, which literally means pull a veil away from what is hidden in the utmost depth and breadth of the psyche. By invoking Self-reflective Intelligence, we thereby also reveal what Heraclitus, the mystical philosopher of change, called the Hidden Harmony, enabling us to unify all dualistic opposites, including self-contradictions, and thus dwell in Stillness in the Eternal Now. This is absolutely essential if we are to heal our existential pain at these end times we live in, in the midst of the sixth mass extinction of species on Earth.

For, in conformity with the fundamental law of the Universe, which I call the paradoxical Principle of Unity, hope and despair are just two sides of the same coin. So, the perfect society—as the union of perfection and imperfection—is one where everyone awarely lives in harmony with Cosmic Law, rejected by axiomatic, deductive mathematics. Such denials are why Utopia is currently as far away from becoming manifest as it has ever been.

Sequences, Series, and Spirals

Introduction

This pdf file contains Chapter 4 of my book *Unifying Mysticism and Mathematics: To Reveal Love, Peace, Wholeness, and the Truth*. This chapter, in particular, demonstrates mathematics as a generative science, irrepressibly emerging directly from the Absolute through the creative power of Life, in contrast to the traditional axiomatic, mechanistic approach. I now plan to write the final chapter of this book titled 'Universal Algebra', explaining how the abstractions of pure mathematics have reached their ultimate level of generality in Integral Relationship Logic, the subject of Chapter 2. All being well, Chapter 5 could be complete by this time next year, when I shall be in my eightieth year.

I first saw the need for this book in the early 1960s, when majoring in mathematics at university, having intuitively realized that what I had been taught in religion, science, economics, mathematics, and logic as an adolescent did not make sense as a coherent whole. However, it was not until 1980 that Life gave me the opportunity to understand what it truly means to be human—in contrast to the other animals and machines, like computers—when I had the idea that the pace of change in society is accelerating exponentially because of the existence of accumulative psychospiritual, mental energies.

I initially thought that unifying these nonphysical energies with the four physical forces recognized by materialists would enable us to rebuild our education and economic systems on the Truth. However, as it has turned out, making such radical changes to the way we live our lives is far beyond the capability of the vast majority of the population, embodied in the political system we call 'democracy'. Even liberal progressives and spiritual seekers are not following Vimala Thakar's exhortation in *Spirituality and Social Action: A Holistic Approach* to awaken to Total Revolution, free of the hindrances of the status quo.

Yet, with Covid-19 spreading rapidly around the world, with global heating accelerating, with the global economy on the brink of collapse, and with increasing racial, sexual, cultural, and ideological tensions, could these existential crises give us the motivation to make the necessary changes in our lives? These are well recognized, for, as David Brooks wrote in an opinion piece in the *New York Times* on 25th June 2020, we've got to have a theory of change if the USA, at least, is to face five epic crises all at once.

For myself, all I can do at these end times we live in is to feel complete with my life's work, having written a dozen books during the past ten years explaining what has been causing the fourteen billion years of evolution to drive itself into chaos, which we are witnessing today. As the narrative passages in this book indicate, the antidote is to heal our fragmented, deluded minds in Wholeness, transforming evolutionary divergence into all-embracing convergence at the Omega Point of evolution, as Pierre Teilhard de Chardin foresaw in *The Human Phenomenon*.

That, in essence, is the experience that has led me to write this book. Even though it is titled *Unifying Mysticism and Mathematics*, I write very little about mysticism, as such, for the mystical experiences that we all share in the utmost depths of being are ineffable. So, as I use Integral Relational Logic to create a coherent conceptual model of mathematics, as a whole, I occasionally intersperse a few insights on what the hidden universal structures of mathematics look like from the Divine, Holoramic perspective of Wholeness. For instance, the section on 'Spirals' explains how sets of spirals—spaced like a Fibonacci sequence, resident in the Cosmic Psyche—generate the sunflowers we enjoy in our gardens.

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The first chapters titled 'Business Modelling', 'Integral Relational Logic' and 'From Zero to Transfinity', together with the Prologue and Epilogue from January 2019, are available on my website at http://mysticalpragmatics.net/documents/unifying_mysticism_and_mathematics_prologue_chapters_1-3_epilogue.pdf and ../documents/unifying_mysticism_and_mathematics-covers.pdf.

Following the completion of Chapter 5, I shall need to make a few revisions to Chapter 3 with the fresh insights I have made by writing Chapters 4 and 5. The Prologue and Epilogue will also need updating to reflect the state of the world in eighteen months' time as I then see it.

From the perspective of applied mathematics, my most important book is *Through Evolution's Accumulation Point: Towards Its Glorious Culmination*, which uses a nonlinear difference equation in systems theory to map the whole of evolution since the most recent big bang, explaining why scientists and technologists are driving the pace of scientific discovery and technological invention at unprecedented exponential rates of accelerating change and why society is degenerating into chaos at the present time.

If nothing else, this book, together with my other major evolutionary book *The Four Spheres: Healing the Split between Mysticism and Science*, could help us to see our lives in perspective, abandoning the blame game which attributes abrupt climate change to human causes. For, when we can see the simple unifying patterns underlying the Universe, we see that human behaviour follows these patterns, albeit in a rather complicated manner, for the Cosmic Psyche, containing all knowledge, including mathematics, is the most complex structure that any of us has the ability to study, much hidden in the unconscious.

4. Sequences, Series, and Spirals

Having seen in the previous chapter how the Divine Power of Life brings the entire number system into existence—from Alpha to Omega, as Zero to Transfinity—there is no better way of demonstrating mathematics as a generative science than through sequences, series, and spirals. A sequence in mathematics is an enumerated collection of objects and a series consists of partial or total sums of such sequences, as an ordered set of terms. As such, a sequence of sums can be further summed in series, a process that can be continued indefinitely.

One fascinating aspect of sequences and series is in the way that they generate spirals in both the biosphere and hylosphere, as we see in asters, sea shells, and in galaxies. So, between the sections on finite discrete series and infinite ones, I have inserted a brief overview of this subject, especially as it concerns phyllotaxis, the arrangement of leaves and florets, for instance. To conclude this chapter, I include a brief section on the sequences underlying objects in multidimensional space, showing that, from a mathematical perspective, these dimensions are just an ordinary domain of values, not special in any way.

It is important to remember here that mathematical objects do not exist in outer space. They live in the Cosmic Psyche, so their study is a branch of psychology. Also, to look at the way these objects grow holistically—in Smuts' original meaning of *holistic*—reveals insights into evolutionary processes. And studying the structure of mathematical objects reveals patterns that demonstrate the underlying structure of the Universe and so are inherent to cosmology.

Most generally, Integral Relational Logic shows that *the underlying structure of the manifest Universe is an infinitely dimensional network of hierarchical relationships*, first defined on page 60. So this chapter on 'Sequences, Series, and Spirals' is a particular instance of this universal principle. We also see the pattern of Plato's universals and particulars in generalized mathematical formulae and particular instances of them.

Naturally, the categories of mathematical objects in this chapter are formed using the universal principle of concept formation in Integral Relational Logic, as a taxonomy of taxonomies. There is nothing mysterious or even mystical about them, as we form meaningful concepts by carefully examining the similar differences and different similarities in the meaningless data patterns that emerge from the Datum of the Universe. What we are doing in this creative process is simply studying *māyā*, as the illusory, superficial aspect of the manifest Universe; much fun.

Finite discrete series

A finite series of discrete terms is not an infinite series that converges to a finite limit. Rather, I mean a series that is the sequence of partial sums of a sequence of mostly natural, counting numbers, mainly monotonically increasing.

We begin with sequences in arithmetic progression, moving on quickly to those that can be arranged geometrically, as figurate numbers, the most basic of which is the sequence of triangular numbers, as the partial sums of the positive integers. From there, we look at the most notable of the variety of other sequences that do not match directly with geometric space, the most significant of which is the Fibonacci series, evidence of which we see in forms of life in the natural environment around us, which we explore further in the section on spirals.

Then we look at multinominals and how their coefficients can be arranged in Pascal's pyramidal simplexes, the simplest being Pascal's triangle, showing how triangular numbers, tetrahedral ones, and those in higher dimensions appear as coefficients of binomials in the binomial theorem.

From there we review the formulae for power series, as preparation for looking at the Riemann function and hypothesis in the third section on infinite series.

Finally, in this section, we look briefly at the aptly named generating functions for sequences, putting them into one subsection so that the relationships between them can be clearly seen. These are formulae, whose evaluation generates polynomials whose coefficients are amazingly the terms in the sequence.

Arithmetic progression

The most fundamental of numerical sequences is arithmetic progression, one in which the difference between consecutive terms is constant. The natural numbers are the most basic of the arithmetic sequences, where the first term is 1 and the constant difference is 1: 1, 2, 3, 4, 5, ….

In general, if the first term is a_1 and the common difference is d, then the nth term (a_n) is given by:

$$a_n = a_1 + (n - 1)d$$

The sum of a finite number of terms in an arithmetic progression is an arithmetic series. This is calculated by taking the average of the first and last terms and multiplying by the number of terms, giving:

$$\frac{n(a_1 + a_n)}{2}$$

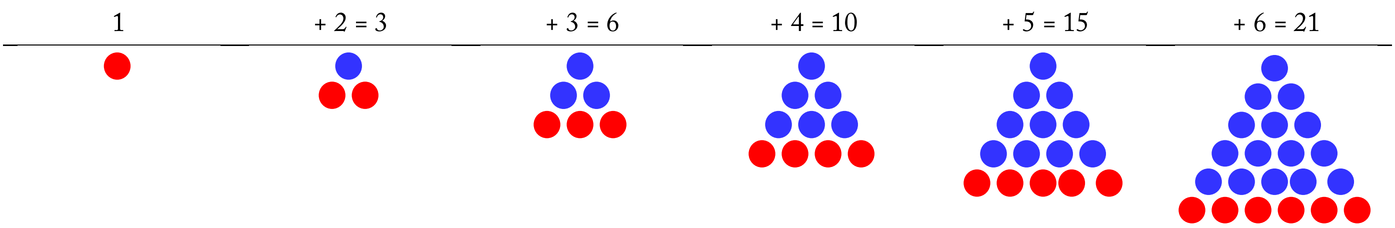
For instance,

$$3 + 10 + 17 + 24 + 31 = \frac{5(3 + 31)}{2} = \frac{5 \times 34}{2} = 85$$

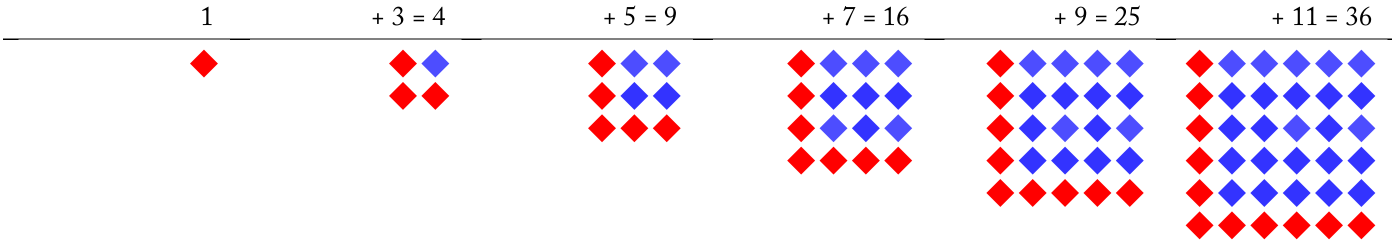
The partial sums of such series themselves form sequences, the most fundamental of which are the natural or counting and triangular numbers, (when d = 0 and 1, respectively), which we now need to generalize.

Figurate numbers

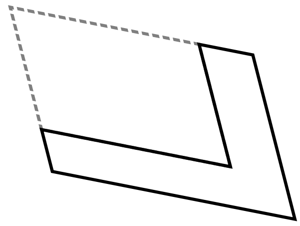
Perhaps the easiest way to visualize the way that sequences and series grow is geometrically, an approach that goes right back to Pythagoras. The ancient Greeks called these 'figured numbers',[1] known as figurate or polytopic numbers today,[2] with different shapes (polygonal numbers) and different dimensions (polyhedral and higher-dimensional polytopic numbers).[3] The most basic of the figurate numbers are the triangular numbers, simply consisting of the sum of the first n natural numbers:



Pythagoras called what is added each time a *gnōmōn*, cognate with *Gnosis*, with an interesting etymology, best explained in terms of the square numbers:



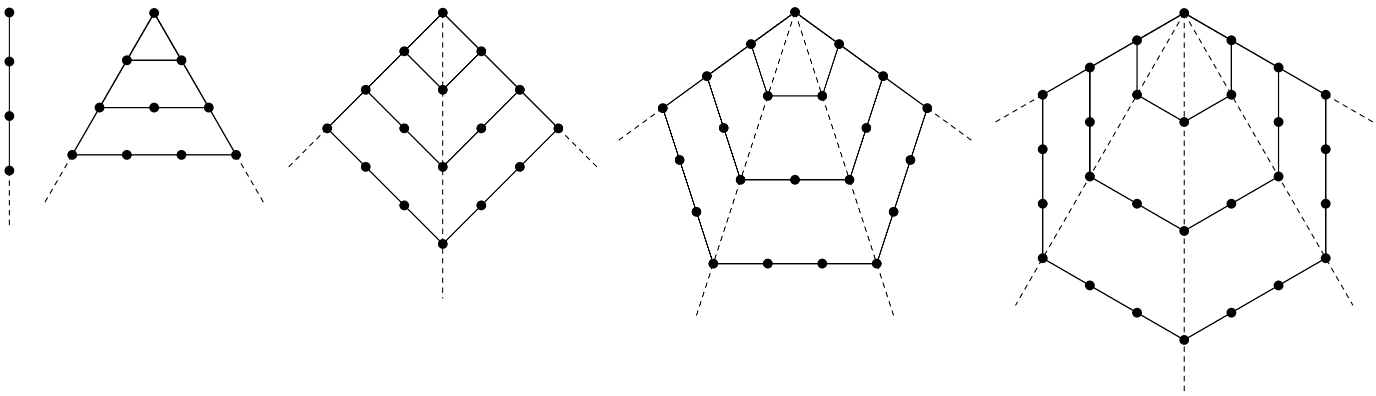
Originally, a gnomon, meaning 'inspector, indicator', was an astronomical instrument for measuring time, as a 'time-knower', consisting of an upright stick, which cast shadows on a surface. This then became generalized as an instrument for drawing right angles, taking the shape of a capital letter L, like a carpenter's square. Pythagoras then noticed that this L-shape was what is added to the square numbers at each step, thereby calling odd numbers 'gnomons', marked red in the above diagram, called gnomic numbers today.[4] Conversely, the L-shape is what is left from a square when a smaller square is removed from it. In *The Elements*, Euclid generalized this notion of gnomon into what is left in a parallelogram, when a smaller, similar parallelogram is removed from it.[5]



Later still, Heron of Alexander defined "a *gnomon* in general as that which, when added to anything, number or figure, makes the whole similar to that which it is added".[6]

As Hypsicles observed around 175 BCE, quoted by Diophantus in *Polygonal Numbers*, the set of polygonal numbers is formed when, starting with 1, the gnomons increase in arithmetic progression. For instance, triangular numbers are formed from the natural numbers, where d is 1. In general, m-gonal numbers are formed when the common difference, d, in the sequence of gnomons is m - 2.[7]

To illustrate the hierarchical growth of polygonal numbers, here are the first five in graphical form, including the natural numbers, in just one dimension.



In numerical terms, polygonal numbers are generated sequences of partial sums of seed sequences, whose terms, after the first, are gnomons, as the differences between the terms of the generated sequences. This table lists the first few of these, where the OEIS IDs are references to *The On-Line Encyclopedia of Integer Sequences*, introduced in the previous chapter:

Type	Seed sequence	OEIS	Generated sequence	OEIS
Natural	1, 1, 1, 1, 1, …	A000012	1, 2, 3, 4, 5, …	A000027
Triangular	1, 2, 3, 4, 5, …	A000027	1, 3, 6, 10, 15, …	A000217
Square	1, 3, 5, 7, 9, …	A005408	1, 4, 9, 16, 25, …	A000290
Pentagonal	1, 4, 7, 10, 13, …	A016777	1, 5, 12, 22, 35, …	A000326
Hexagonal	1, 5, 9, 13, 17, …	A016813	1, 6, 15, 28, 45, …	A000384

As these seed sequences are in arithmetic progression, we can express this as a difference equation, also called a recurrence equation or relation, which is the discrete analogue of a differential equation in the calculus.[8] In this case, the recurrence equation for the seed recurrence sequence is:

$$A_m(n+1) = A_m(n) + (m-2), \qquad A_m(1) = 1$$

Solving, the general formula for $A_m(n)$, generating m-gonal figurate numbers, including the initial value, is given by:

$$A_m(n) = 1 + (m-2)(n-1) = mn - m - 2n + 3$$

This formula is valid for $m = 2$, so we can regard the natural numbers as a degenerate, one-dimensional sequence of polygonal numbers, whose nth terms increase arithmetically, with the common difference being the triangular numbers. Then the nth m-gonal term, $P_m(n)$, is the partial sum of arithmetical sequences, for $m \geq 2$, including the degenerate case:

$$P_m(n) = \sum_{i=1}^{n} A_m(i) = \sum_{i=1}^{n} (1 + (m-2)(i-1))$$

We thus have this difference equation:[9]

$$P_m(n+1) = P_m(n) + A_m(n+1) = P_m(n) + (1 + (m-2)n), \qquad P_m(1) = 1$$

Solving, we then have:[10]

$$P_m(n) = \frac{1}{2}n[(n-1)m - 2(n-2)] = \frac{1}{2}n[(m-2)n + (4-m)]$$

Here are the simplified formulae for the nth terms in the first few polygonal numbers, for $m \geq 2$:

m	Type	Gnomon	Sequence
2	Natural	1	$\frac{1}{2}n \cdot 2 = n$
3	Triangular	n	$\frac{1}{2}n(n+1)$
4	Square	$2n-1$	$\frac{1}{2}n \cdot 2n = n^2$
5	Pentagonal	$3n-2$	$\frac{1}{2}n(3n-1) = \frac{1}{2}(3n^2 - n)$
6	Hexagonal	$4n-3$	$\frac{1}{2}n(4n-2) = \frac{1}{2}2n(2n-1) = 2n^2 - n$

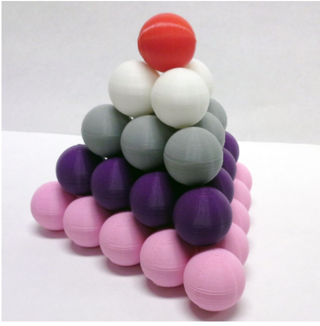
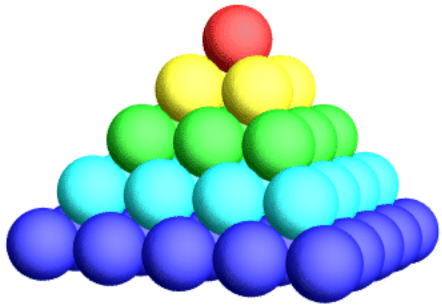
If we now regard each generated sequence as a seed sequence for the next level of generated sequence, where the terms in the seed sequences after the first serve as gnomons, we can generate a collection of pyramidal numbers in three dimensions, extended into four and more, indefinitely. To illustrate the general principle, as triangular numbers generate tetrahedral ones in three dimensions, we have this recurrence equation, where $P_3^2(n)$ is $P_3(n)$, where the superscript denotes the notional spatial dimension.

$$P_3^3(n+1) = P_3^3(n) + P_3^2(n+1) = P_3^3(n) + \frac{1}{2}(n+1)(n+2) \qquad P_3^3(1) = 1$$

Solving, gives:

$$P_3^3(n) = \frac{1}{6}n(n+1)(n+2)$$

The tetrahedral numbers are 1, 4, 10, 20, 35, …. representing the way that balls can be stacked in a triangular pyramid. Similarly, the square numbers generate a square pyramid with this sequence of numbers: 1, 5, 14, 30, 55, …. Here are a couple of pictures of these pyramids borrowed from the web until I find someone to help me with the illustrations:

The higher dimensional *m*-gons do not represent such neatly stacked balls. Nevertheless, further sequences of numbers can be generated from them, called pentagonal pyramidal, hexagonal pyramidal, and so on. The general formula for these 3-dimensional *m*-gonal pyramidal numbers is:

$$P_m^3(n) = \frac{1}{6}n(n+1)[(m-2)n + (5-m)]$$

We can extend these three-dimensional figurate numbers into higher dimensions with this generalized recurrence equation, where *d* denotes the dimension, although it is not generally possible to visualize these geometrically, as the mathematics outlined in the final section describes:

$$P_m^d(n+1) = P_m^d(n) + P_m^{d-1}(n+1) \qquad P_m^d(1) = 1$$

In the case of $m = 3$, the figurate numbers form sequences that can be arranged in higher dimensional simplexes, described on page 316, with the sequences themselves forming Pascal's triangle, pyramid, etc., outlined in the section beginning on page 195. This table gives the first few examples of sequences generated from *m*-gons up to five dimensions:[11]

Type	3-D	OEIS	4-D	OEIS	5-D	OEIS
Tetrahedral	1, 4, 10, 20, 35, …	A000292	1, 5, 15, 35, 70, …	A000332	1, 6, 21, 56, 126, …	A000389
Square pyramidal	1, 5, 14, 30, 55, …	A000330	1, 6, 20, 50, 105, …	A002415	1, 7, 27, 77, 182, …	A005585
Pentagonal pyramidal	1, 6, 18, 40, 75, …	A002411	1, 7, 25, 65, 140, …	A001296	1, 8, 33, 98, 238, …	A051836
Hexagonal pyramidal	1, 7, 22, 50, 95, …	A002412	1, 8, 30, 80, 175, …	A002417	1, 9, 39, 119, 294, …	A034263

The general formula for 4-dimensional *m*-gonal pyramidal numbers is:

$$P_m^4(n) = \frac{1}{24}n(n+1)(n+2)[(m-2)n + (6-m)] = \frac{1}{4}\binom{n}{3}[(m-2)n + (6-m)]$$

The general formula for 5-dimensional *m*-gonal pyramidal numbers is:

$$P_m^5(n) = \frac{1}{120}n(n+1)(n+2)(n+3)[(m-2)n + (7-m)] = \frac{1}{5}\binom{n}{4}[(m-2)n + (7-m)]$$

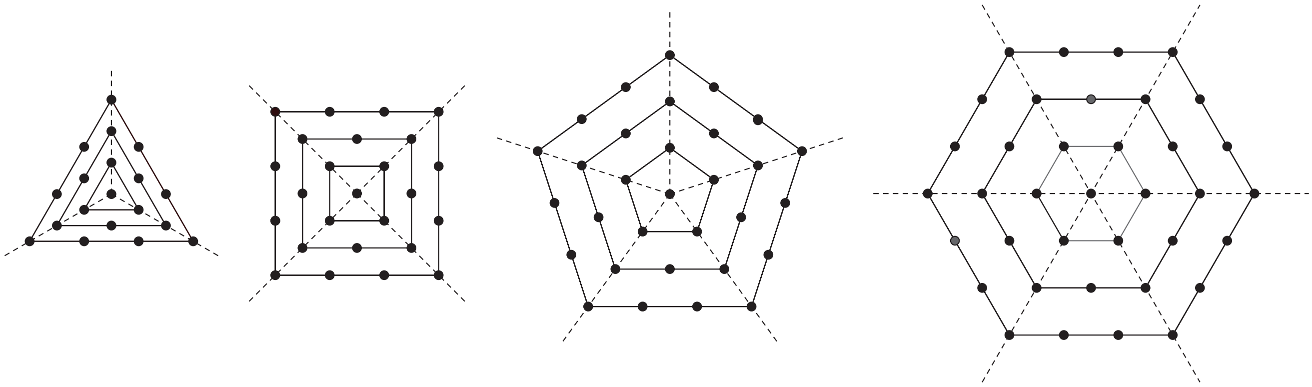
From this pattern, we can see that the general formula for *k*-dimensional *m*-gonal pyramidal numbers is:

$$P_m^d(n) = \frac{1}{d}\binom{n+d-2}{d-1}[(m-2)n + (d+2-m)]$$

And here are some particular instances to illustrate the pattern:

Type	2-D	3-D	4-D	5-D
Triangular	$\frac{1}{2}n(n+1)$	$\frac{1}{6}n(n+1)(n+2)$	$\frac{1}{24}n(n+1)(n+2)(n+3)$	$\frac{1}{120}n(n+1)(n+2)(n+3)(n+4)$
Square	n^2	$\frac{1}{6}n(n+1)(2n+1)$	$\frac{1}{12}n(n+1)^2(n+2)$	$\frac{1}{120}n(n+1)(n+2)(n+3)(2n+3)$
Pentagonal	$\frac{1}{2}n(3n-1)$	$\frac{1}{2}n^2(n+1)$	$\frac{1}{24}n(n+1)(n+2)(3n+1)$	$\frac{1}{120}n(n+1)(n+2)(n+3)(3n+2)$
Hexagonal	$n(2n-1)$	$\frac{1}{6}n(n+1)(4n-1)$	$\frac{1}{6}n^2(n+1)(n+2)$	$\frac{1}{120}n(n+1)(n+2)(n+3)(4n+1)$

In addition to the standard corner polygonal numbers, it is possible to create centred polygonal numbers, in which layers of regular polygons are drawn centred around a dot, each layer having a constant number of sides. Each side of a polygonal layer contains one dot more than a side in the previous layer, so starting from the second polygonal layer each layer of a centred m-gonal number contains m more dots than the previous layer. These diagrams illustrate the first few:



As you can see from the diagrams, the gnomons for the centred m-gonal numbers are successive multiples of m, illustrated in this table:

Type	Seed sequence	OEIS	Generated sequence	OEIS
Triangular	1, 3, 6, 9, 12, …	A008486	1, 4, 10, 19, 31, …	A005448
Square	1, 4, 8, 12, 16, …	A008574	1, 5, 13, 25, 41, …	A001844
Pentagonal	1, 5, 10, 15, 20, …	A008706	1, 6, 16, 31, 51, …	A005891
Hexagonal	1, 6, 12, 18, 24, …	A008458	1, 7, 19, 37, 61, …	A003215

As these centred seed sequences are in multiples of m for $n > 1$, there is no recurrence equation for all $M_m(n)$ that generates them. Rather, we can see directly:

$$M_m(n) = m(n-1), \quad M_m(1) = 1$$

Thus, the nth centred m-gonal number is given by this formula, as the partial sums of the seed sequences:

$$C_m(n) = \sum_{i=1}^{n} M_m(i) = \sum_{i=1}^{n} (1 + m(i-1))$$

From this we have this recurrence equation:[12]

$$C_m(n+1) = C_m(n) + M_m(n+1) = C_m(n) + mn, \quad C_m(1) = 1$$

Solving this equation, the general formula for the partial sums of the seed sequences is:

$$C_m(n) = 1 + \frac{m \cdot n}{2}(n-1) = \frac{mn^2 - mn + 2}{2} = \frac{m}{2}n(n-1) + 1$$

For instance, here are the specific formulae for the first few centred polygonal numbers.[13]

Type	Gnomon	Sequence
Triangular	$3(n-1)$	$\frac{3}{2}n(n-1) + 1$
Square	$4(n-1)$	$2n(n-1) + 1$
Pentagonal	$5(n-1)$	$\frac{5}{2}n(n-1) + 1$
Hexagonal (Hex)[14]	$6(n-1)$	$3n(n-1) + 1 = n^3 - (n-1)^3$

Like the standard polygonal numbers, the centred ones can be used as seed sequences for the next level of generated sequences, as centred pyramidal numbers in three and more dimensions.[15] Here is the general formula:

$$C_m^d(n+1) = C_m^d(n) + C_m^{d-1}(n+1) \qquad C_m^d(1) = 1$$

For instance, the centred tetrahedral numbers are generated from:

$$C_3^3(n+1) = C_3^3(n) + C_3^2(n+1) = C_3^3(n) + \frac{3}{2}n(n+1) + 1 \qquad C_m^3(1) = 1$$

Solving, the formula for 3-dimensional centred pyramidal numbers is:

$$C_3^3(n) = \frac{1}{2}n(n^2+1)$$

And the general formula is:

$$C_m^3(n) = \frac{1}{6}n(mn^2 - m + 6)$$

When $m = 6$, the three-dimensional extension of the hex numbers are cubes.[16]. Feeding the general result into the general recurrence formula gives:

$$C_m^4(n) = \frac{1}{24}n(n+1)(mn^2 + mn - 2m + 12)$$

Moving to the next dimension, we have:

$$C_m^5(n) = \frac{1}{120}n(n+2)(mn^3 + 3mn^2 - mn - 3m + 20n + 20)$$

Out of interest, here is the next one:

$$C_m^6(n) = \frac{1}{720}n(n+3)(mn^4 + 6mn^3 + 7mn^2 - 8m + 30n^2 + 90n + 60)$$

It is therefore not easy to determine a general formula of the *d*-th dimension, *m*-gonal centred 'pyramidal' numbers. Nevertheless, here are the first few sequences with their OEIS references, with the 5-D centred 'pyramidal' numbers not included.

Type	3-D	OEIS	4-D	OEIS	5-D	OEIS
Triangular	1, 5, 15, 34, 65, …	A006003	1, 6, 21, 55, 120, …	A002817	1, 7, 28, 83, 203, …	—
Square	1, 6, 19, 44, 85, …	A005900	1, 7, 26, 70, 155, …	A006325	1, 8, 34, 104, 259, …	A033455
Pentagonal	1, 7, 23, 54, 105, …	A004068	1, 8, 31, 85, 190, …	A006322	1, 9, 40, 125, 315, …	A006414
Hexagonal	1, 8, 27, 64, 125, …	A000578	1, 9, 36, 100, 225, …	A000537	1, 10, 46, 146, 371, …	A024166

Of particular note is that the sum of the first *n* cubes (A000537) is the square of the *n*th triangular number, [17] known as Nichmachus's theorem[18] after Nichmachus of Gerasa (c.60–c.120 CE), who included it in *Introduction to Arithmetic*, more a systematic, mystical treatise than a formal mathematical one, such as Euclid's *Elements*.[19] That is:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

And here are some particular instances of the *n*th terms to illustrate the unknown general pattern:

Type	2-D	3-D	4-D	5-D
Triangular	$\frac{3}{2}n(n-1)+1$	$\frac{1}{2}n(n^2+1)$	$\frac{1}{8}n(n+1)(n^2+n+2)$	$\frac{1}{120}n(n+2)(3n^3+9n^2+17n+11)$
Square	$2n(n-1)+1$	$\frac{1}{3}n(2n^2+1)$	$\frac{1}{6}n(n+1)(n^2+n+1)$	$\frac{1}{30}n(n+1)(n+2)(n^2+2n+2)$
Pentagonal	$\frac{5}{2}n(n-1)+1$	$\frac{1}{6}n(5n^2+1)$	$\frac{1}{24}n(n+1)(5n^2+5n+1)$	$\frac{1}{24}n(n+2)(n^3+3n^2+3n+1)$
Hexagonal	$3n(n-1)+1$	n^3	$\frac{1}{4}n^2(n+1)^2$	$\frac{1}{60}n(n+2)(3n^3+9n^2+7n+1)$

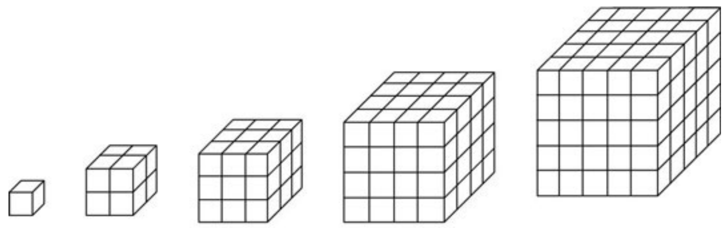
Type	2-D	3-D	4-D	5-D
Heptagonal	$\frac{7}{2}n(n-1)+1$	$\frac{1}{6}n(7n^2-1)$	$\frac{1}{24}n(n+1)(7n^2+7n-2)$	$\frac{1}{120}n(n+2)(7n^3+21n^2+13n-1)$

Rather than generating higher dimensional figurate numbers from polygonal and centred polygonal numbers, polyhedral and higher-dimensional polytopic numbers can be generated directly from more general recurrence equations. The most obvious place to start is with the regular polytopes, such as the five Platonic solids in three dimensions. This is what Hyun Kwang Kim did in 2002 in a paper titled 'On Regular Polytope Numbers', further explored in 2012 in *Figurate Numbers* by Elena Deza and Michel Marie Deza, the long-awaited textbook on the subject. However, the algorithms are rather involved, not fully explained in the Wiki pages of *The On-Line Encyclopedia of Integer Sequences* on 'Platonic numbers' and 'Centered Platonic numbers'.[20] So in this edition of this chapter, I'll mainly present the results rather than how they are generated.

We have already seen three of the Platonic numbers. The first are the tetrahedral numbers, $P_3^3(n)$, which we can denote as $Pl_4(n)$, where the subscript denotes the number of vertices in $Pl_V(n)$. This triangular pyramid is depicted on page 189. Next to this photo is a depiction of the square pyramid, which can be regarded as the top of an octahedron, viewed as a dipyramid. So,

$$Pl_6(n) = P_4^3(n)+P_4^3(n-1) = \frac{1}{6}n(n+1)(2n+1) + \frac{1}{6}(n-1)n(2n-1) = \frac{1}{3}n(2n^2+1) = C_4^3(n)$$

Regarding the cubic numbers, these are the three-dimensional extensions of the centred hexagonal numbers $C_6^3(n)$, as Conway and Guy illustrate.[21] More directly, this is how cubes can be stacked, depicting the cubic number as $Pl_8(n)$:



More generally, Deza and Deza describe the more complex general procedure for generating all five Platonic numbers, as an extension of the diagrams that generate the polygonal numbers. Here is a summary:

Platonic solid	Sequence	OEIS	*n*th term
Tetrahedron	1, 4, 10, 20, 35, …	A000292	$\frac{1}{6}n(n+1)(n+2)$
Octahedron	1, 6, 19, 44, 85, …	A005900	$\frac{1}{3}n(2n^2+1)$
Cube	1, 8, 27, 64, 125, …	A000578	n^3
Icosahedron	1, 12, 48, 124, 255, …	A006564	$\frac{1}{2}n(5n^2-5n+2)$
Dodecahedron	1, 20, 84, 220, 455, …	A006566	$\frac{1}{2}n(3n-1)(3n-2) = \binom{3n}{3}$

Like the ordinary Platonic numbers, the centred Platonic numbers can be generated from a 3-dimensional generalization of the way that the centred polygonal numbers are generated, depicted on page 190. Again, the general recurrence equation for this class of five sequences is rather involved. However, the general formula for the *n*th centred Platonic polyhedron with *V* vertices is:[22]

$$CPl_V(n) = \frac{(2n+1)(k_V n^2 + k_V n + 3)}{3}$$

where $k_V = \{1, 2, 3, 5, 15\}$ for $V = \{4, 6, 8, 12, 20\}$, respectively. Here are the particular terms:

Centred	**Sequence**	**OEIS**	***n*th term**
Tetrahedron	1, 5, 15, 35, 69, …	A005894	$\frac{1}{3}(2n-1)(n^2-n+3)$
Octahedron	1, 7, 25, 63, 129, …	A001845	$\frac{1}{3}(2n-1)(2n^2-2n+3)$
Cube	1, 9, 35, 91, 189, …	A005898	$n^3+(n-1)^3=(2n-1)(n^2-n+1)$
Icosahedron	1, 13, 55, 147, 309, …	A005902	$\frac{1}{3}(2n-1)(5n^2-5n+3)$
Dodecahedron	1, 33, 155, 427, 909, …	A005904	$(2n-1)(5n^2-5n+1)$

Hyun Kwang Kim, using his general recurrence equation for generating the regular polytopic numbers, presents the formulae for their *n*th terms in the six 4-dimensional regular polytopes, where the Schläfli symbol in described in the final section of this chapter:[23]

Polytope	**Schläfli symbol**	**Sequence**	**OEIS**	***n*th term**
5-cell	{3, 3, 3}	1, 5, 15, 35, 70, …	A000332	$\frac{1}{24}n(n-1)(n-2)(n-3)=\binom{n}{4}$
16-cell	{3, 3, 4}	1, 8, 33, 96, 225, 456, 833 …	A014820	$\frac{1}{3}n^2(n^2+2)$
Tesseract	{4, 3, 3}	1, 16, 81, 256, 625, …	A000583	n^4
24-cell	{3, 4, 3}	1, 24, 153, 544, 1425, …	A092181	$n^2(3n^2-4n+2)$
600-cell	{3, 3, 5}	1, 120, 947, 3652, 9985, …	A092182	$\frac{n}{6}(145n^3-280n^2+179n-38)$
120-cell	{5, 3, 3}	1, 600, 4983, 19468, 53505, …	A092183	$\frac{n}{2}(261n^3-504n^2+283n-38)$

After this, only the first three polytopes extend into five dimensions and more, indefinitely. The regular simplexes and hypercubes or measure polytopes, which Harold Scott MacDonald ‘Donald’ Coxeter (1907–2003) designates α_n and γ_n, respectively,[24] are intuitively straightforward. The *n*th term for the *d*-dimensional simplex numbers is simply:

$$\binom{n+d-1}{d}$$

And the *n*th term for the *d*-dimensional hypercube numbers is simply the *d*th power of *n*: n^d. However, as Kim states, “the formulæ for the cross-polytope [β_n] numbers are complicated and look unnatural.” Nevertheless, there is a close relationship between cross- and measure-polytope numbers, involving Eulerian numbers, which we do not need to explore further for the moment.

There is no need to stop here with figurate numbers. There are centred polytopic numbers in higher dimensions. And if we were to consider nonconvex regular (e.g. stellated) polytopic numbers or consider nonregular (e.g. Archimedean solids) polytopic numbers, this “would open the door to a humongous number of possibilities", as a contributor to the OEIS Wiki's page on 'Classifications of figurate numbers' tells us.[25]

To give three examples from Conway and Guy, we can generate figurate numbers by cutting off corners from the regular polyhedra, stellating them, or with duals of the Archimedean polyhedra.

For instance, we can form truncated tetrahedral numbers by chopping off tetrahedral numbers from each corner of a tetrahedral number in this manner:[26]

$$TP_3^3(n) = P_3^3(3n-2) - 4P_3^3(n-1) = \tfrac{1}{2}(3n-2)(3n-1)n - \tfrac{2}{3}(n-1)n(n+1) = \tfrac{1}{6}n(23n^2 - 27n + 10)$$

The truncated tetrahedral numbers generated from this recurrence equation are 1, 16, 68, 180, 375, … (A005906 in OEIS).

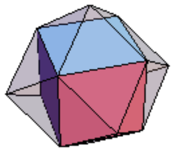
Johannes Kepler, my all-time favourite scientist, is quoted as naming the only stellation of the octahedron the *stella octangular* in 1611 in *De Nive Sexangula* (The Six-Cornered Snowflake).[27] We can form stella octangular numbers by adding tetrahedral numbers to each face of octahedral numbers to form a compound of dual tetrahedral numbers:[28]

$$Stel(n) = Pl_6(n) + 8Pl_4(n-1) = \frac{1}{3}n(2n^2+1) + \frac{4}{3}(n-1)n(n+1)) = n(2n^2-1)$$

The stella octangular numbers generated from this recurrence equation are 1, 14, 51, 124, 245, … (A007588 in OEIS).

The rhombic dodecahedron is another fascinating polyhedron, as the dual of the cuboctahedron, consisting of twelve diamond-shaped faces corresponding to the edges of the cube and octahedron. It appears in nature as a garnet crystal.[29] It is one of three polyhedra that tesselate three-dimensional space, the others being cubes and truncated octahedra, much as triangles, squares, and hexagons tesselate two-dimensional space.

The rhombic dodecahedron can be viewed as a cube inside out, with six internal square pyramids appended to the faces of the cube.[30] A similar construct with the hypercube in four dimensions leads to the 24-cell, which Matt Parker delightfully calls a hyper-diamond, which is regular in four dimensions, consisting of 24 octahedra, 96 triangular faces, 96 edges, and 24 vertices,[31] which exists only in four dimensions.



In three dimensions, rhombic dodecahedral numbers can therefore be formed by adding square pyramidal numbers to the six faces of centred cubic numbers:[32]

$$Rho(n) = CPl_8(n) + 6P_4^3(n-1) = (2n-1)(n^2-n+1) + (n-1)n(2n-1) = (2n-1)(2n^2-2n+1)$$

The rhombic dodecahedral numbers generated from this recurrence equation are 1, 15, 65, 175, 369, … (A005917 in OEIS). Another interesting pattern appears here. Rhombic dodecahedral numbers are the differences of consecutive powers of four, just as hex numbers are the differences between consecutive cubes. For

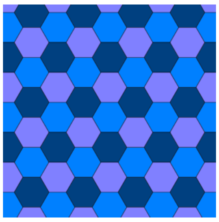
$$Rho(n) = n^4 - (n-1)^4 = n^4 - (n^4 - 4n^3 + 6n^2 - 4n + 1) = (2n-1)(2n^2-2n+1)$$

These similarities are two instances of a class of figurate numbers, known as nexus numbers, built up of the nexus of cells fewer than n steps away from a given cell. The nth d-dimensional nexus number is given by, increasing n by 1 to make the coefficients of the powers of n positive:

$$N_d(n) = \sum_{k=0}^{d} \binom{d+1}{k} = (n+1)^{d+1} - n^{d+1}$$

Here are the first few nexus numbers, to illustrate the general pattern:

d	Name	OEIS	$N_d(n)$	Sequence (from n = 0)
0	Unit	A000012	1	1, 1, 1, 1, 1, …
1	Odd number	A005408	$2n+1$	1, 3, 5, 7, 9, …
2	Hex number	A003215	$3n^2+3n+1$	1, 7, 19, 37, 61, …
3	Rhombic dodecahedral	A005917	$4n^3+6n^2+4n+1$	1, 15, 65, 175, 369, …
4		A022521	$5n^4+10n^3+10n^2+5n+1$	1, 31, 211, 781, 2101, …
5		A022522	$6n^5+15n^4+20n^3+15n^2+6n+1$	1, 63, 665, 3367, 11529, …
6		A022523	$7n^6+21n^5+35n^4+35n^3+21n^2+7n+1$	1, 127, 2059, 14197, 61741, …



Conway and Guy describe the way that the hex numbers are formed by referring to a hexagonal tessellation, where each dot in the figure for the centred hexagonal numbers is replaced by a hexagonal cell, as in this diagram, like a bee's honeycomb. $N_2(n)$ is then the number of hexagons at a distance n or less from the referenced hexagonal cell.

In three dimensions, the situation is somewhat trickier. The base honeycomb consists of truncated octahedra, but the number of truncated octahedra $N_3(n)$ within a nexus of any one cell has the shape of a rhombic dodecahedron.[33] In contrast, in the case of the hex numbers, the base cells and the nexus constructed around the cells have the same shape.

In *Regular Polytopes*, Coxeter refers to honeycombs as 'degenerate' polytopes, designated δ_n.[34] However, are there higher dimensional honeycombs, corresponding to truncated octahedra in three dimensions, as the centre around which the higher dimensional nexus numbers could be formed? Indeed, there are, as we see in the section on permutatopes on page 328. In the meantime, it is now appropriate to end this brief review of the figurate numbers.

Multinomials and Pascal's pyramidal simplexes

Another way that sequences and series are generated is as the coefficients of multinomials, consisting of a polynomial with m variables raised to the power of n. The most basic case are the binomials, whose coefficients can be arranged in what is commonly called Pascal's triangle, named after Blaise Pascal (1623–1662) because he was the first to make any sort of systematic study of the relationships in its internal structure. When the results of his studies in *Traité du triangle arithmétique* (*A Treatise on the Arithmetical Triangle*) were posthumously published in 1665, he began with the triangle on the next page, which I have simplified to highlighted the essential structure.[35]

Pascal was not the first to present what can be also called a *Figurate Triangle*, a *Combinatorial Triangle*, or a *Binomial Triangle*, as A. W. F. Edwards tells us in *Pascal's Arithmetical Triangle*,[36] which provides an excellent history of this evolutionary development. For instance, Michael Stifel (1487–1567) gave a form of the Figurate Triangle in 1544 and 1545, when studying the extraction of roots, extending the triangular numbers to four and more dimensions, long after the Greeks had studied the first few dimensions. Then in 1570, Gerolamo Cardano (1501–1576) presented a form of the Combinatorial Triangle in his *Opus novum*. Regarding the Binomial Triangle, Chu Shih-Chieh presented the binomial coefficients to the eighth power in *Precious Mirror of the Four Elements* in 1303, extending the work of Chia Hsien around 1100.[37] As David M. Burton tells us, "The first triangular arrangement of the binomial coefficients to be printed in European books appeared on the title page of the *Rechnung* (1527) of Peter Apian (1495–1552)".[38] It appeared again in 1545 in *De numeris* by Johann Scheubel (1494–1570), known as Scheubelius, and in 1556 in *General trattato* by Niccolò Fontana Tartaglia (1499/1500–1557).[39]

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	
3	1	3	6	10	15	21	28	36		
4	1	4	10	20	35	56	84	*Parallel ranks*		
5	1	5	15	35	70	126				
6	1	6	21	56	126	*Arithmetical Triangle*				
7	1	7	28	84	*Perpendicular ranks*					
8	1	8	36							
9	1	9								
10	1									

Rather than starting with any of these particular ways to view what Pierre Raymond de Montmort (1678–1719) was to call 'Pascal's Triangle' in 1708, about August 1654 Pascal set out to write a treatise of pure mathematics, starting afresh from a few basic principles. This is an example of evolutionary convergence, which reaches its glorious culmination in Integral Relational Logic, presented in Chapter 2.

Indeed, Pascal begins with the top-left-hand corner number, which he calls the *generator*, extending this into the first row and column, from which the entire triangle can be generated by adding the numbers above and to the left of each cell. He then proceeds to list eighteen corollaries of the relationships between these numbers.[40] In Part II, Pascal then shows how his triangle can be used in the theories of figurate number and combinations and to find the powers of binomial equations.[41] *Traité du triangle arithmétique* thus has a similar structure to Descartes' *Discourse on the Method*, in which *Geometry* is an example of the method in practice, and this book, in which the last three chapters show how Integral Relational Logic can be used to map the whole of mathematics as a generative science of patterns and relationships, lying within the Cosmic Psyche.

Beyond the triangle, binomial coefficients can be arranged in a triangular pyramid, and in higher dimensional polytopes, matching the multidimensional structure of the Universe as a whole, not so easy to visualize geometrically. Nevertheless, let us begin with the most general form of the multinomial, where the coefficients of the variables are 1, which is this monstrous expression:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{t=1}^{m} x_t^{k_t}$$

where

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!\; k_2! \ldots, k_m!}$$

The total number of terms in such a multinomial is m^n, the first few totals being given in this table:

*m**n*	**1**	**2**	**3**	**4**	**5**	**6**
1	1	1	1	1	1	1
2	2	4	8	16	32	64
3	3	9	27	81	243	729
4	4	16	64	256	1,024	4,096
5	5	25	125	625	3,125	15,625
6	6	36	216	1,296	7,776	46,656

Thankfully, these terms are not all unique. For instance, if we denote the variables as *a*, *b*, *c*, *d*, …, the coefficient for $abcd^2$ in $(a + b + c + d)^5$ is

$$\frac{5!}{1!\,1!\,1!\,2!} = \frac{120}{2} = 60$$

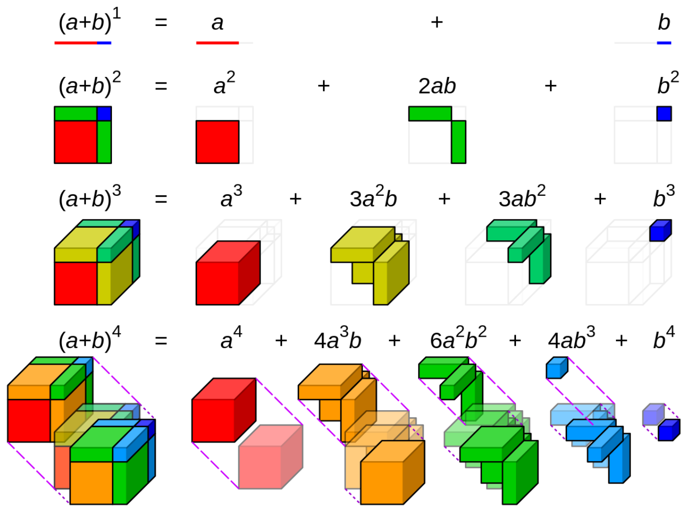
one of 56 distinct terms, totalling $4^5 = 1024$. In general, the number of distinct terms in $(x_1 + x_2 + \cdots + x_m)^n$ is given by:

$$\binom{n+m-1}{m}$$

proved by the ingenious stars-and-bars method in combinatorics,[42] and enumerated in this table:

*m**n*	1	2	3	4	5	6
1	1	1	1	1	1	1
2	2	3	4	5	6	7
3	3	6	10	15	21	28
4	4	10	20	35	56	84
5	5	15	35	70	126	210
6	6	21	56	126	252	462

Considering the much simpler binomials, here are the first few polynomial expansions of $(a+b)^n$, visualized geometrically on the right, where the fourth power extends into the fourth dimension:[43]



$(a+b)^0 = 1$
$(a+b)^1 = a + b$
$(a+b)^2 = a^2 + 2ab + b^2$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 15a^2b^3 + 10ab^4 + b^5$

As can be seen, the number of terms increases by one with each increase in the power, and so can be arranged in a triangle.

If we let $a = 1$ and $b = x$, we have this general expression for the binomial coefficients:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

also known as the number of ways of selecting *k* items from a group of *n* items in combination theory, often pronounced '*n* choose *k*', where *n*! is factorial *n*, defined as the product of all the integers up to *n*, as we saw in Chapter 3. Some alternative notations for binomials, writable in a single line in text, are $C(n,k)$, ${}_nC_k$, and ${}^nC_k$, where *C* stands for *combinations* or *choices*.

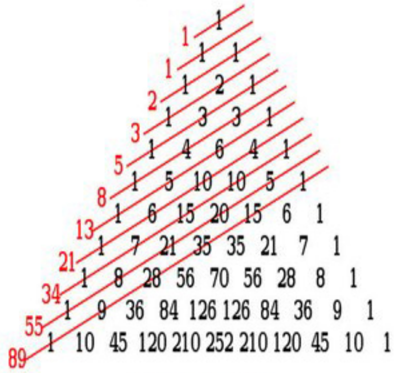
Now, as every term in the triangle in pyramidal form, other than those on the sloping outside edges, is the sum of the two terms above it, we have this recurrence equation, known as Pascal's triangle rule:[44]

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for $n > 0$, $0 \le k \le n$, $C(0,0) = C(n,0) = 1$, and $C(n,-1) = C(n,n+1) = 0$.

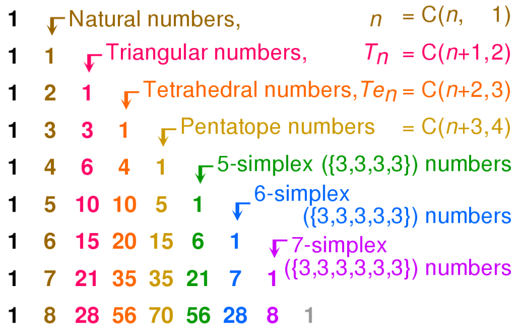
The most obvious property of Pascal's triangle is that the sum of the coefficients on the *nth* row is 2^n, which derives from the second row in the table on page 196. However, what is not clear is that each row can be read as a power of 11. For instance, on the third and fourth rows, $1331 = 11^3$ and $14641 = 11^4$. This formula is not so obvious on the fifth and subsequent rows, for the binomial coefficients are no longer single digits. Nevertheless, the formula does still hold, as we can see from the expansion of $(10+1)^n$:[45]

$$(10+1)^n = 11^n = \sum_{k=0}^{n} \binom{n}{k} 10^{n-k}$$



Rather than summing terms in the horizontal rows, the Fibonacci numbers, which we look at later, appear as the sum of terms in the shallow diagonals of Pascal's triangle, as illustrated here. Because, as Édouard Lucas (1842–1891) derived:[46]

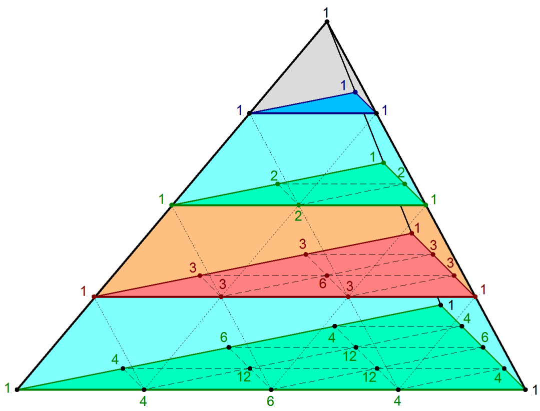
$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$



The most fundamental multidimensional figurate numbers also appear in Pascal's triangle, in the main diagonals, arranged vertically in this representation from Wikipedia. In the notation of the subsection on the Figurate numbers, these sequences are given by:

$$P_3^d(n) = \binom{n+d-1}{d}$$

where d ≥ 2, considering the natural numbers to be in one-dimensional space.



With a trinomial, Pascal's triangle becomes a triangular pyramid, not a tetrahedron, as some sources say, for the four faces are not treated equally. Rather, I would say that Pascal's pyramid is the second of Pascal's pyramidal simplexes, defined as simplexes with a 'base' that is a simplex of one dimension less than that of the simplex. This base represents the levels in the pyramidal simplexes as the powers of *n* increase. With a trinomial, we have a tetrahedron, with the expanding base being a triangle, illustrated here.[47] The first few expansions of the trinomial are:

$(a+b+c)^0 = 1$

$(a+b+c)^1 = a+b+c$

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

$(a+b+c)^3 = a^3+b^3+c^3+3a^2b+3a^2c+3b^2a+3b^2c+3c^2a+3c^2b+6abc$

$(a+b+c)^4$

$= a^4+b^4+c^4+4a^3b+4a^3c+4b^3a+4b^3c+4c^3a+4c^3b+6a^2b^2+6b^2c^2+6c^2a^2+12a^2bc+12b^2ac+12c^2ba$

Each level in the pyramid is a growing sequence of triangles, whose sums are successive powers of 3, illustrated in this table:

Tri 0	Tri 1	Tri 2	Tri 3	Tri 4	Tri 5
1	1	1	1	1	1
	1 1	2 2	3 3	4 4	5 5
		1 2 1	3 6 3	6 12 6	10 20 10
			1 3 3 1	4 12 12 1	10 30 30 10
				1 4 6 4 1	5 20 30 20 5
					1 5 10 10 5 1

With quadrinomials, we need a four-dimensional pyramidal simplex whose growing levels are a sequence of tetrahedra. Here are the first few expansions of the quadrinomial, of rapidly growing complexity:

$(a+b+c+d)^0 = 1$

$(a+b+c+d)^1 = a+b+c+d$

$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$

$(a+b+c+d)^3$
$= a^3+b^3+c^3+d^3+3a^2b+3a^2c+3a^2d+3b^2a+3b^2c+3b^2d+3c^2a+3c^2b+3c^2d+3d^2a+3d^2b+3d^2c$
$+ 6abc+6abd+6acd+6bcd$

$(a+b+c+d)^4$
$= a^4+b^4+c^4+d^4+4a^3b+4a^3c+4a^3d+4b^3a+4b^3c+4b^3d+4c^3a+4c^3b+4c^3d+4d^3a+4d^3b+4d^3c$
$+ 6a^2b^2+6a^2c^2+6a^2d^2+6b^2c^2+6b^2d^2+6c^2d^2+12a^2bc+12a^2bd+12a^2cd+12b^2ac+12b^2ad+12b^2ad$
$+ 12c^2ab+12c^2ad+12c^2bd+12d^2ab+12d^2ac+12d^2bc+24abcd$

Each level in the 4-simplex is a growing sequence of tetrahedra, whose sums are successive powers of 4, illustrated in this table:[48]

Tet 0	Tet 1	Tet 2	Tet 3	Tet 4
1	1	1	1	1
	1	2	3	4
	1 1	2 2	3 3	4 4
		1	3	6
		2 2	6 6	12 12
		1 2 1	3 6 3	6 12 6
			1	4
			3 3	12 12
			3 6 3	12 24 12
			1 3 3 1	4 12 12 4
				1
				4 4
				6 12 6
				4 12 12 1
				1 4 6 4 1

If, rather than setting the x_i in the general multinomial to letters of the alphabet, we set $x_i = x^{i-1}$, we form $(1+x+x^2+\cdots x^{m-1})^n$, where all Pascal's pyramidal simplexes collapse into triangles. Pascal's triangle, itself, does not collapse, for it is already in triangular form. In these multinomial triangles, the number of terms increases arithmetically, as the power increases, with the step each time being m - 1, like the seed sequences for the basic polygonal numbers. This table gives the number of terms in the multinomial expansions for the first few values of m and n.

$m \backslash n$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	2	3	4	5	6	7
3	3	5	7	9	11	13
4	4	7	10	13	16	19
5	5	9	13	17	21	25
6	6	11	16	21	26	31

Each term in an m-nomial triangle is the sum of the m terms symmetrically in the row above it, with the sloping sides being set to 1, with notional zeros outside the edges of the triangles, where necessary.

Another interesting sequence emerges from these multinomial triangles: the sequences represented by their central multinomial coefficients. For instance, the cells marked in yellow below are the central binomial coefficients of the binomial triangle (i.e. Pascal's triangle): 1, 2, 6, 20, 70, 252, 924, … (OEIS A000984).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

The central binomial coefficients are generated from:

$$a(n) = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

Next, the central trinomial triangle, as the collapse of Pascal's Pyramid, is generated from the coefficients of the expansion of $(1 + x + x^2)^n$, the sequence of the central trinomial coefficients being the largest in each row: 1, 1, 3, 7, 19, 51, 141, 393, … (OEIS A002426):

1
1 1 1
1 2 3 2 1
1 3 6 7 6 3 1
1 4 10 16 19 16 10 4 1
1 5 15 30 45 51 45 30 15 5 1
1 6 21 50 90 126 141 126 90 50 21 6 1
1 7 28 77 161 266 357 393 357 266 161 77 28 7 1
1 8 36 112 266 504 784 1016 1107 1016 784 504 266 112 36 8 1
1 9 45 156 414 882 2304 2304 2907 3139 2907 2304 2304 882 414 156 45 9 1

In turn, the central quadrinomial triangle is generated from the expansion of the coefficients of the expansion of $(1 + x + x^2 + x^3)^n$, the sequence of the central quadrinomial coefficients being the largest in each alternate row, like the binomial triangle and all even-termed multinomials: 1, 4, 44, 580, 8092, 116304, … (OEIS A005721):

1
1 1 1 1
1 2 3 4 3 2 1
1 3 6 10 12 12 10 6 3 1
1 4 10 20 31 40 44 40 31 20 10 4 1
1 5 15 35 65 101 135 155 155 135 101 65 35 15 5 1
1 6 21 56 120 216 336 456 546 580 546 456 336 216 120 56 21 6 1

The *On-line Encyclopedia of Integer Sequences* calls the next sequence central *pentanomial* coefficients. But this is mixing Greek and Latin roots, which philologists tell us should be avoided whenever possible. So, what I prefer to call the sequence of central *quinquenomial* coefficients is 1, 1, 5, 19, 85, 381, 1751, … (OEIS A005191), highlighted in this quinquenomial triangle:

1
1 1 1 1 1
1 2 3 4 5 4 3 2 1
1 3 6 10 15 18 19 18 15 10 6 3 1
1 4 10 20 35 52 68 80 85 80 68 52 35 20 10 4 1
1 5 15 35 70 221 185 255 320 365 381 365 320 255 185 221 70 35 15 5 1
1 6 21 56 126 246 426 666 951 1246 1506 1686 1751 1686 1506 1246 951 666 426 246 126 56 21 6 1

There is, of course, an infinite sequence of multinomial triangles, the next few being variously named *sextinomial*, *sexanomial*, or *hexanomial*; *septinomial* or *heptanomial*; and *octonomial*, but there is no need to explore these any further now that the general pattern is clear, albeit of increasing complexity.

Lucas, Fibonacci, and Pell numbers

Difference equations, which generate the figurate numbers, appear in many other branches of mathematics, not the least in nonlinear systems theory and combinatorics. I introduce the former in my 2016 book *Through Evolution's Accumulation Point: Towards Its Glorious Culmination*, where I describe how the logistic map can model all 13.8 billion years of evolution since the most recent big bang, explaining why all these aeons of evolution are degenerating into political, that is, psychological chaos at the moment.

In that book, as an introduction to nonlinear difference equations, I briefly describe the famous Fibonacci sequence, generated from a second-order linear recurrence equation. In this subsection, I revisit this sequence, as the Fibonacci numbers, as a special case of what are today called Lucas sequences, leading to the so-called Pell equation for estimating the square roots of non-square integers.

In 1878, Édouard Lucas (1842–1891), a professor at Lycée Charlemagne in Paris, wrote an extensive paper titled '*Théorie des fonctions numériques simplement périodiques*' (The Theory of Simply Periodic Numerical Functions),[49] whose purpose was "to study the symmetric functions of the roots of a quadratic equation, and their application to the theory of prime numbers".[50] We saw in Chapter 3 'From Zero to Transfinity' how Lucas had developed a method two years earlier for determining which Mersenne numbers are prime or composite. In this paper, he went further, introducing a general second-order linear recurrence equation:

$$x_n = P \cdot x_{n-1} - Q \cdot x_{n-2}$$

where P and Q are the relatively prime coefficients of this quadratic equation:

$$x^2 - Px + Q = 0$$

This is called a *characteristic polynomial* in linear algebra[51] (which we look at in Chapter 5, to be written in the next year), whose roots a and b are the eigenvalues of this matrix M, with determinant Q:

$$M = \begin{pmatrix} P & -Q \\ 1 & 0 \end{pmatrix}$$

giving

$$a = \frac{P + \sqrt{D}}{2} \qquad \text{and} \qquad b = \frac{P - \sqrt{D}}{2}$$

where $D = P^2 - 4Q = (a-b)^2$, called the *discriminant* of the polynomial. Expressing P and Q in terms of a and b gives, as for any quadratic equation:

$$P = a + b \qquad \text{and} \qquad Q = ab$$

Lucas then defined a pair of complementary sequences, similar to ones that Jacques Philippe Marie Binet (1786–1856) had defined in 1843:[52]

$$U_n = \frac{a^n - b^n}{a - b}, \quad V_n = a^n + b^n$$

We can define V_n in terms of U_n in this way, for $a \neq b$ and $n > 0$:

$$V_n = a^n + b^n = \frac{(a^n + b^n)(a^n - b^n)}{a^n - b^n} = \frac{a^{2n} - b^{2n}}{a^n - b^n} = \frac{U_{2n}}{U_n}$$

Also, for $a \neq b$ and $n > 0$:

$$(a-b)V_n = (a-b)(a^n + b^n) = a^{n+1} + ab^n - ba^n - b^{n+1} = a^{n+1} - b^{n+1} - ab(a^{n-1} - b^{n-1})$$

So, when $Q = ab = -1$, we have:

$$V_n = a^n + b^n = \frac{a^{n+1} - b^{n+1}}{a - b} + \frac{a^{n-1} - b^{n-1}}{a - b} = U_{n+1} + U_{n-1}$$

Setting $n = 0$ and 1 in the formulae for U_n and V_n gives this general pair of second-order linear recurrence equations, generating Lucas sequences:[53]

$$U_n = P \cdot U_{n-1} - Q \cdot U_{n-2} \qquad U_0 = 0, \quad U_1 = 1$$
$$V_n = P \cdot V_{n-1} - Q \cdot V_{n-2} \qquad V_0 = 2, \quad V_1 = P$$

Looking now at some particular cases, when $(P, Q) = (1, -1)$, we have the famous Fibonacci sequence (F_n) and its complementary sequence of what are today called Lucas numbers (L_n), with these simple recurrence equations:

$$F_n = F_{n-1} + F_{n-2} \qquad F_0 = 0, \quad F_1 = 1$$
$$L_n = L_{n-1} + L_{n-2} \qquad L_0 = 2, \quad L_1 = 1$$

Here are the first few terms generated from these equations:

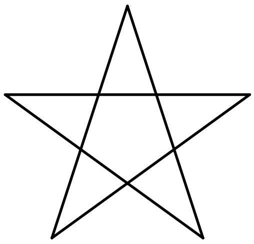
OEIS	0	1	2	3	4	5	6	7	8	9	10
A000045	0	1	1	2	3	5	8	13	21	34	55
A000032	2	1	3	4	7	11	18	29	47	76	123

In this case, the discriminant *D* is 5 and the roots *a* and *b* of the characteristic equation are:

$$\frac{1 \pm \sqrt{5}}{2}$$

The positive root is the familiar *Golden Ratio*, also called *Golden Mean*[54] or *Golden Section*, influenced by Euclid's definition: "A straight line is said to have been **cut in extreme and mean ratio** when, as the whole line is to the greater segment, so is the greater to the less,"[55] illustrated here:

Influenced by what the Italian painter Piero della Francesca (c. 1415–1492) called the *Divine Proportion*,[56] in 1509 the Italian mathematician Luca Pacioli (c. 1447–1517) published a book titled *De Divina Proportione*, illustrated by his friend Leonardo da Vinci (1452–1519), describing thirteen properties.[57] For instance, the Divine Proportion is evident in the pentagram, known to the Pythagoreans, although it wasn't until 1835 that Martin Ohm—the brother of George Simon Ohm, who gave his name to Ohm's law in electromagnetism—applied the honorific epithet *Golden* to the Divine Proportion.[58] In mathematical terms, the Golden Section is simply defined by this equation:



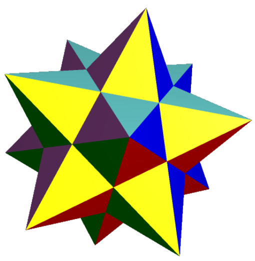
$$\frac{x+1}{x} = \frac{x}{1}$$

giving

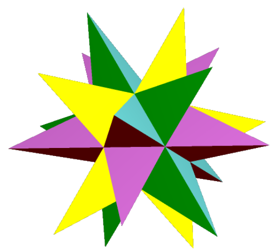
$$x^2 = x + 1$$

which is Lucas's characteristic equation for $(P, Q) = (1, -1)$.

In professional mathematical literature, the Golden Ratio is denoted by *tau* (τ), from Greek *tomos* 'a cutting, section', from *temnein* 'to cut', root of *atom* (not-cut), *tome*, and *tomography*. However, at the beginning of the twentieth century, Mark Barr (1871–1950) denoted the Golden Ratio with *phi* (φ), the initial letter of the sculptor Phidias (c. 490–430 BCE),[59] and this is how it is most popularly symbolized today. The negative root is 1 - φ, often called *psi* (ψ).



Johannes Kepler was well aware of the Divine Proportion, helping him greatly in his quest to discover the unifying harmony that underlies geometry, music, poetry, architecture, and astronomy, enabling him to discover the small and great stellated dodecahedra in 1619 in *Harmonices Mundi* (*The Harmony of the World*).[60]



Earlier, Kepler had become became fascinated by the Divine Proportion in 1594, when he was appointed District Mathematician in Graz despite studying theology at Tübingen University.[61] During the next three years, he wrote *Mysterium Cosmographicum* (*The Secret of the Universe*),[62] saying, "Geometry has two great treasures; one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."[63]

Then in *The Six-Cornered Snowflake* in 1611, Kepler wondered why in most trees and bushes their flowers unfold in a five-sided pattern, with five petals, turning again to the Divine Proportion to explain this natural phenomenon. In particular, he noticed that the ratio of consecutive terms in the sequence 1, 1, 2, 3, 5, 8, 13, 21 approaches a constant value, saying, "It is in the likeness of this self-developing series that the faculty of propagation is, in my opinion, formed," foreseeing mathematics as a generative science of patterns and relationships emerging directly from the Divine Origin of the Universe, as this book is demonstrating. Kepler noted, "it is impossible to provide a perfect example in round numbers. However, the further we advance from the number one, the more perfect the example becomes."[64]

However, it was to take another century or two before mathematicians discovered:

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2}=\varphi$$

From this and the definition of U_n, we have this monstrous expression:

$$F_n=\frac{\phi^n-\psi^n}{\phi-\psi}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

This formula is usually credited to Binet, although it was well understood more than a century earlier than 1843 by Abraham de Moivre (1667–1754), Daniel Bernouilli (1700–1782), and Leonhard Euler (1707–1783).[65] It is a rather strange formula, for F_n is an integer, and the right-hand side does not appear to be so. However, the root-fives cancel out for all values of *n*. So all is well.

From this, we can determine the *n*th Lucas number:

$$L_n=F_{n-1}+F_{n+1}=\phi^n+\psi^n=\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n$$

The Fibonacci sequence is so named because Lucas attributed *Leonardo Fibonacci* to the Lucas sequence for (P, Q) = (1, -1) in his seminal paper.[66] However, Leonardo Pisano was not the first to discover this sequence. It seems that it first appeared in *Chandahshāstra* 'The Art of Prosody', by the Sanskrit grammarian Pingala, sometime between 450 and 200 BCE,[67] later being associated with a number of other Indian thinkers, all of whom lived and worked before Leonardo.[68]

Fibonacci's famous puzzle about rabbits is contained in the extensive Chapter 12 of *Liber Abbaci* (*Book of Calculation*) from 1228, the only edition that is extant, written to illustrate the power of the Indo-Arabic notation for numbers, as described in Chapter 3. Fibonacci presented 259 worked examples in this chapter,[69] the problem that led to *Fibonacci* becoming widely known in popular culture lying between a technique to find perfect numbers and a puzzle about the numbers of *denari* four men each hold, when the totals of three of them are known. Regarding the former, having found the first three perfect numbers, Leonardo wrote, "always doing thus you will be able to find perfect numbers without end,"[70] not realizing that the method breaks down with the fifth prime number, as we see in Chapter 3.

Fibonacci rather whimsically asked 'How Many Pairs of Rabbits Are Created by One Pair in One Year', beginning his investigation with this statement, in L. E. Sigler's translation:[71]

> A certain man put one pair of rabbits in a certain place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.

David M. Burton provides this translation,[72] similar to a number of others I have seen in the literature:

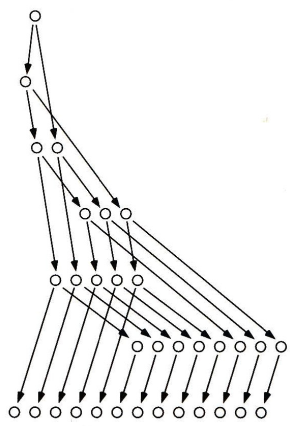
> A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair bears a new pair which from the second month on becomes productive?

Fibonacci provided a table of his calculations in the margin of his book, telling us how he calculated the number of rabbits in the field after twelve months of breeding.[73] As you can see, at the end of the first

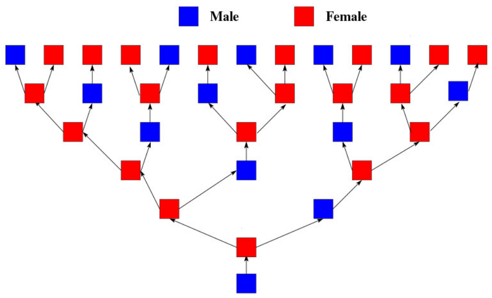
beginning	1
first	2
second	3
third	5
fourth	8
fifth	13
sixth	21
seventh	34
eighth	55
ninth	89
tenth	144
eleventh	233
end	377

month, he thought that there would be two rabbits, while there would only be one, unless the rabbits had begun to breed one month before being put into the field. So, at the end of the twelfth month, there should have been 233 rabbits in total. Furthermore, what we call F_2 today was the zeroth term in his sequence. So 377 is actually F_{14} rather than F_{12}. One other unique property of the Fibonacci sequence is that because of the way it is formed, the partial sum of the first n terms is $F_{n+2} - 1$.[74] This is almost as if the sequence contains its own gnomons within it.

However, as the way that Fibonacci formulated the problem cannot be extended indefinitely in a biological sense, Conway and Guy suggested that another way of looking at the problem of breeding rabbits is by wondering how many pairs of rabbits would be produced in the nth generation, starting from a single pair and supposing that any pair of rabbits of one generation produces one pair of rabbits for the next generation and one for the generation after that, and then they die, providing the hierarchical schema on the right.[75]



Alternatively, in *The Divine Proportion* and *The Golden Ratio*, respectively, H. E. Huntley and Mario Livio suggest that a more appropriate biological metaphor is of that of honey bees, in which males, as drones, are born from unfertilized eggs, and so have only the queen as a parent, while female bees, including workers, have both female and male parents.[76] The number of female, male, and all bees in each generation follows a Fibonacci sequence. T. C. Scott and P. Marketos provide this ancestor tree of this model, pointing out that beeswax was a major commodity in Bugia (Béjaïa in modern Algeria),[77] where Leonardo moved as a youth to be with his father, acting as a 'public official', as he tells us in his brief autobiography.[78] However, whether Fibonacci had this reproductive system of bees in his subconscious mind when formulating his puzzle seems rather unlikely.



Be that as it may, these complementary graphs, as illustrations of the underlying structure of the Cosmos, are examples of evolutionary hierarchies in Integral Relational Logic, defined in Chapter 2. Fibonacci did not envisage rabbits emerging from our Divine Source, as Emptiness. Nevertheless, by Lucas defining $F_0 = 0$, this is just how mysticism and mathematics can be seen as an inseparable pair of opposites.

We'll later look further at the Fibonacci sequence in their appearance in spirals. But first, let us look at another pair of Lucas sequences. When $(P, Q) = (2, -1)$, what are called Pell (P_n) and Pell-Lucas (Q_n) sequences are generated:

OEIS	0	1	2	3	4	5	6	7	8	9	10
A000129	0	1	2	5	12	29	70	169	408	985	2,378
A002203	2	2	6	14	34	82	198	478	1,154	2,786	6,726

In this case, the discriminant D is 8 and the roots a and b of the characteristic equation are:

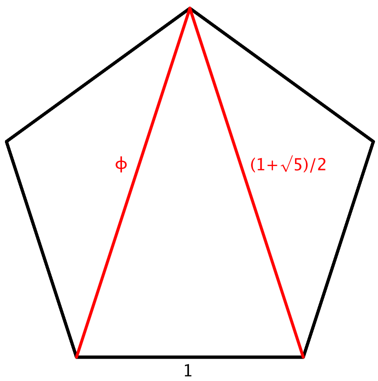
$$1 \pm \sqrt{2}$$

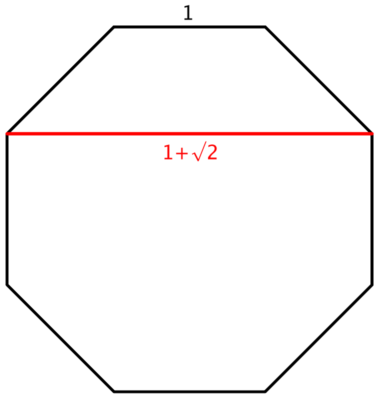
giving

$$P_n = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right]$$

$$Q_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$$

The positive root is called the Silver Ratio, denoted by δ_S. Here are a couple of diagrams illustrating the Golden and Silver Ratios geometrically, as the ratios of diagonals to the sides of regular pentagons and octagons, respectively. The upright isosceles triangle in the pentagon is called the Golden Triangle and the other two triangles are Golden Gnomons.[79]





Now, while Pell's numbers provide a convergent sequence to the value of $1+\sqrt{2}$ through this limit:

$$\lim_{n\to\infty}\frac{P_{n+1}}{P_n} = 1+\sqrt{2}$$

to obtain a convergent sequence to the value of $\sqrt{2}$ itself, we need to set $d = 2$ in Pell's equation, which has an ancient history, long before Euler misattributed Pell's name to this Diophantine equation, whose integer solutions lead to approximations for $\sqrt{d}$:

$$x^2 - dy^2 = 1$$

$\sqrt{2}$ is known today as Pythagoras's Constant because it is the measure of the hypotenuse of an isosceles right-angled triangle whose equal sides are of length 1.[80] Leonard Eugene Dickson tells us in *History of the Theory of Numbers* that the Indians and Greeks found approximations to $\sqrt{2}$ and other surds as early as 400 BCE.[81] However, it was to take about a thousand years before Brahmagupta found a method for finding a sequence of solutions to Pell's equation in 628 CE and another 1000 years before Fermat, known today as the founder of modern number theory,[82] issued a challenge to European mathematicians in 1657 to find general solutions, not aware of Brahmagupta's brilliant work and that of other Indian mathematicians who followed him.[83]

William Brouncker (1620–1684), who was to become the first president of the Royal Society of London, quickly found a solution using continued fractions, which John Wallis (1616–1703) published in 1658. In the same year, Johann Rahn (1622–1676), who was the first to use the symbol ÷ for division,[84] published an algebra book with the assistance of John Pell (1611–1685),[85] which contained an example of Pell's equation. This is only known connection between Pell and the equation that has been named after him. It is generally believed that Euler gave it that name around 1732 because he confused Brouncker and Pell, thinking that the major contributions which Wallis had reported on as due to Brouncker were in fact the work of Pell.[86] As H. W. Lenstra Jr. says, "attempts to change the terminology introduced by Euler have always proved futile." [87]

Turning now to solutions of Pell's equation, Lenstra says that because we can rewrite Pell's equation as

$$(x + y\sqrt{d}) \cdot (x - y\sqrt{d}) = 1$$

finding a solution comes down to finding a nontrivial unit of the ring $\mathbb{Z}[\sqrt{d}]$ of norm 1. This makes intuitive sense, even though, at the time of writing, I don't fully understand how to interpret $\mathbb{Z}[\sqrt{d}]$. Nevertheless, viewing $(x + y\sqrt{d})$ and $(x - y\sqrt{d})$ as norms of value 1 in linear algebra, it is not difficult to see that the nth solution (a_n, b_n) can be expressed in terms of the fundamental solution (a_1, b_1), with (a_0, b_0) being the trivial one (1, 0), thus:

$$a_n + b_n\sqrt{d} = (a_1 + b_1\sqrt{d})^n$$

Derek Smith provides an elegant proof by induction of this relationship,[88] showing that

$$a_{n+1} = a_1a_n + b_1b_nd$$
$$b_{n+1} = b_1a_n + a_1b_n$$

In this way

$$\sqrt{d} = \lim_{n\to\infty}\frac{a_n}{b_n}$$

Wolfram *MathWorld* gives the smallest integer solutions to Pell's equation as (a_1, b_1), which is sometimes the same as the first convergent of the continued fraction expansion of $\sqrt{d}$, using the algorithm for finding the continued fractions of square roots, described in Chapter 3. For $d = 2$, Pell's recurrence equations give these sequences for (a_n, b_n), which converge much faster than the convergents of the continued fraction for $\sqrt{2}$:

OEIS	0	1	2	3	4	5	6	7	8	9	10
A001541	1	3	17	99	577	3363	19601	114243	665857	3880899	22619537
A001542	0	2	12	70	408	2378	13860	80782	470832	2744210	15994428

Here is a table of the first few convergents of the first few values of $\sqrt{d}$, with $\sqrt{7}$ having a better starting value (8/3) than 3/1, as the first convergent of the continued fraction for $\sqrt{7}$.

$\sqrt{d}$	OEIS	1	2	3	4	5	6
$\sqrt{2}$	A001541 / A001542	$\frac{3}{2}$	$\frac{17}{12}$	$\frac{99}{70}$	$\frac{577}{408}$	$\frac{3363}{2378}$	$\frac{19601}{13860}$
$\sqrt{3}$	A001075 / A001353	$\frac{2}{1}$	$\frac{7}{4}$	$\frac{26}{15}$	$\frac{97}{56}$	$\frac{362}{209}$	$\frac{1351}{780}$
$\sqrt{5}$	A023039 / A060645	$\frac{9}{4}$	$\frac{161}{72}$	$\frac{2889}{1292}$	$\frac{51841}{23184}$	$\frac{930249}{416020}$	$\frac{16692641}{7465176}$
$\sqrt{6}$	A001079 / A001078	$\frac{5}{2}$	$\frac{49}{20}$	$\frac{485}{198}$	$\frac{4801}{1960}$	$\frac{47525}{19402}$	$\frac{470449}{192060}$
$\sqrt{7}$	A001081 / A001080	$\frac{8}{3}$	$\frac{127}{48}$	$\frac{2024}{765}$	$\frac{32257}{12192}$	$\frac{514088}{194307}$	$\frac{8193151}{3096720}$
$\sqrt{8}$	A001541 / A001109	$\frac{3}{1}$	$\frac{17}{6}$	$\frac{99}{35}$	$\frac{577}{204}$	$\frac{3363}{1189}$	$\frac{19601}{6930}$
$\sqrt{10}$	A078986 / A084070	$\frac{19}{6}$	$\frac{721}{228}$	$\frac{27379}{8658}$	$\frac{1039681}{328776}$	$\frac{39480499}{12484830}$	$\frac{1499219281}{474094764}$

If you are a young man in a hurry, we also can define:

$$a_{n+1} + b_{n+1}\sqrt{d} = (a_n + b_n\sqrt{d})^2$$

giving:

$$a_{n+1} = a_n^2 + b_n^2d$$
$$b_{n+1} = 2a_nb_n$$

Here are the first few convergents for $\sqrt{2}$ and $\sqrt{3}$ with this algorithm:

$\sqrt{d}$	OEIS	1	2	3	4	5
$\sqrt{2}$	A001601 / A051009	$\frac{3}{2}$	$\frac{17}{12}$	$\frac{577}{408}$	$\frac{665857}{470832}$	$\frac{886731088897}{627013566048}$

$\sqrt{d}$	OEIS	1	2	3	4	5
$\sqrt{3}$	A002812 / A071579	$\frac{2}{1}$	$\frac{7}{4}$	$\frac{97}{56}$	$\frac{18817}{10864}$	$\frac{708158977}{408855776}$

These sequences arise from Newton's iteration, an algorithm for computing $\sqrt{n}$, as an application of Newton's method, also called the Newton-Raphson method, for finding approximate roots for polynomials from a guesstimate of the first value.[89] In this case, the convergents are not particularly sensitive to the initial values; they converge just as fast.

Returning to Lucas sequences, so far, we have looked at two examples of $(P, Q) = (k, -1)$, with k = 1 and 2, giving the Golden and Silver Ratios. Naturally, this gives rise to another sequence of sequences, which Ron Knott calls *silver means*,[90] related to simple continued fractions where $a_n = k$ for all n. For, in this case, we can rewrite the characteristic polynomial as

$$x = k + \frac{1}{x} = k + \frac{1}{k + \frac{1}{x}} = k + \frac{1}{k + \frac{1}{k + \frac{1}{\ddots}}}$$

The positive root a of the characteristic polynomial is then:

$$a = \frac{k + \sqrt{k^2 + 4}}{2} = [k; \bar{k}]$$

with the negative of the negative root b being the fractional part of a, for

$$-b = a - k = \frac{1}{a}$$

The discriminant $D = k^2 + 4$ has this recurrence equation for $k \geq 0$:

$$d_{n+1} = d_n + n + 1 \qquad d_0 = 4$$

giving this sequence for the discriminant, including the degenerate case when k = 0:

OEIS	0	1	2	3	4	5	6	7	8	9	10	11	12
A087475	4	5	8	13	20	29	40	53	68	85	104	125	148

Vera W. de Spinadel (1929–2017) has called the sequence of positive roots a formed from discriminant D a family of metallic means, from Golden and Silver to Bronze, Copper, Nickel and so on.[91] However, even though there is a Wikipedia page on the subject,[92] it is uncertain to what extent these terms are generally acceptable within the mathematical community.

In general, when $(P, Q) = (k, -1)$, we have these three convergent limits:

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{V_{n+1}}{V_n} = a$$

$$\lim_{n\to\infty} \frac{V_n}{U_n} = a - b = a + \frac{1}{a}$$

Also, when $(P, Q) = (k, -1)$, there is a close relationship between successive terms in a primary Lucas sequence and the matrix form M of the characteristic equation, expressing such sequences as powers of matrices:[93]

$$\begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix} = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}^n = M^n$$

Here are the matrices for the first five values of n, whose determinants are all 1, for that of the initial matrix is 1:[94]

$$\begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} \Bigg| \begin{pmatrix} k^2+1 & k \\ k & 1 \end{pmatrix} \Bigg| \begin{pmatrix} k(k^2+2) & k^2+1 \\ k^2+1 & k \end{pmatrix} \Bigg| \begin{pmatrix} k^4+3k^2+1 & k(k^2+2) \\ k(k^2+2) & k^2+1 \end{pmatrix} \Bigg| \begin{pmatrix} k(k^4+4k^2+3) & k^4+3k^2+1 \\ k^4+3k^2+1 & k(k^2+2) \end{pmatrix}$$

Here is a table for the next few values of k:

k	OEIS	0	1	2	3	4	5	6	7	8	9	10
3	A006190	0	1	3	10	33	109	360	1,189	3,927	12,970	42,837
	A006497	2	3	11	36	119	393	1,298	4,287	14,159	46,764	15,4451
4	A001076	0	1	4	17	72	305	1,292	5,473	23,184	98,209	416,020
	A014448	2	4	18	76	322	1,364	5,778	24,476	103,682	439,204	1,860,498
5	—	0	1	5	26	135	701	3,640	18,901	98,145	509,626	2,646,275
	A087130	2	5	27	140	727	3,775	19,602	101,785	528,527	2,744,420	14,250,627

Briefly looking at other values of Q, the most significant is $(P, Q) = (3, 2)$, giving $D = 1$, $a = 2$, and $b = 1$. We then have the first example that Lucas included in his seminal paper, generating sequences that relate to the Mersenne and Fermat numbers:

OEIS	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
A000225	0	1	3	7	15	31	63	127	255	511	1023	2047	4095	8191	16383	32767	65535	131071
A000051	2	3	5	9	17	33	65	129	257	513	1025	2049	4097	8193	16385	32769	65537	131073

Here, the nth Mersenne number is $2^n - 1$, which is called a Mersenne prime when n is prime (OEIS A001348), mostly marked in yellow, although, as we saw in Chapter 3, very few of the Mersenne primes are actually prime (OEIS A000688), the first exception being $n = 11$, marked in pink.

The nth term in the complementary sequence is $2^n + 1$, marked in green, which is a subset of the Fermat numbers, which have the form, where $m = n - 1$ for $n > 0$:

$$F_m = 2^{2^m} + 1$$

Now, when $P^2 = 4Q$, $D = 0$, and the Lucas sequences are degenerate, with $a = b = S$, giving $P = 2S$ and $Q = S^2$. The nth terms of the sequences then become:

$U_n(P, Q) = U_n(2S, S^2) = nS^{n-1}$

$V_n(P, Q) = V_n(2S, S^2) = 2S^n$

So, when $S = 1$, $U_n(2, 1)$ are the noninteger natural numbers and $V_n(2, 1)$ is just a sequence of twos.

Catalan sequence

Next, we look at Catalan numbers, which appear frequently in enumerative combinatorics, named after Eugène Charles Catalan (1814–1894), although he wasn't the first to discover them, any more than Pell solved Pell's equation or Binet found Binet's formula, one of many examples of Stigler's Law of Eponymy, which states that that no scientific or mathematical discovery is named after its original discoverer.[95]

Catalan numbers, in particular, and enumerative combinatorics, in general, typify mathematics as a generative science of patterns and relationships, appearing in many different guises. As these patterns are ubiquitous, it is therefore not easy to define what combinatorics actually is or how it has evolved as a distinct branch of mathematics. Indeed, *combinatorics* is not mentioned in Carl B. Boyer's *A History of Mathematics* or in David M. Burton's *The History of Mathematics*. However, in *A History of Mathematics*, Victor J. Katz tells us that the earliest recorded statements of combinatorial rules appeared in India as early as the sixth century BCE, with a sixth century CE work by Varāhamihira calculating the binomial coefficient $^{16}C_4$, the number of different perfumes that could be created using four ingredients chosen from sixteen. Then, in the ninth century, Mahāvīra found the general formula for ^nC_k.[96]

After the Arabs later studied the ways to count combinations[97] and Leonardo Pisano introduced Indo-Arabic numerals into Europe, the first mention of combinatorics, as such, was a youthful 1666-dissertation that Gottfried Wilhelm Leibniz (1646–1716) wrote titled *Dissertatio de arte combinatoria* (Dissertation on the Art of Combinations). This work illustrates the fundamental nature of Leibniz's thinking, as it contains the germ of the plan for a universal characteristic and logical calculus,[98] a notion much more extensive and profound than what is regarded as combinatorics today.[99] This led to Leibniz's lifetime quest to develop a universal language that could embrace the whole of human reasoning, as I briefly mention in my 2013 book *The Theory of Everything: Unifying Polarizing Opposites in Nondual Wholeness*, explored further in Chapter 5.

In the event, George Boole, initially not knowing about Leibniz's musings, wrote *Laws of Thought*, which laid down the foundation of mathematical logic, published in 1854, which led to the invention of the stored-program computer in the late 1940s. In turn, this led to Ted Codd's relational model of data in 1970, the inspiration for the development of Integral Relational Logic, as the taxonomy of taxonomies, emphasizing the conceptual models or cognitive maps that underlie language, dwelling in the Cosmic Psyche, as I describe in Chapters 1 and 2 in this book.

So what is combinatorics today, and how has it evolved since Percy A. MacMahon (1854–1929) wrote an early major treatise on *Combinatory Analysis*, published in two volumes in 1915 and 1916? MacMahon introduced combinatory analysis as lying between algebra and higher arithmetic,[100] later explaining that the meaning of the latter is what was called in 1958 the *Theory of Numbers*, today simply *Number Theory*. This was the explanation of John Riordan (1903–1988) in *An Introduction to Combinatorial Analysis* in 1958, saying that while combinatorial analysis is a well-recognized part of mathematics, it seems to have a poorly defined range and position, it not being clear at the time "to what is and what is not combinatorial". He resolved his problem by saying, "anything enumerative is combinatorial; that is … on finding the *number of ways there are* of doing some well-defined operation."[101]

Nevertheless, in 1963 in *Combinatorial Mathematics*, Herbert John Ryser (1923–1985) still felt the need to begin this comparatively short monograph by asking the question "What is combinatorial mathematics?" As he said, "combinatorial mathematics cuts across many subdivisions of mathematics, and this makes a formal definition difficult." However, he did distinguish two basic characteristics in the literature: first to determine the existence of a prescribed configuration and secondly to enumerate the classification of these configurations according to types.[102] In other words, sets are a more fundamental notion than numbers, and until the definitions of the former are established, there is nothing meaningful to count.

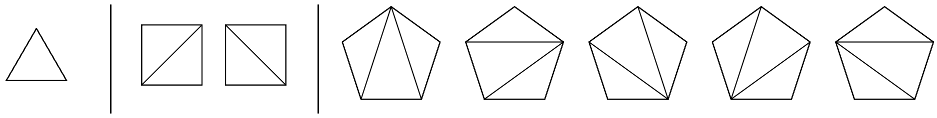
Then, in 1997, Richard P. Stanley began his definitive 2-volume, 1200-page textbook on *Enumerative Combinatorics* by answering the question "What is enumerative combinatorics?" in this way: "The basic problem of enumerative combinatorics is that of counting the number of elements of a finite set." But, as the basic building blocks of mathematics are sets and numbers, isn't that what all of mathematics is about? Well, from a combinatorist's perspective, things are not quite that simple. As Stanley tells us, there is no definitive answer to the question, "What does it mean to 'count' the number of elements of [set] S_i?" As he says, it is only through experience that one can "develop an idea of what is meant by a 'determination' of a counting function $f(i)$".

Experience, then is essential. Just as one cannot understand what philosophy is by watching philosophers think or even by reading their writings, as Antony Flew (1923–2010) pointed out in *Philosophy: An Introduction*,[103] to understand combinatorics it is necessary to engage in the practices of combinatorists. Thus, we could say that combinatorics is what combinatorists do, just as philosophy is what philosophers do.

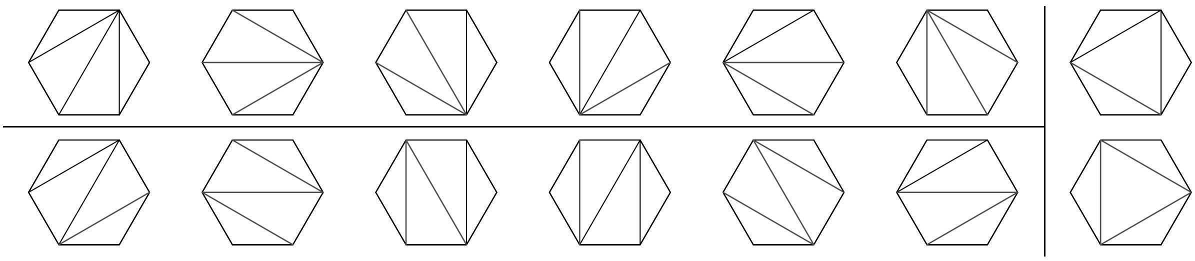
For myself, not having studied combinatorics or any other branch of mathematics in any detail since I graduated in 1964, all I can do to paint a coherent picture of mathematics as a whole in the context of all knowledge is to explore the various constructs that combinatorists study and their relationships to similar structures in graph theory, geometry, abstract algebra and group theory, linear algebra and matrix theory, category theory, and even topology, for instance. For all these branches of mathematics, as branches, and the constructs that they study, are aspects or instances of the underlying structure of the Cosmos, as a multidimensional network of hierarchical relationships.

The challenge I face in studying combinatorics, or any other subject for that matter, is that specialists are not generally aware that they are implicitly using Integral Relational Logic to form concepts and organize their ideas in semantic networks, mathematical graphs, and tables, such as relations and matrices. So, as this picture is emerging from the Datum of the Cosmos in the psyche, as a cognitive map or conceptual model, after I write Chapter 5 on 'Universal Algebra', I will probably need to return to this chapter, making any clarifying revisions, as necessary.

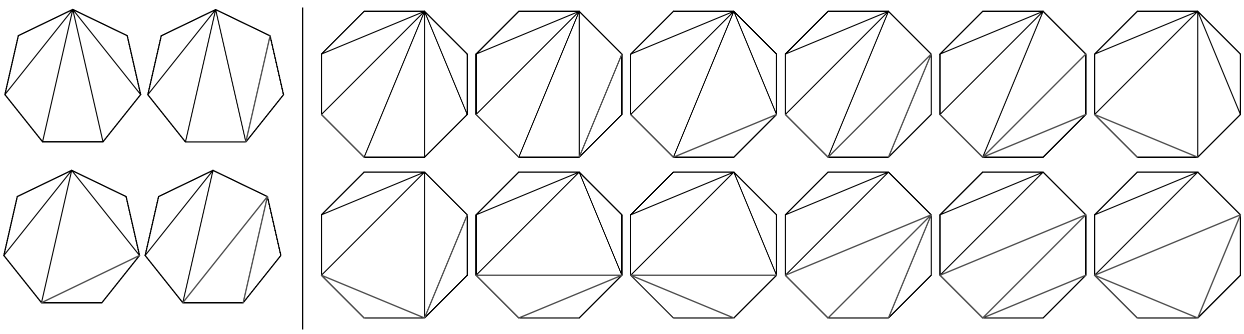
In the meantime, let us take a look at the fascinating Catalan numbers, which Igor Pak tells us had a rather chaotic history until they were understood reasonably well.[104] Although it is today acknowledged that Ming Antu (c. 1692–c. 1763), a Mongolian scholar working in China during the Qing dynasty, discovered the Catalan numbers (in infinite series) around 1730,[105] they first appeared in Europe as 'Euler's Polygon Division Problem' about twenty years later. This asks in how many ways E_n can a planar convex polygon of n sides be divided into triangles by nonintersecting diagonals, where $E_n = C_{n-2}$, as the Catalan number. For instance, with n = 3, 4, and 5, E_n is 1, 2, and 5, as these diagrams illustrate.



In those cases, they are all essentially the same because of rotational symmetry. However, with hexagons, there are three basic types, discounting rotational and reflective symmetry, giving E_6 = 14.



To keep the diagrams as simple as possible, when n = 7 and 8, there are 4 and 12 distinct patterns, and E_n = 42 and 132.



But what is the pattern here? Well, this was not immediately obvious to Euler in 1751, when he wrote to Christin Goldbach (1690–1764) about the problem. Rather laboriously, Euler found values for E_n up to n = 10, as in this table (OEIS A000108), "probably the longest entry in the OEIS",[106] as these numbers "are probably the most frequently occurring combinatorial numbers after the binomial coefficients":[107]

n	3	4	5	6	7	8	9	10
E_n	1	2	5	14	42	132	429	1,430

From this sequence of numbers, Euler surmised that the general formula is given by:

$$E_n = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \ldots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \ldots (n-1)}$$

Then in the late 1750s, Euler wrote to Johann Andreas von Segner (1704–1777), originally a physician, about the problem, who published a recurrence equation in Latin in 1758 for generating values of E_n, setting $E_2 = 1$, for the degenerate polygon as a line in two dimensions:[108]

$$E_n = E_2E_{n-1} + E_3E_{n-2} + \cdots + E_{n-1}E_2 = \sum_{k=2}^{n+1} E_kE_{n-k+1}$$

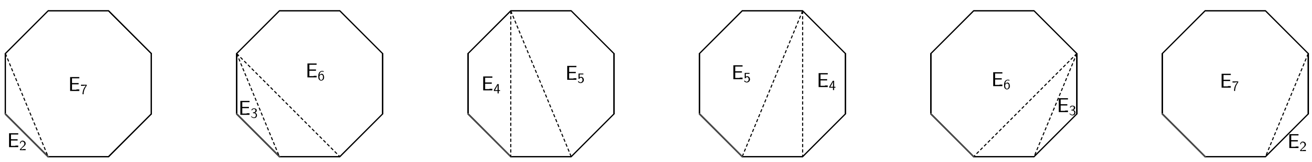
Here is a 'Catalan triangle', showing how the Catalan numbers are generated from this recurrence equation:

C_0	E_2	1	= 1
C_1	E_3	1·1	= 1
C_2	E_4	1·1 + 1·1	= 2
C_3	E_5	1·2 + 1·1 + 2·1	= 5
C_4	E_6	1·5 + 1·2 + 2·1 + 5·1	= 14
C_4	E_7	1·14 + 1·5 + 2·2 + 5·1 + 14·1	= 42
C_6	E_8	1·42 + 1·14 + 2·5 + 5·2 + 14·1 + 42·1	= 132
C_7	E_9	1·132 + 1·42 + 2·14 + 5·5 + 14·2 + 42·1 + 132·1	= 429
C_8	E_{10}	1·429 + 1·132 + 2·42 + 5·14 + 14·5 + 42·2 + 132·1 + 429·1	= 1430

Omitting the single 1 above the vertex, each element in the triangle $T(n, k)$ is the product of two Catalan numbers (OEIS A078391), where $n \geq 0$ and $0 \leq k \leq n$:

$$T(n,k) = C_n \times C_{n-k}$$

We can illustrate how this recurrence equation is formed from the way that the dissections of an octagon can be shown in terms of previous values of E_n. These diagrams show six ways of dividing the octagon into smaller polygons, with either a line or triangle between them. From these, we can calculate E_8.



1·42=42 1·14=14 2·5=10 5·2=10 14·1=14 42·1=42

This is where the matter rested for the next eighty years, until Olry Terquem (1782–1862) asked Joseph Liouville (1809–1882) in 1838 if he knew a simple way to derive Euler's product formula from Segner's recurrence equation. Liouville circulated the question among his geometric associates, soon receiving an elegant solution in a letter from Gabriel Lamé (1795–1870), which Liouville promptly published in *Journal de Mathématiques Pures et Appliquées*,[109] which he had founded two years earlier.

Lamé, professor of physics at the prestigious École Polytechnique in Paris, solved this problem by looking at polygon triangular divisions in two ways: in terms of the edges and vertices of the polygon. The former gives Segner's recurrence equation, as we see in the example above, which Lamé wrote as:[110]

$$E_3E_{n-1} + E_4E_{n-3} + \cdots + E_{n-2}E_4 + E_{n-1}E_3 = E_{n+1} - 2E_n$$

When looking at the problem from the perspective of a vertex, noting that solutions were counted multiple times, Lamé derived this formula:

$$E_3E_{n-1} + E_4E_{s-3} + \cdots + E_{n-2}E_4 + E_{n-1}E_3 = \frac{2n-6}{n}E_n$$

Hence:

$$E_{n+1} - 2E_n = \frac{2n-6}{n} E_n$$

Which proves this simple recurrence equation, a proof that had eluded mathematicians for nearly a century:

$$E_{n+1} = \frac{4n-6}{n} E_n \qquad\qquad E_2 = 1$$

At this, the flood gates were opened. Later that year, Olinde Rodrigues (1795–1851), a banker, produced an inductive proof of the recurrence equation and most significantly, Catalan, before himself obtaining a post at École Polytechnique and before he had received a degree, found another simple formula for the polygon division problem.[111] First, by expanding E_{n+1} as:

$$E_{n+1} = \frac{n(n+1)(n+2)\ldots(2n-2)}{2 \cdot 3 \cdot 4 \cdot \ldots \cdot n}$$

he immediately saw that

$$E_{n+2} = \frac{1}{n+1}\binom{2n}{n} = \left(1 - \frac{n}{n+1}\right)\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{(n+1)!\,n!} = \frac{1}{n+1}\binom{2n}{n}$$

For Catalan also noticed at once that as $\binom{2n}{n}$ is an expression of the central binomial coefficients in Pascal's triangle, defined on page 199, if these (1, 2, 6, 20, 70, 252, 924, …) are divided by the natural numbers (1, 2, 3, 4, 5, 6, 7, …), one obtains: 1, 1, 2, 5, 14, 42, 132, …. So, setting $n = n + 2$ in Lamé's recurrence equation, the recurrence equation for the Catalan numbers is:[112]

$$C_{n+1} = \frac{2(2n+1)}{n+2} C_n \qquad\qquad C_0 = 1$$

In summary, here is a table of the first few Catalan numbers and their relationships to Catalan's formula. The first row is the sequence of central binomial coefficients (T_n), denoting the total of all ways of choosing n from $2n$ possibilities, while the second (X_n) denotes the number of instances that are excluded, by the definition of the constructs. The table, which becomes clearer when we look at other constructs that generate Catalan numbers, lists ratios between these sequences as irreducible fractions, plus, for completeness, the associated unique polygon triangulation numbers disregarding rotational and reflective symmetries for regular (n+2)-gons.

OEIS	ID	0	1	2	3	4	5	6	7	8	9	10
A000984	T_n	1	2	6	20	70	252	924	3,432	12,870	48,620	184,756
A001791	X_n	0	1	4	15	56	210	792	3,003	11,440	43,758	167,960
A000108	$C_n = T_n - X_n$	1	1	2	5	14	42	132	429	1,430	4,862	16,796
—	X_n/T_n	$\frac{0}{1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{10}{11}$
—	C_{n+1}/C_n	1	2	$2\frac{1}{2}$	$2\frac{4}{5}$	3	$3\frac{1}{7}$	$3\frac{1}{4}$	$3\frac{1}{3}$	$3\frac{2}{5}$	$3\frac{5}{11}$	$3\frac{1}{2}$
A000207	U_n	1	1	1	3	4	12	27	82	228	733	2,282

As you can see, the proportion of instances that are invalid grows as n/n+1, giving the maximum growth rate of the Catalan numbers as:

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \lim_{n\to\infty} \frac{2(2n+1)}{n+2} = 4$$

So, while the sequence of Catalan numbers converges to infinity, its rate of growth is limited to a small finite value, an example of evolution under constraint. Another such example is the discrete logistic map, which I used in my 2016 book *Through Evolution's Accumulation Point* to explain why 13.8 billion years of evolution are degenerating into political chaos at the moment. It is interesting to note that the maximum rate of growth of this nonlinear difference equation is also four. Is this just a coincidence?

I ask this question because Catalan went even further in his 1838 paper. He showed that the same sequence arises from what is called the bracketing or parenthesization problem: how to find the number of different ways B_n that completely parenthesize a product of n letters, so that there are two factors inside each set of parentheses. For example, for the four factors a, b, c, and d, there are five possibilities:

$$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd)).$$

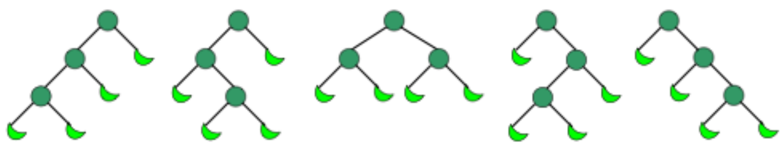
Such strings of characters are called 'well-formed' in computer science, for when writing or parsing expressions, opening parentheses must precede closing ones, there must never be more closing parentheses that opening ones in the string, and at the end, the parentheses must balance.

Thus $B_4 = 5 = E_5 = C_3$. So this way of generating the Catalan numbers is better defined in terms of $n + 1$ characters, as an example of our bifurcating Universe, one of many that have been discovered since then. So a humble schoolteacher, with the invaluable assistance of Lamé, was able to see patterns in these numbers that had eluded leading mathematicians for nearly a century. This is how the Catalan numbers came to be so named. Having previously been called Segner or Euler-Segner numbers, in *Combinatorial Identities* in 1968, John Riordan mentioned the term *Catalan number* seven times,[113] which led this term to be adopted by the mathematical community.[114]

Then, in the 1970s, when Richard P. Stanley began teaching enumerative combinatorics, he became aware of the ubiquity of this sequence and started to collect combinatorial interpretations of Catalan numbers. He published 66 examples in the second volume of his monumental *Enumerative Combinatorics* in 1999, listing 214 in 2015 in *Catalan Numbers*, as an extension of his Catalan Addendum web page.[115] He considered six to be the most fundamental and gave bijective proofs between them, showing their equivalence, pointing out that bijective proofs between all the elements of sets of instances would require $214 \cdot 213 = 45{,}582$ bijections in total![116]

Such an exercise would, of course, be made much simpler by finding projections to just a few examples, like currency exchange rates defined in terms of a few standards, like dollars and sterling, where the projections between them are thoroughly proved. Stanley's six are Euler's polygon triangulations, Catalan's parenthesization problem, binary trees, plane trees, ballot sequences, and Dyck paths, named after Walther von Dyck (1856–1934).[117] We have already seen the first two. So here are the other four, plus a couple of other interesting examples.

Binary trees are much used in computer science today. They are a type of graph, in which each node has at most two children, which are referred to as the left child and the right child. Stanley gives two examples, both of which are illustrated in this diagram from Wikipedia.



First, the circles show the five possible binary trees for $n = 3$, effectively following the diagonals of a square lattice. Secondly, the diagrams as a whole show the number of complete binary trees, that is those in which every vertex has either zero or two children, giving $2n + 1$ vertices, with $n + 1$ endpoints, marked as green moons.

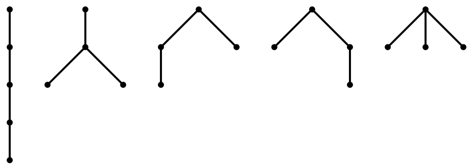
As you can see, the central binary tree is structurally different from the other four, giving rise to another sequence of structurally different binary trees D_n,[118] different from the symmetries in the polygon triangular numbers. So there is not necessarily a bijection between the similar sets of Catalan sequences that are generated from different constructs. For reference, here are the first few terms of D_n:

OEIS	ID	0	1	2	3	4	5	6	7	8	9	10
A000108	C_n	1	1	2	5	14	42	132	429	1,430	4,862	16,796
A001190	D_n	1	1	1	2	3	6	11	23	46	98	207

Many other examples of trees generate Catalan numbers, where a plane or ordered tree T is defined recursively as a finite set of vertices such that:[119]

a. One specially designated vertex is called the *root* of T, and
b. The remaining vertices (excluding the root) are put into an *ordered* partition (T_1, …, T_m) of $m \geq 0$ pairwise disjoint, nonempty sets T_1, …, T_m, each of which is a plane tree.

Here are the diagrams for plane trees with $n + 1$ vertices and $n = 3$:



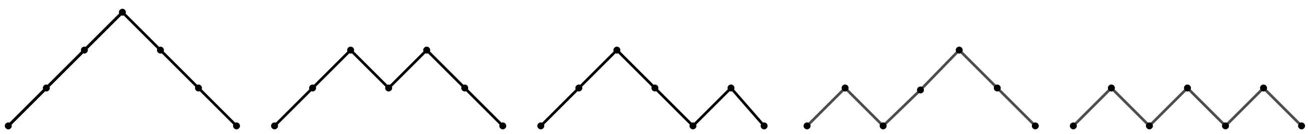
The simplest form of the ballot problem concerns two candidates A and B for office, where n vote for A and n for B. In how many ways can the ballots be counted so that B is never ahead of A? To put this another way, if there is an urn with n white and black balls, in how many ways can the balls be drawn so that there are never more black balls drawn than white. Here is this solution for $n = 3$ once again:

AAABBB AABABB AABBAB ABAABB ABABAB

We can obtain such a ballot sequence directly from a bracketing, by first putting parentheses around the entire expression. We then replace every left parenthesis by A and each character except the last with a B, deleting all the right parentheses and the last character.[120] Here, A is not-B, so A and B could be represented by any pair of opposites, such as up and down arrows (↑↓). So, we could write the above solution for the ballot problem like this:

↑↑↑↓↓↓ ↑↑↓↑↓ ↑↑↓↑↓ ↑↓↑↑↓↓ ↑↓↑↓↑↓

This representation takes us directly to the most basic of the Dyck paths, which asks in how many ways can a path of length $2n$ cross a lattice from (0, 0) to ($2n$, 0) with steps (1, 1) and (1, -1), never falling below the x-axis. This can be presented as a mountain range, as a bijection of the ballot problem, thus:



The Dyck paths lead to another version of a Catalan triangle (OEIS A001263), one of several in the OEIS, counting the number of diagrams with the same number of peaks:[121]

C_1	1	= 1
C_2	1 + 1	= 2
C_3	1 + 3 + 1	= 5
C_4	1 + 6 + 6 + 1	= 14
C_4	1 + 10 + 20 + 10 + 1	= 42
C_6	1 + 15 + 50 + 50 + 15 + 1	= 132
C_7	1 + 21 + 105 + 175 + 105 + 21 + 1	= 429
C_8	1 + 28 + 196 + 490 + 490 + 196 + 28 + 1	= 1430

This triangle contains Narayana numbers, named after T. V. Narayana (1930–1987), who rediscovered them in 1955, after they were first mentioned in the two-volume *Combinatory Analysis* in 1916 in another context by MacMahon, the first major treatise on combinatorics, regarded as a classic. Narayana numbers $N(n, k)$ are given by, where $n \geq 1$ and $1 \leq k \leq n$:

$$N(n,k) = \frac{1}{k}\binom{n}{k}\binom{n}{k-1}$$

The Dyck paths provide the representation of a construct that directly illustrates Catalan's formula as the difference between two binomial coefficients. The number of paths from (0, 0) to (6, 0) is 20, while 15 of them cross the x-axis, leaving just 5 paths that remain on or above the x-axis. An alternative way of depicting Dyck paths, which correspond to Dyck words, as in the ballot problem, is the number of paths from (0, 0) to (n, n) that stay either above or below the line $x = y$, not crossing it.

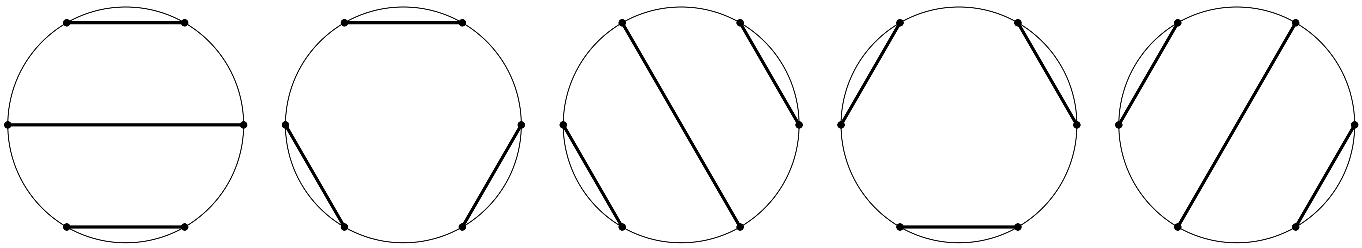
If instead of using the letters A and B in Dyck words, we denote these opposites with 1 and 0 in binary notation, we obtain a sequence of binary numbers (OEIS A063171) and hence their corresponding decimal numbers (OEIS A014486) for all possible constructs that participate in Catalan sequences of whatever length the constructs might be. Here are the numbers for the first four Catalan sets.

Catalan	C_0	C_1	C_2		C_3				
OEIS n	0	1	2	3	4	5	6	7	8
Binary	0	10	1010	1100	101010	101100	110010	110100	111000
Decimal	0	2	10	12	42	44	50	52	56

And here are the next terms in the sequences for Dyck words in C_4 for n = 9 to 22.

10101010	10101100	10110010	10110100	10111000	11001010	11001100	11010010	11010100	11011000	11100010	11100100	11101000	11110000
170	172	178	180	184	202	204	210	212	216	226	228	232	240

Antti Karttunen has created a file for these sequences up to n = 625, to C_7,[122] as an extension of the OEIS Wiki page on 'Combinatorial interpretations of Catalan numbers',[123] including bijections between Dyck words, mountain ranges, binary trees, plane trees, and one other that we have not looked at yet. This is another geometric example, where the Catalan numbers denote the number n of nonintersecting chords that join $2n$ points on the circumference of a circle, sometimes called handshaking, although you would need very long arms as n increases!



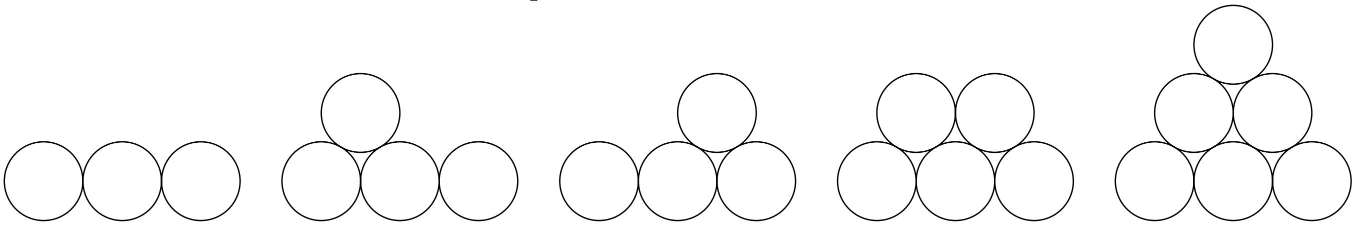
This example well illustrates the link between Catalan numbers and probability theory. If chords are drawn randomly across the circle, the number of chords is given by $\binom{2n}{n}$ of which $\binom{2n}{n-1}$ cross. So the probability of chords not crossing is given by:

$$\Pr(n) = \frac{\binom{2n}{n} - \binom{2n}{n-1}}{\binom{2n}{n}} = \frac{1}{n}$$

One other pattern arises from this example. As you can see, there are two distinct patterns, excluding rotations and reflections, different from the distinct patterns in Euler's original triangular polygon problem. This sequence begins 1, 1, 2, 3, 6 and looks as it is OEIS A001405, with the next terms 10, 20, 35, 70, 126 given by, although I have not seen a proof of this:

$$a(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

As a final example, these diagrams show the number of ways to stack coins in the plane, the bottom row consisting of n consecutive coins.



This interpretation of the Catalan numbers generates another 'Catalan triangle' (OEIS A080936), giving the number of constructs with a height of 1 to *n*. More formally, the OEIS states that $T(n, k)$ is the number of Dyck paths of semilength *n* and height *k*, where $1 \leq k \leq n$. In this case, the triangle is not symmetric, containing some familiar and not so familiar sequences in the diagonals.

C_1	1	= 1
C_2	1 + 1	= 2
C_3	1 + 3 + 1	= 5
C_4	1 + 7 + 5 + 1	= 14
C_5	1 + 15 + 18 + 7 + 1	= 42
C_6	1 + 31 + 57 + 33 + 9 + 1	= 132
C_7	1 + 63 + 169 + 132 + 52 + 11 + 1	= 429
C_8	1 + 127 + 482 + 484 + 247 + 75 + 13 + 1	= 1430

Browsing through the OEIS sequences that mention Catalan numbers (2891 in September 2019), their underlying simplicity is quickly overwhelmed by complexity, which requires much study over many years to become assimilated in consciousness. Not only are there many interrelationships within the interpretations of the Catalan sequences, there are also relationships with other sequences, which I have not studied in any depth, knowing that they are just instances of the fundamental structure of the Cosmos, which I present in terms of mathematical graphs, relations, and matrices in Integral Relational Logic, as we see in Chapter 2.

To illustrate these instances, two fascinating ways of arranging Catalan constructs as lattices and one way of presenting Catalan numbers in matrices have caught my eye. When I first came across these constructs, I thought that the lattices referred to meant a regularly spaced array of points, such as the square lattices in Dyck paths, although I could not see their relevance in this context. So, on checking Wolfram *MathWorld* and Wikipedia to clarify this concept, I discovered that such lattices should be more properly called 'point lattices', to distinguish them from the primary meaning of *lattice* in mathematics, which I was not aware of until I came to write this section in this book.

What I have found is another astonishing story in the evolution of mathematics. Garrett Birkhoff (1911–1996), co-author with Saunders Mac Lane (1909–2005) of *A Survey of Modern Algebra*, which I partially studied as an undergraduate in the early 1960s, wrote the seminal book *Lattice Theory*, published in three editions in 1940, 1948, and 1965. In the second edition, available on the Web, Birkhoff tells us that lattice theory originally evolved from Boolean algebra and the algebra of relations, which Charles Sanders Peirce and Ernst Schröder developed in the 1800s,[124] a neglected chapter in the history of logic, as Geraldine Brady points out, mentioned in Chapter 3.

The second stage in the development of lattice theory took place in the 1930s, building on *Moderne Algebra* by Bartel Leendert van der Waerden (1903–1996), a systematized treatise on the abstract algebra of groups, rings, and fields, perhaps the first treatise to treat the subject as a comprehensive whole.[125] As George Grätzer tells us in *Lattice Theory: First Concepts and Distributive Lattices* from 1971, the first of an ever-expanding sequence of books on the subject, Birkhoff wrote a brilliant series of papers in which "he demonstrated the importance of lattice theory and showed that it provides the unifying framework for hitherto unrelated developments in many mathematical disciplines."[126] In 1948, Birkhoff envisaged that while lattice theory may never equal its 'elder sister group theory', he believed "that it will achieve a comparable status."[127]

What I am seeing here is an attempt to take the abstractions of human reasoning to the utmost level of generality from within mathematics, not unlike the way that scientists have been attempting to solve the ultimate problem of human learning from within physics since the 1920s. For Mac Lane went on to co-found category theory and Grätzer wrote a book on *Universal Algebra*, following in the footsteps of Arthur North Whitehead, who wrote a book with a similar title in 1898.

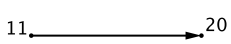
Yet, the ultimate taxonomy of taxonomies, as the coordinating framework for the Unified Relationships Theory, as the Theory of Everything, is Integral Relational Logic, as Chapter 2, explains. To demonstrate this beyond all reasonable doubt, even among the sceptics, I still need to write Chapter 5, which I have provisionally titled 'Universal Algebra'. This will show how all these abstract branches of mathematics, up to quantum computation, can be represented in this universal system of thought.

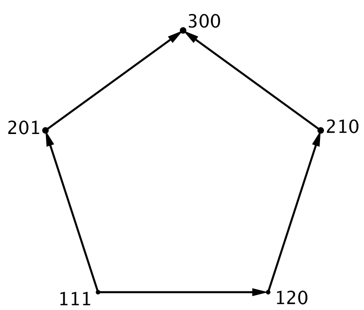
In the meantime, let us look at Tamari lattices, named after Dov Tamari (1911–2006), formerly Bernhard Teitler, a fascinating character, as Folkert Müller-Hoissen and Hans-Otto Walther tell us in a book to celebrate the hundredth anniversary of his birth.[128] Inevitably, a background in abstract algebra is needed to fully understand the 1951 doctoral thesis in which Tamari presented his ideas on associative binary operations, later published in 1962 in the Netherlands as 'The algebra of bracketings and their enumeration'.[129]

Nevertheless, as I currently understand the situation, Tamari saw that the bracketings that Catalan introduced could be arranged in chains according to a rightward application of the usual associativity law as partially ordered sets, which Birkhoff called *posets* in *Lattice Theory*.[130] However, the notion of poset did not, in itself, prove that Tamari's construct is actually a lattice. For this requires there to be a top and bottom element in the structure, known as the least upper bound or supremum and greatest lower bound or infimum. In the event, Tamari developed the necessary proof with Samuel Huang, published in 1972.[131]

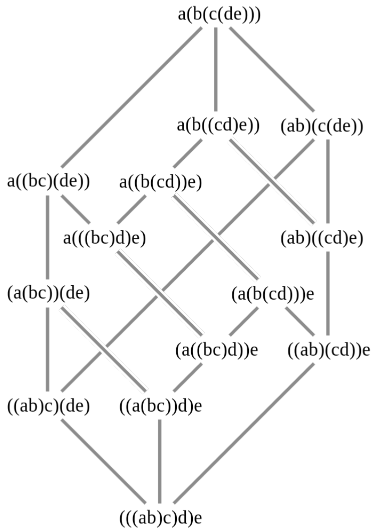
In order to order the Catalan binary bracketings as posets, Tamari numbered them in this way in his original 1951 thesis,[132] omitted from the publication of his thesis in 1954,[133] also giving codings for the fourteen bracketings in C_4 with five characters.[134] He then showed how these lattices could be displayed in one to three dimensions, such as these two diagrams:

Code	Brackets
1	*ab*
11	*a*(*bc*)
20	(*ab*)*c*
111	*a*(*b*(*cd*))
120	*a*((*bc*)*d*)
201	(*ab*)(*cd*)
210	(*a*(*bc*))*d*
300	((*ab*)*c*)*d*





Wikipedia provides a diagram of the Tamari lattice T_4 as a Hasse diagram, named after Helmut Hasse (1898–1979) because of the effective use Hasse made of them.[135] What is particularly interesting about this model, as a lattice, is that the chains from one end to the other are not of equal length, ranging, in this



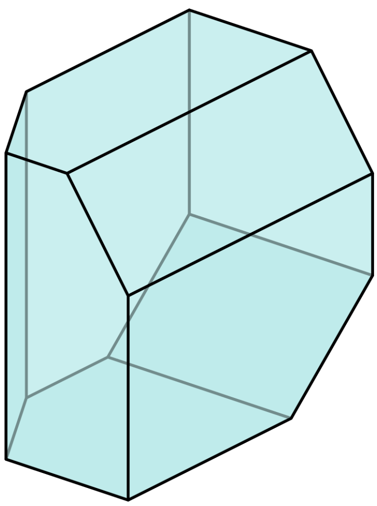
case, from four points and three binary operations, to seven points and six binary operations. Using Tamari's encoding for the vertices, the nine chains are:

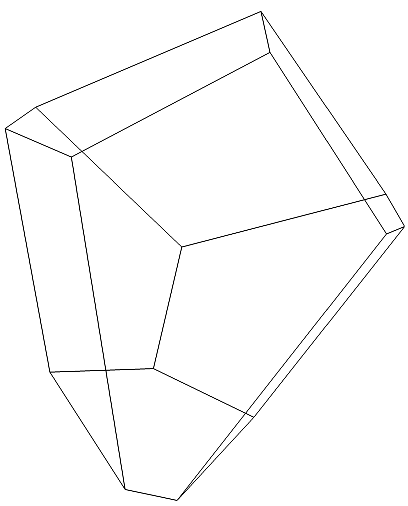
1111→1120→1210→1300→2200→3100→4000
1111→1120→1210→2110→3010→4000
1111→1120→2020→3010→4000
1111→1201→2101→3001→4000
1111→2011→3001→4000
1111→1120→1210→2110→2200→3100→4000
1111→1201→1300→2200→3100→4000
1111→2011→2020→3010→4000
1111→1201→2101→3100→4000

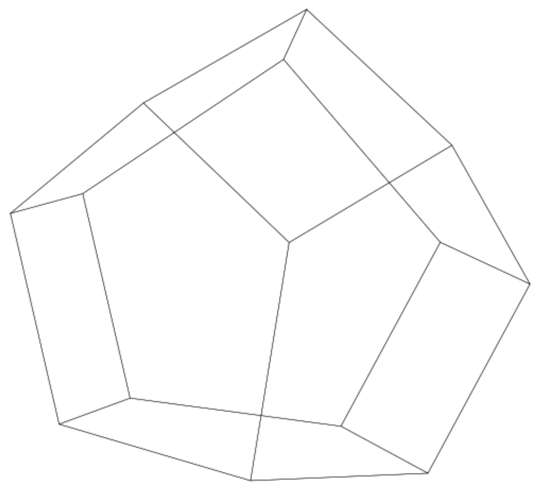
However, as Tamari observed in his original thesis, the lattices for the C_n bracketings can be extended into three and more dimensions as polytopes. Then, in 1963, Jim Stasheff rediscovered this sequence of polytopes when studying something called homotopy, not being aware of developments in combinatorics because of the compartmentalism of mathematics. Like me, he did not know initially whether parentheses should be moved to the left or the right. Be that as it may, such polytopes are generally called associahedra or Stasheff polytopes, today, although Stasheff says that they should really be called Tamari or Tamari-Stasheff polytopes.[136]

The term *associahedron*, which Mark Haiman introduced in 1984,[137] seems a little confusing to me, for I associate *polyhedron* with three dimensions, while a polytope has any number of dimensions, the 3-polytope being a polyhedron. So maybe *associahedron* should really be a 3-*associatope*. Be that as it may, the three-dimensional associahedron, which Stasheff calls K_5 , has 14 vertices, 21 edges, and 9 faces, in conformity with Euler's Polyhedron Formula: $V - E + F = 2$, which generalizes into higher dimensions, as we see on page 315 in the final section on 'Spatial dimensions'.

There are many ways of constructing the associahedron as a cardboard model, three of which are illustrated below. The first appears to be an orthogonal representation, presented in Wikipedia,[138] with the vertices at the intersections of a 3-dimensional grid in a cube of side three. In the centre, is the dual of the triaugmented triangular prism,[139] a prism with pyramids stuck on each square face, also known as a tetrakaidecadeltahedron, 14-face deltahedron,[140] or J51 in the set of Johnson solids, named after Norman Johnson (1930–2017), who in 1966 enumerated 92 convex non-uniform polyhedra with regular faces.[141] The enneahedron on the right, apparently consisting of six regular pentagons and three squares,[142] is not one of them. It is a 'near-miss Johnson solid',[143] not possible to construct without a little distortion.

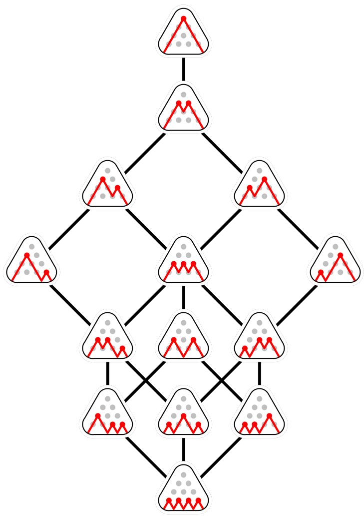






It is illuminating to construct the model on the right, as I did from a net that I found on the Web. For this clearly shows that the two vertices where three pentagons meet are not the supremum and infimum in the Tamari lattice. Rather, these vertices are a((bc)(de)) and ((ab)(cd))e in the top-left and bottom-right. The actual model also clearly shows the 'short-cut' through the vertices from the lowest to the highest and also the longest chains.

Jean-Louis Loday tells us that it is possible to generalize the construction of associatopes in higher dimensions, but in a way that is slightly more involved than with similar constructs with simplexes and hypercubes.[144] It will thus be more appropriate to look at this possibility in the final section in this chapter on 'Spatial dimensions'.



Another way of arranging C_n constructs in a graph is the Dyck lattice, illustrated by this example from Wikipedia. Unlike the Tamari lattice, all fifteen chains are of equal length, in this case seven with six links.[145] I'm currently unsure what the binary operator in the algebra is here, for this lattice seems to be less studied than the Tamari lattice. However, yet another 'Catalan triangle' emerges from this construct (OEIS A227543). The OEIS says that the antidiagonal of this triangle sums to the number of fountains of n coins (OEIS A005169), related to the stacking of coins in the example above: 1, 1, 1, 2, 3, 5, 9, 15, 26, 45, 78, 135, …. However, I cannot see what 'antidiagonal' refers to here.

C_n	L	$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	**1**	0	1																												
1	**1**	1	1																												
2	**2**	2	1	1																											
5	**4**	3	1	2	1	1																									
14	**7**	4	1	3	3	3	2	1	1																						
42	**11**	5	1	4	6	7	7	5	5	3	2	1	1																		
132	**16**	6	1	5	10	14	17	16	16	14	11	9	7	5	3	2	1	1													
429	**22**	7	1	6	15	25	35	40	43	44	40	37	32	28	22	18	13	11	7	5	3	2	1	1							
1430	**29**	8	1	7	21	41	65	86	102	115	118	118	113	106	96	85	73	63	53	42	34	26	20	15	11	7	5	3	2	1	1

The length of each chain $L(n)$ (OEIS A152947) is given by, for $n \geq 1$:

$$L(n) = 1 + \frac{1}{2}(n-2)(n-1)$$

We can also arrange the terms in the Catalan sequence in a Hankel matrix, which has this form, named after Hermann Hankel (1839–1873), which has a surprising result, unique to the Catalan sequence.

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{n-1} \\ a_1 & a_2 & & & \iddots & \vdots \\ a_2 & & & & & \vdots \\ \vdots & & & & & a_{2n-4} \\ \vdots & \iddots & & & a_{2n-4} & a_{2n-3} \\ a_{n-1} & \cdots & \cdots & a_{2n-4} & a_{2n-3} & a_{2n-2} \end{pmatrix}$$

If we set $a_0 = C_0$ and $a_k = C_k$, as in these first five matrices, their determinants are all 1. Furthermore, if $a_0 = C_1$ and so on, the determinants are still 1.

$$(1)\quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 14 & 25 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 2 & 5 & 14 \\ 1 & 2 & 5 & 14 & 42 \\ 2 & 5 & 14 & 42 & 132 \\ 5 & 14 & 42 & 132 & 429 \\ 14 & 42 & 132 & 429 & 1430 \end{pmatrix}$$

After this, setting $a_0 = C_2, C_3, C_4, \ldots$, the values of the determinants grow as in this table, for matrices of size m:

OEIS	$a_0 \backslash m$	1	2	3	4	5	6
A000108	C_0	1	1	2	5	14	42
A005700	C_1	1	1	3	14	84	594
A006149	C_2	1	1	4	30	330	4719
A006150	C_3	1	1	5	55	1001	26026
A006151	C_4	1	1	6	91	2548	111384

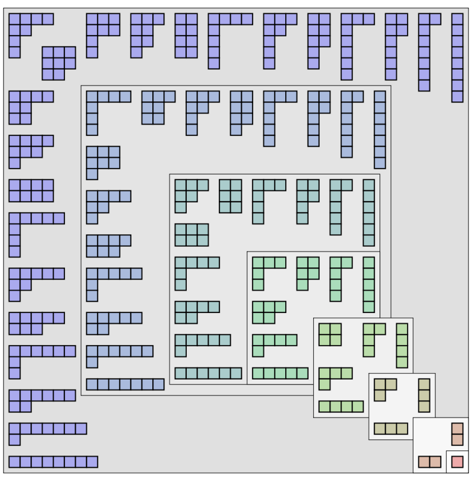
I'm not sure how to determine the meaning of the sequences in the rows and columns in this table. The fourth column appears to be the sequence of square pyramidal numbers (OEIS A000330) and the OEIS describes the last three rows as 'Number of Dyck paths', but this is normally regarded as the Catalan sequence itself. So relationships with other structures need further clarification, perhaps through their recurrence equations and corresponding generating functions.

Integer partitions

Another way in which mathematicians like to cut mathematical objects into bits and count the pieces is with the natural numbers themselves, in a notion called *partitions*. The first record of partitions in the literature is a letter that Leibniz wrote to Johann Bernoulli (1667–1748) wondering if it were possible to enumerate them, which Leibniz thought to be difficult, but important.[146]

However, it was Euler who really kicked off their study, after receiving a letter on 4th September 1740 from Philippe Naudé the younger (1684-1747), a Fellow of the Royal Society.[147] Naudé asked Euler, "how many ways can the number 50 be written as a sum of seven different positive integers?"[148] Euler responded to Naudé within a few weeks, apologizing for his tardiness, giving the answer 522.[149] Seven months later, Euler gave this solution in an ingenious paper titled *Observationes analyticae variae de combinationibus*, E158 in the Eneström index, which he presented at St. Petersburg Academy, but not published until 1751.[150] In the meantime, he gave a clearer solution in 1748 in *Introductio in analysin infinitorum*, "one of the world's truly great mathematics books".[151]

Euler later wrote six other papers on this intricate subject,[152] but its modern study really took off when Srinivasa Ramanujan (1887–1920) began investigating it, as dramatized in *The Man Who Knew Infinity*, a 2015 biopic about his relationship with G. H. Hardy (1877–1947) and John Littlewood (1885–1977) at the University of Cambridge.



In number theory and combinatorics, a partition of a positive integer n, also called an integer partition, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. The partition function $p(n)$ thus represents the number of possible partitions of a nonnegative integer n, setting $p(0) = 1$, given that "the empty sequence forms the only partition of zero".[153] For instance, 4 can be partitioned in five distinct ways: $4 = 3 + 1 = 2 + 2 = 2 + 1 +$

$1 = 1 + 1 + 1 + 1$. The growth in the sequence of integer partitions (OEIS A000041) can best be visualized graphically in schemas introduced by Norman Macleod Ferrers (1829–1903) and Alfred Young (1873–1940), as shown in the above diagram from Wikipedia.

For instance, in a Ferrers diagram, which J. J. Sylvester introduced in 1882, from an idea that Ferrers had given him thirty years earlier,[154] one of the partitions of 14 that could be represented as an arrangement of asterisks, in a pair of conjugate ways, representing 6 + 6 + 4 and 3 + 3 + 3 + 3 + 2 + 2:

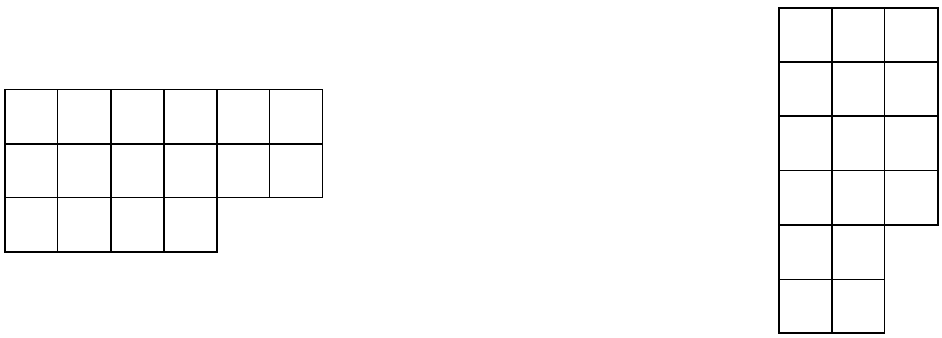
```
*  *  *  *  *  *          *  *  *
*  *  *  *  *  *          *  *  *
*  *  *  *                *  *  *
                           *  *  *
                           *  *
                           *  *
```

These are two examples of partitions for $p(14)$=135. They are also two examples of the function $P(n, k)$, which is the number of partitions of n into k summands. In these cases, $P(14, 3) = 16$ and $P(14, 6) = 20$. These partitions are examples of other sets of partitions, those with a maximum or minimum summand, in these cases, 6 and 3 and 4 and 2, respectively, examples of many conditions that can be placed on partitions.

The number of unrestricted partitions of n can thus be calculated with this formula:

$$p(n) = \sum_{k=1}^{n} P(n,k)$$

Young's method for depicting partitions in squares arose from a paper he wrote in 1900,[155] addressing a more general combinatorial situation in which symbols can be placed within the boxes, not necessary for partitions:



Nowadays, it seems from the literature that Young's depiction of partitions is that which is preferred by professional mathematicians, sometimes used to illustrate their properties. For instance, they can be used to create a recurrence equation for $P(n, k)$, which Louis Comtet (1933–2012) gives in *Advanced Combinatorics* as, with slightly different initial values:[156]

$$P(n,k) = P(n-1,k-1) + P(n-k,k) \qquad P(n,1) = P(k,k) = 1, \quad P(n,k) = 0, \;\; k > n$$

You can see that this recurrence equation has some similarities with Pascal's triangle rule on page 197. The first terms are the same, while the second goes much further back in the hierarchy. This, we can trace to some extent with the hierarchical evolutionary structure in Integral Relational Logic, such as a family tree, depicting parents, grandparents, etc. For instance, the following triangle indicates the 'ancestors' of $P(20, 5) = 84$, where $P(10, 3) = 8$ is darker because this 'great great grandparent' appears twice. Further back, the predecessors overlap far more, with different generations mixing, making the patterns difficult to discern, just like our family trees. For we are all cousins of each other, many times over. Another feature of this recurrence equation is that the first non-zero *k* terms in the *k*th column are the same as those in the (*k* - 1)th column, with the first change in each column being marked in **red**.

$n\backslash k$	$p(n)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1																			
2	2	1	1																		
3	3	1	1	1																	
4	5	1	2	1	1																
5	7	1	2	2	1	1															
6	11	1	3	3	2	1	1														
7	15	1	3	4	3	2	1	1													
8	22	1	4	5	5	3	2	1	1												
9	30	1	4	7	6	5	3	2	1	1											
10	42	1	5	8	9	7	5	3	2	1	1										
11	56	1	5	10	11	10	7	5	3	2	1	1									
12	77	1	6	12	15	13	11	7	5	3	2	1	1								
13	101	1	6	14	18	18	14	11	7	5	3	2	1	1							
14	135	1	7	16	23	23	20	15	11	7	5	3	2	1	1						
15	176	1	7	19	27	30	26	21	15	11	7	5	3	2	1	1					
16	231	1	8	21	34	37	35	28	22	15	11	7	5	3	2	1	1				
17	297	1	8	24	39	47	44	38	29	22	15	11	7	5	3	2	1	1			
18	385	1	9	27	47	57	58	49	40	30	22	15	11	7	5	3	2	1	1		
19	490	1	9	30	54	70	71	65	52	41	30	22	15	11	7	5	3	2	1	1	
20	627	1	10	33	64	84	90	82	70	54	42	30	22	15	11	7	5	3	2	1	1

However, this recurrence equation does not answer the question that Naudé asked Euler in 1740. The former wanted to know in how many ways the number 50 could be written as a sum of seven *distinct* positive integers. Comtet provides this formula for deriving the number of distinct partitions *Q*(*n*, *k*) from unrestricted partitions:[157]

$$Q(n,k) = P\left(n - \binom{k}{2}, k\right)$$

This formula has the effect of pushing down the columns in the above partition triangle by the sequence of natural numbers, the position of the initial value of each column then being given by the triangular numbers, 1, 3, 6, 10, 15, 21, 36, 45, …. In other words, the triangle of distinct partitions contains the same values as the triangle of unrestricted partitions, giving this recurrence equation:

$$Q(n,k) = Q(n-k,k-1) + Q(n-k,k) \qquad Q(n,1) = Q\left(\binom{k+1}{2}, k\right) = 1, \qquad Q(n,k) = 0, \quad \binom{k+1}{2} > n$$

The total number of partitions with distinct summands, sometimes called 'strict partitions', is denoted by *q*(*n*) (OEIS A000009):

$$q(n) = \sum_{k=1}^{n} Q(n,k)$$

This extended table, to *n* = 50, depicts the first few 'ancestors' of *Q*(50, 7) = 522, giving the answer to Naudé's query.

$n\backslash k$	$q(n)$	1	2	3	4	5	6	7	8	9
1	1	1								
2	1	1								
3	2	1	1							
4	2	1	1							
5	3	1	2							
6	4	1	2	1						
7	5	1	3	1						
8	6	1	3	2						
9	8	1	4	3						
10	10	1	4	4	1					

$n\backslash k$	$q(n)$	1	2	3	4	5	6	7	8	9
11	12	1	5	5	1					
12	15	1	5	7	2					
13	18	1	6	8	3					
14	22	1	6	10	5					
15	27	1	7	12	6	1				
16	32	1	7	14	9	1				
17	38	1	8	16	11	2				
18	46	1	8	19	15	3				
19	54	1	9	21	18	5				
20	64	1	9	24	23	7				
21	76	1	10	27	27	10	1			
22	89	1	10	30	34	13	1			
23	104	1	11	33	39	18	2			
24	122	1	11	37	47	23	3			
25	142	1	12	40	54	30	5			
26	165	1	12	44	64	37	7			
27	192	1	13	48	72	47	11			
28	222	1	13	52	84	57	14	1		
29	256	1	14	56	94	70	20	1		
30	296	1	14	61	108	84	26	2		
31	340	1	15	65	120	101	35	3		
32	390	1	15	70	136	119	44	5		
33	448	1	16	75	150	141	58	7		
34	512	1	16	80	169	164	71	11		
35	585	1	17	85	185	192	90	15		
36	668	1	17	91	206	221	110	21	1	
37	760	1	18	96	225	255	136	28	1	
38	864	1	18	102	249	291	163	38	2	
39	982	1	19	108	270	333	199	49	3	
40	1113	1	19	114	297	377	235	65	5	
41	1260	1	20	120	321	427	282	82	7	
42	1426	1	20	127	351	480	331	105	11	
43	1610	1	21	133	378	540	391	131	15	
44	1816	1	21	140	411	603	454	164	22	
45	2048	1	22	147	441	674	532	201	29	1
46	2304	1	22	154	478	748	612	248	40	1
47	2590	1	23	161	511	831	709	300	52	2
48	2910	1	23	169	551	918	811	364	70	3
49	3264	1	24	176	588	1014	931	436	89	5
50	3658	1	24	184	632	1115	1057	522	116	7

However, Euler did not use Comtet's recurrence equation or any other to answer Naudé's question. Rather, he introduced another technique that is widely used in combinatorics and number theory, that of generating functions, which we look at the end of this section, starting on page 243. Specifically, in the case of integer partitions, Euler noticed that the coefficient of $k^m x^n$ in the expansion of this expression would give the number of ways in which n can be written in m distinct summands.

$$(1+kx)(1+kx^2)(1+kx^3)(1+kx^4)(1+kx^5)\ldots$$

So, to answer Naudé's question, he needed to find the coefficient of k^7x^{50} in this expansion of the product, which is as far as he took it in *Introductio*:[158]

$$\prod_{j=1}^{\infty}(1+kx^j) = 1 + k(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \cdots)$$

$$+k^2(x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + 5x^{11} + \cdots)$$
$$+k^3(x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + 10x^{14} + \cdots)$$
$$+k^4(x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + 11x^{17} + 15x^{18} + \cdots)$$
$$+k^5(x^{15} + x^{16} + 2x^{17} + 3x^{18} + 5x^{19} + 7x^{20} + 10x^{21} + 13x^{22} + 18x^{23} + \cdots)$$

$$+k^6(x^{21} + x^{22} + 2x^{23} + 3x^{24} + 5x^{25} + 7x^{26} + 11x^{27} + 14x^{28} + 20x^{29} + \cdots)$$
$$+k^7(x^{28} + x^{29} + 2x^{30} + 3x^{31} + 5x^{32} + 7x^{33} + 11x^{34} + 15x^{35} + 21x^{36} + \cdots)$$
$$+k^8(x^{36} + x^{37} + 2x^{38} + 3x^{39} + 5x^{40} + 7x^{41} + 11x^{42} + 15x^{43} + 22x^{44} + \cdots)$$

etc.

Although Euler did not give the coefficient of k^7 up to x^{50} in *Introductio*, he did provide a rectangular table for $n \leq 69$ and $k \leq 11$, as a realignment of the two triangular representations of $Q(n, k)$ and $P(n, k)$, above, enabling him to say that the number 50 can be expressed in unequal and either equal or unequal numbers in 522 and 8946 ways.[159] So, with this table he was able to say in E158 that the fiftieth coefficient of k^7 is 522, thereby bringing Naudé's problem to a 'most perfect solution'.[160] For, as you can see, the sequences of coefficients of the powers of x associated with the powers of k are the columns in the triangle for $Q(n, k)$. To find the total number of strict partitions $q(n)$, Euler set $k = 1$, giving:[161]

$$\prod_{j=1}^{\infty}(1 + x^j) = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \cdots$$

To find the number of partitions $P(n, k)$, free of the condition that the summands be distinct, Euler realized that the coefficient of $k^m x^n$ in the expansion of this expression would give him the required numbers:

$$\frac{1}{(1 - kx)(1 - kx^2)(1 - kx^3)(1 - kx^4)(1 - kx^5)\ldots}$$

Accordingly, in *Introductio*, he gave this expansion:[162]

$$\prod_{j=1}^{\infty}\frac{1}{(1 - kx^j)} = 1 + k(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \cdots)$$
$$+k^2(x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + 4x^9 + 5x^{10} + \cdots)$$
$$+k^3(x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + 8x^{10} + 10x^{11} + \cdots)$$
$$+k^4(x^4 + x^5 + 2x^6 + 3x^7 + 5x^8 + 6x^9 + 9x^{10} + 11x^{11} + 15x^{12} + \cdots)$$
$$+k^5(x^5 + x^6 + 2x^7 + 3x^8 + 5x^9 + 7x^{10} + 10x^{11} + 13x^{12} + 18x^{13} + \cdots)$$
$$+k^6(x^6 + x^7 + 2x^8 + 3x^9 + 5x^{10} + 7x^{11} + 11x^{12} + 14x^{13} + 20x^{14} + \cdots)$$
$$+k^7(x^7 + x^8 + 2x^9 + 3x^{10} + 5x^{11} + 7x^{12} + 11x^{13} + 15x^{14} + 21x^{15} + \cdots)$$
$$+k^8(x^8 + x^9 + 2x^{10} + 3x^{11} + 5x^{12} + 7x^{13} + 11x^{14} + 15x^{15} + 22x^{16} + \cdots)$$

etc.

For instance, he could read immediately from this formula that 13 can be written as the sum of 5 whole numbers in 18 ways. Setting $k = 1$ once again, Euler obtained this generating function for the integer partitions:

$$\prod_{j=1}^{\infty}\frac{1}{(1 - x^j)} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \cdots$$

As ever, Euler did not stop there. He proved a number of theorems with various restrictions on the partitions, including this: *The number of different ways a given number can be expressed as the sum of different whole numbers is the same as the number of ways in which the same number can be expressed as the sum of odd numbers, whether the same or different.*[163] For instance, 6 has four such subsets, given in this table.[164] This is just one of many identities that mathematicians have found when enumerating subsets of partitions.

Odd	**Distinct**
5 + 1	5 + 1
3 + 3	6
3 + 1 + 1 + 1	3 + 2 + 1
1 + 1 + 1 + 1 + 1 + 1	4 + 2

However, while Euler created an extensive table of values, he was still in search of a recurrence equation in which to generate partition numbers. To this end, he noted that the expansion of the reciprocal of the function that generates the unrestricted partition numbers is:

$$(1-x)(1-x^2)(1-x^3)\ldots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots$$

where the exponents 1, 2, 5, 7, 12, ... on the right-hand side are given by the formula

$$g_k = \frac{1}{2}k(3k-1)$$

This we recognize as the *k*th pentagonal number, normally defined for positive values of *k*. However, in this expression, *k* = 1, −1, 2, −2, 3, -3, …, giving the generalized pentagonal numbers, which are the exponents in the pentagonal number theorem, relating the product and series representations of the Euler function:[165]

$$\prod_{k=1}^{\infty}(1-x^k) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2} = 1 + \sum_{k=0}^{\infty}(-1)^k\left(x^{k(3k+1)/2} + x^{k(3k-1)/2}\right)$$

Now, the coefficients of this generating function are the differences between partition numbers with even and odd different summands, as this table indicates:

OEIS	*n*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A000009	*q*(*p*)	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64
A067661	**even**	1	0	0	1	1	2	2	3	3	4	5	6	7	9	11	13	16	19	23	27	32
A067659	**odd**	0	1	1	1	1	1	2	2	3	4	5	6	8	9	11	14	16	19	23	27	32
A010815	**diff**	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0	-1	0	0	0	0	0

Rather surprisingly, Euler discovered that this pattern occurs in an apparently unrelated subject. When exploring the divisor function that enabled him to find 59 amicable pairs, additional to the three already found, as mentioned in Chapter 3, he wondered if it were possible to find any pattern in the sums of divisors of each natural number, today often called the sigma function. To this end, he presented the values of the sigma function up to 100 in tabular form, which George Pólya (1887–1985) simplified thus, in his translation and commentary of 'Discovery of an extraordinary law of numbers in relation to the sum of their divisors':[166]

	1	2	3	4	5	6	7	8	9	10
0	1	3	4	7	6	12	8	15	13	18
1	12	28	14	24	24	31	18	39	20	42
2	32	36	24	60	31	42	40	56	30	72
3	32	63	48	54	48	91	38	60	56	90
4	42	96	44	84	78	72	48	124	57	93
5	72	98	54	120	72	120	80	90	60	168
6	62	96	104	127	84	144	68	126	96	144
7	72	195	74	114	124	140	96	168	80	186
8	121	126	84	224	108	132	120	180	90	234
9	112	168	128	144	120	252	98	171	156	217

When looking at this table of numbers, Euler said, "we are almost driven to despair. We cannot hope to discover the least order. The irregularity of the primes [marked in red] is so deeply involved in it that we must think it impossible to disentangle any law governing this sequence, unless we know the law governing the sequence of the primes itself."[167] Nevertheless, he did find a recurring sequence linking all these numbers:

$$\begin{aligned}\sigma(n) = {} & \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) \\ & + \sigma(n-12) + \sigma(n-15) - \sigma(n-22) - \sigma(n-26) \\ & + \sigma(n-35) + \sigma(n-40) - \sigma(n-51) - \sigma(n-57) \\ & + \sigma(n-70) + \sigma(n-77) - \sigma(n-92) - \sigma(n-100) \\ & + \cdots\end{aligned}$$

Although he admitted that he was unable to give this formula a 'rigorous demonstration', he noted that the exponents of x in the formula $(1-x)(1-x^2)(1-x^3)\dots$ are the same as in this function, with the signs + and - similarly arising twice in succession. This could not be coincidence, enabling him to eventually prove the pentagonal number theorem,[168] establishing the validity of this amazing recurrence equation for the partition numbers $p(n)$, setting $p(n-k)=0$, when $n<k$:[169]

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - p(n-26) - \cdots$$

There is one other fascinating recurrence equation between the sums of divisors and integer partitions,[170] given here:

$$p(n) = \frac{1}{n}\sum_{k=0}^{n-1} \sigma(n-k)p(k) \qquad p(0) = 1$$

where $\sigma(k)$ is the sum of the divisors of k (OEIS A000203), also known as $\sigma_1(k)$, as we saw in the previous chapter. This is a fascinating relationship, showing a connection between divisions of integers in terms of both multiplicative factors and partitions by addition, giving this 'Partition triangle', not dissimilar to Segner's recurrence equation for the Catalan numbers:

p_0	1	= 1	= 1
p_1	1·1	= 1 /1	= 1
p_2	3·1 + 1·1	= 4 /2	= 2
p_3	4·1 + 3·1 + 1·2	= 9 /3	= 3
p_4	7·1 + 4·1 + 3·2 + 1·3	= 20 /4	= 5
p_5	6·1 + 7·1 + 4·2 + 3·3 + 1·5	= 35 /5	= 7
p_6	12·1 + 6·1 + 7·2 + 4·3 + 3·5 + 1·7	= 66 /6	= 11
p_7	8·1 + 12·1 + 6·2 + 7·3 + 4·5 + 3·7 + 1·11	= 105 /7	= 15
p_8	15·1 + 8·1 + 12·2 + 6·3 + 7·5 + 4·7 + 3·11 + 1·15	= 176 /8	= 22

Leonard James Rogers (1862–1933) found two other identities, as further subsets of $q(n)$, as particular cases of more general theorems, not immediately associating them with partitions, publishing their proofs in 1894 in *The Proceedings of the London Mathematical Society*.[171] Not unlike Euler, he showed that an infinite series could be represented as an infinite product of terms in two ways:

$$1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots = \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})(1-q^{14})\dots}$$

$$1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)} + \cdots = \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})(1-q^{13})\dots}$$

As an obituary, published by the Royal Society, points out, the indices of the powers of q in the numerators of the two series are n^2 and $n(n+1)$ and the indices of the powers of q in the two products are of the form $5k+1$ or $5k+4$ and $5k+2$ or $5k+3$, respectively. However, the mathematical community completely ignored these remarkable identities, perhaps because Rogers had no desire for recognition, being more interested in music than in exploring the work of other mathematicians.[172]

In the event, Ramanujan independently discovered these beautiful identities without proof in 1913, when studying partitions, which MacMahon published in 1916 in volume II of *Combinatory Analysis*.[173] Then, in 1917, Ramanujan accidently discovered Rogers's paper, subsequently publishing his own work with Rogers in 1919. In the meantime, Issai Schur (1875–1941), working in Germany during the First World War, unaware of developments in England, also independently discovered these identities in 1917.[174] Such is the stumbling way in which human knowledge is discovered, lying at the heart of what George E. Andrews and Kimmo Eriksson call the 'romance of mathematics' in their introductory book *Integer Partitions*—full of life stories and anecdotes of an astonishing nature.[175]

Partition	$G_1(n)$	$G_2(n)$	$H_1(n)$	$H_2(n)$
10		✓		✓
9 1	✓	✓		
8 2		✓	✓	✓
7 3		✓	✓	✓
6 4	✓	✓		✓
6 3 1		✓		
6 1 1 1 1	✓			
4 4 1 1	✓			
4 1 1 1 1 1 1	✓			
3 3 2 2			✓	
2 2 2 2 2			✓	
1 1 1 1 1 1 1 1 1 1	✓			
Total	**6**	**6**	**4**	**4**

What is today called the first Rogers–Ramanujan identity, $G(n)$, shows that the number of partitions of n into summands that are all of the form $5k \pm 1$ is equal to the number of partitions into distinct parts, where the difference between successive parts is ≥ 2. The Rogers–Ramanujan second identity, $H(n)$, shows that the number of partitions of n into addends that are all of the form $5k \pm 2$ is equal to the number of partitions into distinct parts, where the difference between successive parts is ≥ 2 and the smallest term is ≥ 2. Here are the examples of these identities that MacMahon gave for $n = 10$, saying that the second condition excludes repetitions and sequences.[176]

In summary, here are the first 21 terms of $G(n)$ and $H(n)$, which Wikipedia tells us are the coefficients of infinite polynomials in q, as expansions of the Rogers–Ramanujan identities.[177]

OEIS	n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A003114	$G(n)$	1	1	1	1	2	2	3	3	4	5	6	7	9	10	12	14	17	19	23	26	31
A003106	$H(n)$	1	0	1	1	1	1	2	2	3	3	4	4	6	6	8	9	11	12	15	16	20

Now, while there are several recurrence equations for integer partitions, it was not until 1918 that Ramanujan, with the help of Hardy, found and proved this highly complex asymptotic approximation:[178]

$$p(n) = \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

which Hans Rademacher(1892–1969) later completed and perfected.[179]

Ramanujan also discovered these remarkable congruences for unrestricted partitions:

$p(5k+4) \equiv 0 \pmod 5$

$p(7k+5) \equiv 0 \pmod 7$

$p(11k+6) \equiv 0 \pmod{11}$

In other words, if n is a member of the sets {4, 9, 14, …}, {5, 12, 19, …}, or {6, 17, 28, …}, then the number of partitions is {5, 30, 135, …}, {7, 77, 490, …}, or {11, 297, 3718, …}, all divisible by 5, 7, or 11, respectively.

Ramanujan proved two of them in 1919, with Hardy completing the proof of the other after Ramanujan died.[180] It might appear that these congruences are the beginning of many other similar ones. However, Ramanujan stated, "It appears there are no equally simple properties for any moduli involving primes other than these".[181]

In the event, it was not until 2011 that mathematicians were able to explain why only these three simple congruences exist. In the meantime, from the 1960s on, A. O. L. Atkin (1925–2008), a wartime colleague of Alan Turing, and his successors, found other, more complex, congruences, for small prime moduli, such as:

$$p(11^3 \cdot 13k + 237) \equiv 0 \pmod{13}$$
$$p(23^3 \cdot 17k + 2623) \equiv 0 \pmod{17}$$
$$p(101^4 \cdot 19k + 815655) \equiv 0 \pmod{19}$$

Indeed, it has now been proved that there is a similar congruence for all partitions with a prime modulus. During this century, Ken Ono, Karl Mahlburg, Rhiannon L. Weaver, and Fredrik Johansson found other algorithms, generating over 22 billion congruences, apparently related to the Hardy–Ramanujan–Rademacher formula for $p(n)$, another large example being: [182]

$$p(999959^4 \cdot 29k + 28995221336976431135321047) \equiv 0 \pmod{29}$$

There seems to be no end to the amazing patterns that mathematicians find among the partitions.

As an aside, rather than partitioning integers by addition, we can also explore their composition by multiplication. On this point, we saw in the previous chapter that Ramanujan defined a highly composite number as one that has more divisors than any number smaller than it (OEIS A002182). The number of divisors of the *n*th highly composite number is OEIS A002183 and the number of prime factors is OEIS A112778.

However, Euler went even further with his studies of prime factorization. He defined a function, denoted $\phi(n)$ or $\varphi(n)$, which J. J. Sylvester termed *totient*, as the number of integers k in the range $1 \le k \le n$ for which the greatest common divisor gcd(n, k) is equal to 1. For instance, for $n = 9$, the six numbers 1, 2, 4, 5, 7, and 8 are totients of 9 for they are all relatively prime to 9. So, $\varphi(9) = 6$. The totient function thus just counts the totients of n and doesn't add the elements, like partitions, or multiply them, like prime factors.

However, there is a simple formula to calculate the totient of any number. If it is a prime p, then, fairly obviously, $\varphi(p) = p - 1$. On the other hand, if it is a composite number $n = p^\alpha q^\beta r^\gamma$ ..., where p, q, and r are distinct primes, then we need to eliminate all multiples of p, q, and r, etc. from the set {1, 2, 3, …, n} to find $\varphi(n)$. The function for doing this is:[183]

$$\varphi(n) = n(1 - {}^1/_p)(1 - {}^1/_q)(1 - {}^1/_r) \ldots$$

For instance,

$$\varphi(12) = \varphi(2^2 \cdot 3) = 12(1 - ½)(1 - ⅓) = 4$$

counting the prime factor 2 just once, not twice.[184]

One other important property that Euler discovered is that if a and n are coprime, then:

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

When n is prime, $\varphi(n) = n - 1$, and Euler's totient function becomes Fermat's little theorem, which we met in Chapter 3, and which Euler proved in various ways over a period of nearly thirty years.[185]

Exploring the totient function further is more than I wish to go in this book, other than to give the first twenty terms of the sequence (OEIS A000010), which Euler lists in '*Theoremata arithmetica nova methodo demonstrate*', published in 1763, but the only one of Euler's papers on Fermat's little theorem not translated into English:[186]

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\boldsymbol{\varphi(n)}$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16	6	18	8

Stirling numbers

We next come to the Stirling cycle and set numbers, also known as the Stirling numbers of the first and second kind. These numbers are named after James Stirling (1692–1770), who described them in 1730 in *Methodus Differentials: Sive Tractatus De Summatione Et Interpolatione Seriesum Infinitarum* (The method of Differences: or a Treatise on Summation and Interpolation of Infinite Series). Stirling is most famous for this celebrated approximation for $n!$, although this is as much the work of Abraham de Moivre (1667–1754):[187]

$$n! \approx \sqrt{2\pi n} n^n e^{-n}$$

In keeping with the central theme of this book, I prefer to begin describing these numbers as generative numerical data patterns, rather than with their modern interpretations in terms of combinatorics and abstract algebra. But first, a word about notation, about which there is not universal consensus among mathematicians, as Eric W. Weisstein tells us on Wolfram *MathWorld*. Perhaps the simplest way of denoting them is as $s(n, k)$ and $S(n, k)$, like $C(n, k)$ for the binomial coefficients. However, there is an added complication here, for there are two versions of the Stirling numbers of the first kind: signed and unsigned. Regarding the latter, which Stanley denotes with $c(n, k)$,[188] the Stirling numbers of the first kind are also denoted as, inspired by the notation $\binom{n}{k}$ for the binomial coefficients:[189]

$$\begin{bmatrix} n \\ k \end{bmatrix}$$

with the Stirling numbers of the second kind being denoted as:

$$\begin{Bmatrix} n \\ k \end{Bmatrix}$$

The unsigned Stirling numbers of the first kind are generated in triangular form, like Pascal's triangle. But rather than adding the numbers to the left and right in the previous row, the number to the right is first multiplied by $(n - 1)$, the sum of each row then being n! The suffixed superscripts in this table denote the multiplication factor for each cell in the triangle, with the factorials being OEIS A000142.

n	**Unsigned Stirling numbers of the first kind**										*n*!
1	1										1
2	1^{1}	1^{1}									2
3	2^{2}	3^{2}	1^{2}								6
4	6^{3}	11^{3}	6^{3}	1^{3}							24
5	24^{4}	50^{4}	35^{4}	10^{4}	1^{4}						120
6	120^{5}	274^{5}	225^{5}	85^{5}	15^{5}	1^{5}					720
7	720^{6}	1764^{6}	1624^{6}	735^{6}	175^{6}	21^{6}	1^{6}				5040
8	5040^{7}	13068^{7}	13132^{7}	6769^{7}	1960^{7}	322^{7}	28^{7}	1^{7}			40320
9	40320^{8}	109584^{8}	118124^{8}	67284^{8}	22449^{8}	4536^{8}	546^{8}	36^{8}	1^{8}		362880
10	362880^{9}	1026576^{9}	1172700^{9}	723680^{9}	269325^{9}	63273^{9}	9450^{9}	870^{9}	45^{9}	1^{9}	3628800

To generate unsigned Stirling numbers of the first kind, Pascal's triangle rule, defined on page 197, becomes:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}$$

with $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$, $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n+1 \end{bmatrix} = 0$, and $1 \le k \le n$.

The (signed) Stirling numbers of the first kind are then generated from this recurrence equation:

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k)$$

with $s(1, 1) = 1$, $s(n, 0) = s(n, n + 1) = 0$, and $1 \le k \le n$.

The sum of $s(n, k)$ on each row is 0. Here is the Stirling triangle of the first kind (OEIS A008275).

n	**Stirling numbers of the first kind**									
1	1									
2	-1	1								
3	2	-3	1							
4	-6	11	-6	1						
5	24	-50	35	-10	1					
6	-120	274	-225	85	-15	1				
7	720	-1764	1624	-735	175	-21	1			
8	-5040	13068	-13132	6769	-1960	322	-28	1		
9	40320	-109584	118124	-67284	22449	-4536	546	-36	1	
10	-362880	1026576	-1172700	723680	-269325	63273	-9450	870	-45	1

The recurrence equation for the Stirling numbers of the second kind is similar to that for the unsigned Stirling numbers of the first kind, except that the number on the right is first multiplied by k, the value of the position of the cell in the row:

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

with $\left\{ {1 \atop 1} \right\} = 1$, $\left\{ {n \atop n+1} \right\} = 0$, and $1 \le k \le n$.

Here is the Stirling triangle of the second kind (OEIS A008277), again with suffixed superscripts denoting the multiplication factor for the right-hand term in the previous row. The totals of each row b_n are Bell numbers (OEIS A000110), named after Eric Temple Bell (1883–1960).

n	Stirling numbers of the second kind										b_n
1	1										1
2	1^1	1^2									2
3	1^1	3^2	1^3								5
4	1^1	7^2	6^3	1^4							15
5	1^1	15^2	25^3	10^4	1^5						52
6	1^1	31^2	90^3	65^4	15^5	1^6					203
7	1^1	63^2	301^3	350^4	140^5	21^6	1^7				877
8	1^1	127^2	966^3	1701^4	1050^5	266^6	28^7	1^8			4140
9	1^1	255^2	3025^3	7770^4	6951^5	2646^6	462^7	36^8	1^9		21147
10	1^1	511^2	9330^3	34105^4	42525^5	22827^6	5880^7	750^8	45^9	1^{10}	115975

Of particular interest here is that if we view the Stirling triangles as lower triangular matrices, that is with zeroes above and to the right of the main diagonal, the matrices are inverses of each other. This means that their product is the identity matrix, as in this example for n = 5:

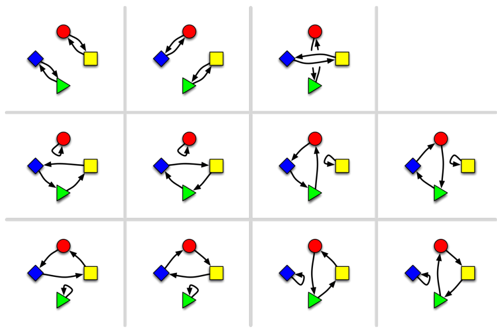
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 \\ 24 & -50 & 35 & -10 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

As there is no reference to the Stirling numbers in MacMahon's *Combinatory Analysis* from 1915 and 1916, it seems that it was not until the emergence of modern algebra that mathematicians began to interpret these triangular sequences combinatorially. For instance, in *An Introduction to Combinatorial Analysis* in 1958, dedicated to E. T. Bell, Riordan introduced Stirling numbers fairly early on, with the first chapter reviewing the algebra of both combinations and permutations. However, five years later, Ryser does not mention Stirling in his comparatively short monograph on *Combinatorial Mathematics*.

Interpreting the Stirling numbers, $c(n, k)$ are called the Stirling cycle numbers, the count of the permutations of n objects that have just k cycles and $S(n, k)$ are called Stirling set numbers, the number of groupings of n distinct things into exactly k groups. Such objects can be anything whatsoever. So, I'll illustrate them with letters of the alphabet, although they can also be illustrated graphically, like some of the interpretations of the Catalan numbers, as Robert M. Dickau shows on his web site of Math Figures.

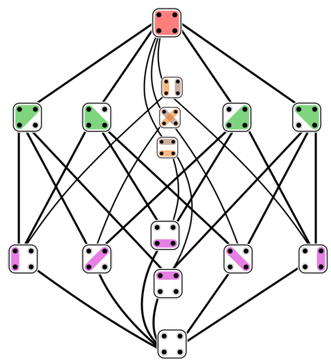
The concept of cycle in the Stirling cycle numbers derives from group theory, which we look at in the next chapter. Here is an example that links combinatorics with abstract algebra, where cycles of permutations are called orbits, shown in the diagram on the right.[190]

c(4, 1)=6	*c*(4, 2)=11	*c*(4, 3)=6	*c*(4, 4)=1
{a,b,c,d}	{a,b,c} {d}	{a,b} {c} {d}	{a} {b} {c} {d}
{a,b,d,c}	{a,c,a} {d}	{a,c} {b} {d}	
{a,c,b,d}	{a,b,d} {c}	{a,d} {b} {c}	
{a,c,d,b}	{a,d,b} {c}	{b,c} {a} {d}	
{a,d,b,c}	{a,c,d} {b}	{b,d} {a} {b}	
{a,d,c,b}	{a,d,c} {b}	{c,d} {a} {b}	
	{b,c,d} {a}		
	{b,d,c} {a}		
	{a,b} {c,d}		
	{a,c} {b,d}		
	{a,d} {b,c}		



Here is an example of the fourth row in the Stirling triangle of the second kind, where the cycles in the first triangle are not distinguishable. It also shows how these partitions of sets can be arranged in a Hasse diagram. If the crossed set in the diagram is removed, omitting similar structures in higher values of n, the Stirling triangle reduces to the Narayana triangle on page 214.[191]

S(4, 1)=1	*S*(4, 2)=7	*S*(4, 3)=6	*S*(4, 4)=1
{a,b,c,d}	{a,b,c} {d}	{a,b} {c} {d}	{a} {b} {c} {d}
	{a,b,d} {c}	{a,c} {b} {d}	
	{a,c,d} {b}	{a,d} {b} {c}	
	{b,c,d} {a}	{b,c} {a} {d}	
	{a,b} {c,d}	{b,d} {a} {b}	
	{a,c} {b,d}	{c,d} {a} {b}	
	{a,d} {b,c}		



The total of the rows, in the second Stirling triangle, the Bell numbers, are also called exponential numbers:[192] the total number of ways to partition a set of n labelled elements (OEIS A000110). Naturally, the centre columns of the Stirling triangles form sequences, like the central binomial in Pascal's triangle. The centre column of the first unsigned Stirling triangle is the number of permutations of $2n$-1 objects with exactly n cycles (OEIS A129505) and the centre column of the second Stirling triangle is the number of partitions of a {$2n$-1}-set into n nonempty subsets (OEIS A129506).

As well as the lower triangular matrix form of the Stirling numbers of the first and second kind being inverses of each other, Ivo Lah (1896–1979), an actuary, found another relationship between these two triangular sequences in a 1955 paper titled 'A new kind of numbers, their properties and applications in mathematical statistics'.[193]

Like the Stirling numbers of the first kind, Lah numbers, as Riordan named them,[194] have signed and unsigned versions, the former being initially defined by Riordan and Comtet as $L_{n,k}$. However, as the unsigned numbers are most useful in combinatorics, I'll use Wikipedia's notation of $L'(n,k)$ to denote the original Lah numbers and $L(n,k)$ as those most commonly used.

The recurrence equation that generates the signed Lah numbers is not unlike those that generate the Stirling numbers of the first and second kinds:

$$L'(n+1,k) = -L'(n,k-1) - (n+k)L'(n,k)$$

with $L'(1,1) = -1, L'(n,0) = 0, L'(n,k) = 0, k > n$

From this, we obtain this formula for each term in the Lah triangle of numbers:

$$L'(n,k) = (-1)^n \binom{n-1}{k-1} \frac{n!}{k!}$$

And here is the relationship between the Lah numbers and the Stirling numbers:[195]

$$L'(n,k) = \sum_{j=k}^{n} (-1)^j s(n,j) S(j,k)$$

For instance,

$$L'(6,3) = (-1)(-225) \cdot 1 + 85 \cdot 6 + (-1)(-15) \cdot 25 + 1 \cdot 90 = 1200$$

These functions generate this lower triangular matrix of Lah numbers (OEIS A008297):

*n**k*	1	2	3	4	5	6	7	8	9	10	Σ
1	-1										-1
2	2	1									3
3	-6	-6	-1								-13
4	24	36	12	1							73
5	-120	-240	-120	-20	-1						-501
6	720	1800	1200	300	30	1					4051
7	-5040	-15120	-12600	-4200	-630	-42	-1				-37633
8	40320	141120	141120	58800	11760	1176	56	1			394353
9	-362880	-1451520	-1693440	-846720	-211680	-28224	-2016	-72	-1		-4596553
10	3628800	16329600	21772800	12700800	3810240	635040	60480	3240	90	1	58941091

Lah numbers

In combinatorics, the positive, unsigned Lah numbers are used, defined as $(-1)^n L'(n,k)$,[196] giving OEIS A105278. The recurrence equation then becomes:

$$L(n+1,k) = L(n,k-1) + (n+k)L(n,k)$$

with $L(1,1) = 1, L(n,0) = 0, L(n,k) = 0, k > n$

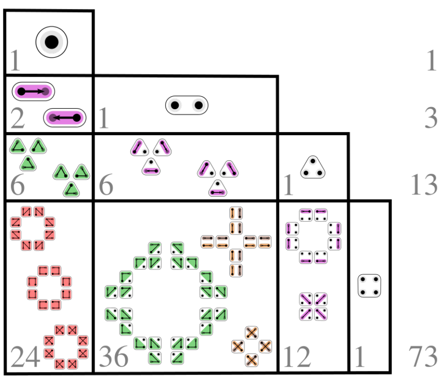
To see why the Lah numbers grow so must faster than the Stirling numbers, here is a triangular table of them showing the way that each term is the sum of the upper-left term and the upper-right term multiplied by the suffixed superscript, as the sum of those used in the Stirling numbers of the first and second kind.

n	Unsigned Lah numbers																	Σ
1									1									1
2								2^2		1^3								3
3							6^3		6^4		1^5							13
4						24^4		36^5		12^6		1^7						73
5					120^5		240^6		120^7		20^8		1^9					501
6				720^6		1800^7		1200^8		300^9		30^{10}		1^{11}				4051
7			5040^7		15120^8		12600^9		4200^{10}		630^{11}		42^{12}		1^{13}			37633
8		40320^8		141120^9		141120^{10}		58800^{11}		11760^{12}		1176^{13}		56^{14}		1^{15}		394353
9	362880^9		1451520^{10}		1693440^{11}		846720^{12}		211680^{13}		28224^{14}		2016^{15}		72^{16}		1^{17}	4596553

There is also a recurrence equation defining each column in terms of the previous one:

$$L(n,k+1) = \frac{n-k}{k(k+1)} L(n,k) \qquad L(n,1) = n!$$

So, setting k = 1, 2, 3, etc., we can determine the nth term of the sequences for the columns in the Lah triangle, given with their generating functions on page 253. The diagonals don't have such an obvious pattern, except the sequence after the first, which are called oblong numbers (OEIS A002378), whose nth term is:



$$L(n,n-1) = n(n-1) \qquad n > 1$$

In this context, the Lah numbers count the number of ways to partition a set of n elements into k nonempty linear queues, illustrated in this diagram from Wikipedia. The totals of each row are OEIS A000262, defined as the "number of 'sets of lists': number of partitions of {1, 2, 3, …, n} into any number of lists, where a list means an ordered subset." Enumerating all ordered subsets in this way generates a sequence that grows very fast, but not as rapidly as the number of elements in higher dimensional permutatopes, whose generating number of vertices is similarly $n!$, as we see on page 331. A000262 has this recurrence equation:

$$a(n) = (2n-1)a(n-1) - (n-1)(n-2)a(n-2) \quad a(1) = 1, a(2) = 3$$

Unsigned Lah numbers are also coefficients expressing rising factorials in terms of falling factorials and vice versa,[197] which may be the way that Lah first found them. Rising and falling factorials are defined as

$$x^{(n)} = x^{\overline{n}} = x(x+1)(x+2)\ldots(x+n-1)$$

and

$$(x)_n = x^{\underline{n}} = x(x-1)(x-2)\ldots(x-n+1)$$

Then, the unsigned Lah numbers are the coefficients of these expressions:

$$x^{\overline{n}} = \sum_{k=1}^{n} L(n,k)\, x^{\underline{n}}$$

and

$$x^{\underline{n}} = \sum_{k=1}^{n} (-1)^{n-k} L(n,k)\, x^{\overline{n}}$$

For instance, the red coefficients here are based on positive Lah numbers, adjusting for the sign in the second example.

$$x(x+1)(x+2) = 6x + 6x(x-1) + 1x(x-1)(x-2)$$

$$x(x-1)(x-2)(x-3) = -24x + 36x(x+1) - 12x(x+1)(x+2) + 1x(x+1)(x+2)(x+3)$$

Comtet defines the Stirling and Lah numbers as special cases of Bell polynomials,[198] which it is not necessary to investigate further at the moment.

Eulerian numbers

Eulerian numbers arise from another way of generating a triangular array from permutations. However, this interpretation came later. Euler discovered these numbers in 1736[199] and further developed them in 1755 in Volume II of *Institutiones calculi differentialis* (Foundations of Differential Calculus), which is concerned with applications of the differential calculus, Volume I being concerned with theory.[200] Having explored generalized series in Chapter 5 of Volume II, including Bernoulli numbers, which we look at next, in Chapter 7 he further generalized these series in a somewhat more complex way, as he admitted, by combining a general series with a geometric one. This led him to define this amazing sequence of expressions:[201]

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1\cdot 2(p-1)^2}$$

$$\gamma = \frac{p^2+4p+1}{1\cdot 2\cdot 3(p-1)^3}$$

$$\delta = \frac{p^3+11p^2+11p+1}{1\cdot 2\cdot 3\cdot 4(p-1)^4}$$

$$\varepsilon = \frac{p^4+26p^3+66p^2+26p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5(p-1)^5}$$

$$\zeta = \frac{p^5+57p^4+302p^3+302p^2+57p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6(p-1)^6}$$

$$\eta = \frac{p^6+120p^5+1191p^4+2416p^3+1191p^2+120p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7(p-1)^7}$$

etc.

Euler also noted that the general form of these formulae could generate them arbitrarily far:

$$\frac{p^{n-2} + Ap^{n-3} + Bp^{n-4} + Cp^{n-5} + Dp^{n-6} + \cdots}{1 \cdot 2 \cdot 3 \cdot \ldots (n-1)(p-1)^{n-1}}$$

where the *n*th terms for *A*, *B*, *C*, *D*, etc. are:

$$A = 2^{n-1} - n$$

$$B = 3^{n-1} - n \cdot 2^{n-1} + \frac{n(n-1)}{1 \cdot 2}$$

$$C = 4^{n-1} - n \cdot 3^{n-1} + \frac{n(n-1)}{1 \cdot 2} 2^{n-1} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$$

$$D = 5^{n-1} - n \cdot 4^{n-1} + \frac{n(n-1)}{1 \cdot 2} 3^{n-1} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} 2^{n-1} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

etc.

We need to be careful with the indexing here, which did not seem to concern Euler explicitly. The powers of *p* in the polynomial in the denominator (*p* – *k*) end when *k* = *p*. So the sequence begins with *n* = 2, rather than 1 or 0, as is more common. Furthermore, the initial values of *A*, *B*, *C*, *D*, etc. as 1 are set when *n* = 3, 4, 5, 6, etc. When *n* is less than these values, all the terms in the formulae miraculously cancel themselves out and *A*, *B*, *C*, *D*, etc. become 0.

The polynomials in the numerators of Euler's formulae are called Euler's polynomials today and the coefficients of p^k form what is known as Euler's number triangle (OEIS A008292), as Riordan and Comtet presented it in their classic books *Combinatorial Analysis*[202] and *Advanced Combinatorics*[203] in 1968 and 1970, respectively. Eulerian numbers are generated by another modification of Pascal's triangle rule. But unlike the Stirling numbers of the second kind, the numbers to the left in the previous row are multiplied by a factor (*n* – *k* + 1), indicated by the prefixed superscripts in the triangle. As *MathWorld* puts it, "The Eulerian numbers represent a sort of generalization of the binomial coefficients where the defining recurrence relation weights the sum of neighbours by their row and column numbers, respectively."[204] The totals of each row are *n*!, like Stirling numbers of the first kind.

n	**Euler's number triangle**	*n*!
1	1	1
2	${}^{2}1^{1}\quad {}^{1}1^{2}$	2
3	${}^{3}1^{1}\quad {}^{2}4^{2}\quad {}^{1}1^{3}$	6
4	${}^{4}1^{1}\quad {}^{3}11^{2}\quad {}^{2}11^{3}\quad {}^{1}1^{4}$	24
5	${}^{5}1^{1}\quad {}^{4}26^{2}\quad {}^{3}66^{3}\quad {}^{2}26^{4}\quad {}^{1}1^{5}$	120
6	${}^{6}1^{1}\quad {}^{5}57^{2}\quad {}^{4}302^{3}\quad {}^{3}302^{4}\quad {}^{2}575^{5}\quad {}^{1}1^{6}$	720
7	${}^{7}1^{1}\quad {}^{6}120^{2}\quad {}^{5}1191^{3}\quad {}^{4}2416^{4}\quad {}^{3}1191^{5}\quad {}^{2}120^{6}\quad {}^{1}1^{7}$	5040
8	${}^{8}1^{1}\quad {}^{7}247^{2}\quad {}^{6}4293^{3}\quad {}^{5}15619^{4}\quad {}^{4}15619^{5}\quad {}^{3}4293^{6}\quad {}^{2}247^{7}\quad {}^{1}1^{8}$	40320
9	${}^{9}1^{1}\quad {}^{8}502^{2}\quad {}^{7}14608^{3}\quad {}^{6}88234^{4}\quad {}^{5}156190^{5}\quad {}^{4}88234^{6}\quad {}^{3}14608^{7}\quad {}^{2}502^{8}\quad {}^{1}1^{9}$	362880

The recurrence equation for the numbers in Euler's number triangle is thus:

$$A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k)$$

with $A(1, 1) = 1$, $A(n, 0) = A(n, n + 1) = 0$, and $1 \le k \le n$.

Euler's formulae for *A*, *B*, *C*, *D*, etc. form sequences as the diagonals of the triangle, corresponding to the pyramidal simplexes in Pascal's triangle, highlighted in Wolfram *MathWorld*'s article on the Eulerian numbers. The first few sequences are naturally defined in the OEIS, with Euler's offset reduced by one:

OEIS	*n*th term	Offset	Sequence
A000012	1^n	1	1, 1, 1, 1, 1, …
A000295	$2^n - (n + 1)$	2	1, 4, 11, 26, 57, …

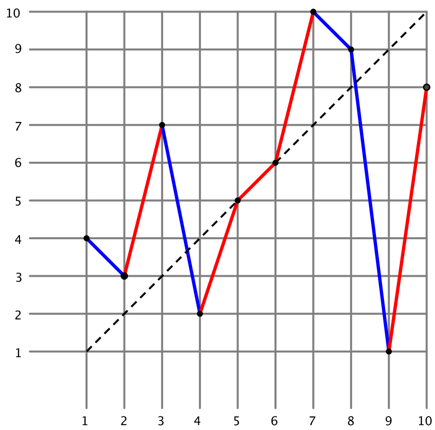
OEIS	*n*th term	Offset	Sequence
A000460	$3^n - 2^n(n+1) + \frac{1}{2}n(n+1)$	3	1, 11, 66, 302, 1191, …
A000498	$4^n - 3^n(n+1) + \frac{1}{2}2^n n(n+1) - \frac{1}{6}(n-1)n(n+1)$	4	1, 26, 302, 2416, 15619, …
A000505	$5^n - 4^n(n+1) + \frac{1}{2}3^n n(n+1) - \frac{1}{6}2^n(n-1)n(n+1) + \frac{1}{24}(n-2)(n-1)n(n+1)$	5	1, 57, 1191, 15619, 156190…

Then, of course, there are the central Eulerian numbers (OEIS A025585):

n	1	2	3	4	5	6	7	8
Ce_n	1	4	66	2416	156190	15724248	2275172004	447538817472

However, when it comes to interpreting the Eulerian numbers in terms of combinatorics, there is some confusion in the literature, which bothered me for a time. Doing my best to sort out this confusion, not being a professional mathematician, I can best begin with Comtet's approach.

He defined a rise [or a fall] in a permutation σ of n elements when $\sigma(i) < \sigma(i+1)$ [or $\sigma(i) > \sigma(i+1)$]. For instance, there are 5 rises and 4 falls in this diagram of a permutation of 10 elements, with the dotted line joining those elements that do not change in the permutation.[205] Using the notation for permutations that Augustin-Louis Cauchy (1789–1857) introduced in 1815:[206]



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 3 & 7 & 2 & 5 & 6 & 10 & 9 & 1 & 8 \end{pmatrix}$$

Comtet then wrote, "The number $a(n,k)$ of permutations of $[n]$ with k rises satisfies the following recurrence relations:

$$a(n,k) = (n-k)a(n-1,k-1) + (k+1)a(n-1,k)$$

for $n, k \geq 1$, with $a(n, 0)=1$ for $n \geq 0$, and $a(0, k) = 0$ for $k \geq 1$".

However, this is not how he defined the Eulerian numbers, which he had earlier defined in terms of an exponential generating function, a concept that we'll look at later. For in the Eulerian triangle, $1 \leq k \leq n$, whereas, in his definition of a rise, $0 \leq k \leq n - 1$. To connect the two recurrence equations, he showed that[207]

$$a(n,k-1) = A(n,k) = A(n,n-k+1)$$

But here there is some confusion in the literature. For instance, Wolfram *MathWorld* calls Comtet's adjusted rise a permutation run[208] and his original rise a permutation ascent,[209] which is denoted as:

$$\left\langle {n \atop k} \right\rangle$$

like the binomial coefficients and Stirling numbers, where < and > denote rise and fall.

However, the Eulerian triangle then becomes asymmetric (OEIS A173018), as defined in *Concrete Mathematics* by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik,[210] with

$$\left\langle {0 \atop 0} \right\rangle = 1 \quad \text{and} \quad \left\langle {n \atop n} \right\rangle = 0$$

The notion of run is implemented as `runs` in the Combinatorica extension to the Mathematica language,[211] which would return this for the permutation σ above:

{{4}, {3,7}, {2, 5, 6, 10}, {9}, {1, 8}}

To unravel this confusion, I look at the way the sets of Eulerian numbers grow as consecutive numbers are added to the base that is permuted. In these permutations, the vertical line | marks each point in the permutation where there is a fall in the sequence of numbers.

<table><tr><td>1</td></tr></table>

<table><tr><td>1 2</td><td>2 | 1</td></tr></table>

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</table>

Then there are two ways of interpreting these permutations, with runs on the left and rises/ascents on the right in this example for n = 4, where the runs are all unique, but the rises are not.

$A(4,1)=1$	$A(4,2)=11$	$A(4,3)=11$	$A(4,4)=1$
{1, 2, 3, 4}	{2}, {1, 3, 4}	{3}, {2}, {1, 4}	{4}, {3}, {2}, {1}
	{1, 3}, {2, 4}	{2}, {1, 4}, {3}	
	{2, 3}, {1, 4}	{3}, {1, 4}, {2}	
	{3}, {1, 2, 4}	{3}, {2, 4}, {1}	
	{1, 2, 4}, {3}	{1, 4}, {3}, {2}	
	{1, 3, 4}, {2}	{2, 4}, {3}, {1}	
	{2, 3, 4}, {1}	{3, 4}, {2}, {1}	
	{1, 4}, {2, 3}	{4}, {2}, {1, 3}	
	{2, 4}, {1, 3}	{4}, {1, 3}, {2}	
	{3, 4}, {1, 2}	{4}, {2, 3}, {1}	
	{4}, {1, 2, 3}	{4}, {3}, {1, 2}	

$a(4,3)=1$	$a(4,2)=11$	$a(4,1)=11$	$a(4,0)=0$
{1, 2}, (2, 3), {3, 4}	{1, 3}, {3, 4}	{1, 4}	—
	{1, 3}, {2, 4}	{1, 4}	
	{2, 3}, {1, 4}	{1, 4}	
	{1, 2}, {2, 4}	{2, 4}	
	{1, 2}, {2, 4}	{1, 4}	
	{1, 3}, {3, 4}	{2, 4}	
	{2, 3}, {3, 4}	{3, 4}	
	{1, 4}, {2, 3}	{1, 3}	
	{2, 4}, {1, 3}	{1, 3}	
	{3, 4}, {1, 2}	{2, 3}	
	{1, 2}, {2, 3}	{1, 2}	

There are clearly some further patterns appearing here, which there is no need to explore further. However, it is interesting to note that the total number of rises in all permutations of order n is:[212]

$$r_n = \sum_{k=0}^{n-1} k \cdot a(n,k) = \frac{1}{2}(n-1)n!$$

giving this sequence (OEIS A001286), the unsigned Lah numbers $L(n\text{-}1, 2)$, for $n > 1$:

n	1	2	3	4	5	6	7	8	9	10
r_n	0	1	6	36	240	1800	15120	141120	1451520	199584000

Sums of powers and Bernoulli numbers

When cataloguing the figurate numbers, we found formulae for the nth terms of the partial sums of the natural numbers, squares, and cubes:

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1) = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(2n^3 + 3n^2 + n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2 = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

But what is the polynomial expansion of the general power series?[213]

$$\sum_{k=1}^{n} k^m$$

Can we find a general formula for this sequence of sequences, the first ten partial sums of powers k^m:

m	OEIS	Sequence
0	A000027	1, 2, 3, 4, 5, 6, …
1	A000217	1, 3, 6, 10, 15, 21, …
2	A000330	1, 5, 14, 30, 55, 91, …
3	A000537	1, 9, 36, 100, 225, 441, …
4	A000538	1, 17, 98, 354, 979, 2275, …
5	A000539	1, 33, 276, 1300, 4425, 12201, …

m	OEIS	Sequence
6	**A000540**	1, 65, 794, 4890, 20515, 67171, …
7	**A000541**	1, 129, 2316, 18700, 96825, 376761, …
8	**A000542**	1, 257, 6818, 72354, 462979, 2142595, …
9	**A007487**	1, 513, 20196, 282340, 2235465, 12313161, …
10	**A023002**	1, 1025, 60074, 1108650, 10874275, 71340451, …

Well, this is a question that has a long history, from the ancient Greeks, through Indian and Arabic mathematicians, to the Europeans in the sixteenth and seventeenth centuries, as Janet Beery explains in her Web pages on 'Sums of Powers of Positive Integers'.[214] In particular, it fascinated Johann Faulhaber (1580–1635), a German cossist or early algebraist who collaborated with Kepler and influenced Descartes with his Rosicrucian thinking. For the Rosicrucians were "a brotherhood combining elements of mystical beliefs with an optimism about the ability of science to improve the human condition",[215] also at the heart of Comenius's pansophic vision, rejected by the founders of the Royal Society, leading to the mess that the world is in today.

After Faulhaber had found formulae for the sums of powers up to the seventh and twelfth in 1614 and 1617,[216] in 1631, he presented formulae for values of m up to 17 in *Academia Algebræ*, written in German despite its Latin title with a rather strange cossist notation. Helpfully interpreting this paper, Donald E. Knuth tells us that Faulhaber first found formulae for odd m in terms of $N = (n^2 + n)/2$, giving the sum of the cubes as N^2, for instance. Then knowing the relationship between the sums of even and odd powers, he was able to interpolate the even powers too.[217] Going from m = 0 to 9, this table from Ken Ward's website gives the coefficients for each of the powers of n:[218]

Power	m+1	m	m-1	m-2	m-3	m-4	m-5	m-6	m-7
0	1/1								
1	1/2	1/2							
2	1/3	1/2	1/6						
3	1/4	1/2	1/4						
4	1/5	1/2	1/3		-1/30				
5	1/6	1/2	5/12		-1/12				
6	1/7	1/2	1/2		-1/6		1/42		
7	1/8	1/2	7/12		-7/24		1/12		
8	1/9	1/2	2/3		-7/15		2/9		-1/30
9	1/10	1/2	3/4		-7/10		1/2		-3/20

There seems to be a pattern here, but what on earth is it? The coefficients total one, the first being $1/(m+1)$, the second ½, and the third seems to be $m/12$. After this, alternating coefficients are zero and the other coefficients alternate between minus and plus. But does this pattern continue indefinitely and what is the pattern that underlies the coefficients? Such a puzzle is not unlike the intelligence tests that teachers set children at school or those that Mensa sets as entry to their exclusive club. Well, like Tycho Brahe, measuring the positions of the stars and planets, Faulhaber did not find the underlying pattern. It was left to Jakob Bernoulli (1654/55–1705), acting like Kepler to Tycho, to find a generalized expression for these coefficients.

Bernoulli found the solution to this problem after reading *Arithmetica Infinitorum* by John Wallis (1616–1703), providing the first adequate proof of the binomial theorem for positive integral powers, presenting an array that is the substantially the same as Pascal's triangle.[219] This proof is contained in Part Two, titled 'The Doctrine of Permutations and Combinations' of *Ars Conjectandi* (*The Art of Conjecturing*), saying in the Introduction:

The Art called *Combinatorics* should be judged, as it merits, most useful, because it remedies this defect of our mind and teaches us how to enumerate all possible ways in which several things can be combined, transposed, or joined with each other, so that we may be sure that we have omitted nothing that can contribute to our purpose.[220]

Ars Conjectandi, which was incomplete when Bernoulli died, despite writing it on and off for twenty years, was published by his nephew Nicolaus Bernoulli (1687–1759), also nephew to Jakob's younger brother Johann, in 1711, laying down the foundations of modern probability theory.[221] To unravel what some call the 'important and quite mysterious role' that Bernoulli's numbers play in mathematics,[222] I find it useful to begin with a video that Burkard Polster posted on his Mathologer YouTube channel,[223] often an entertaining and clarifying way of revealing the inner secrets of mathematics, counterbalancing the general trend, for "mathematicians like to make things complicated", as Robbert Dijkgraaf, Director of the Institute for Advanced Study, said in another YouTube video on the same day in October 2019.[224]

Mathologer began by noting that the sum of cubes is equal to the square of the sum of the natural numbers, or triangular numbers. Labelling each sum of powers S_m, this suggests that we could find a recurrence equation for S_m in terms of previous values of S_m. To find this relationship in terms of the powers of n, we begin with this expression for the binomial formula:

$$(x-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k}$$

For instance, to find S_4, we use:

$$(x-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5\text{x} - 1$$

which can be rewritten:

$$5x^4 - 10x^3 + 10x^2 - 5\text{x} + 1 \ = x^5 - (x-1)^5$$

Now setting $x = 1$ to n in this formula, we obtain n equations whose sum is:

$$5S_4 - 10S_3 + 10S_2 - 5S_1 + S_0 \ = n^5$$

for the right-hand expressions are a telescoping series in which terms from consecutive equations cancel each other out. As S_3, S_2, S_1, and S_0 are already known, we therefore obtain:

$$S_4 = \sum_{k=1}^{n} n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

We could continue calculating S_m indefinitely for consecutive values of m, but this is a little tedious. We need a general formula. To find this, we recognize that what we are doing in developing a formula for S_4, for instance, is solving a set of five linear equations, which can be solved with one matrix equation in linear algebra, which we look at in Chapter 5 on 'Universal algebra'.

$$\begin{pmatrix} n^1 \\ n^2 \\ n^3 \\ n^4 \\ n^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 \\ 1 & -5 & 10 & -10 & 5 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix}$$

To solve this equation for all values of S_m simultaneously, we take the inverse of the 5 by 5 matrix to obtain:

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} n^1 \\ n^2 \\ n^3 \\ n^4 \\ n^5 \end{pmatrix}$$

Here, we see the coefficients of powers of *n* in the expressions for S_m, for $m = 0$ to 4, which are obtained directly by multiplying the vector of powers of *n* by the matrix. As the inverse matrix is formed directly from Pascal's triangle, we have found a simple way of calculating all the coefficients of the powers of *n* in the expressions for S_m. Of course, Bernoulli did not know of the elegant way of solving the sums-of-powers problem, because matrices were not discovered until the 1800s. Nevertheless, he did base his reasoning on Pascal's triangle in a somewhat similar fashion.

First, acknowledging the contributions that Faulhaber, Wallis, and others had made in their attempts to find the hidden pattern in the sums of powers of the integers, Bernoulli noted that while figurate numbers are generated by addition, powers are generated by multiplication. So, as the sums of figurate numbers are known, "the sums of powers can be investigated with no more difficulty that with which Wallis derived the former from the latter." Then, in just three paragraphs describing this generative process, Bernoulli provided a table of the first ten sums of powers, similar to Faulhaber's formulæ.[225] Most significantly, he arranged the sums of powers in columns,[226] somewhat like this, to which I have added S_0, merging Bernoulli's presentation with that of Mathologer, correcting the mistake that Bernoulli made.[227]

$$S_0 = \sum_{k=1}^{n} k^0 = \frac{1}{1}n$$

$$S_1 = \sum_{k=1}^{n} k^1 = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$S_2 = \sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S_3 = \sum_{k=1}^{n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 + 0n$$

$$S_4 = \sum_{k=1}^{n} k^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 + 0n^2 - \frac{1}{30}n$$

$$S_5 = \sum_{k=1}^{n} k^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 + 0n^3 - \frac{1}{12}n^2 + 0n$$

$$S_6 = \sum_{k=1}^{n} k^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 + 0n^4 - \frac{1}{6}n^3 + 0n^2 + \frac{1}{42}n$$

$$S_7 = \sum_{k=1}^{n} k^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 + 0n^5 - \frac{7}{24}n^4 + 0n^3 + \frac{1}{12}n^2 + 0n$$

$$S_8 = \sum_{k=1}^{n} k^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 + 0n^6 - \frac{7}{15}n^5 + 0n^4 + \frac{2}{9}n^3 + 0n^2 - \frac{1}{30}n$$

$$S_9 = \sum_{k=1}^{n} k^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 + 0n^7 - \frac{7}{10}n^6 + 0n^5 + \frac{1}{2}n^4 + 0n^3 - \frac{3}{20}n^2 + 0n$$

$$S_{10} = \sum_{k=1}^{n} k^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 + 0n^8 - 1n^7 + 0n^6 + 1n^5 + 0n^4 - \frac{1}{2}n^3 + 0n^2 + \frac{5}{66}n$$

When Bernoulli looked at the law of progression in these formulae, he realized immediately that those I have marked in cyan are special, giving this general expression for the sum of powers:[228]

$$\sum n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c(c-1)(c-2)}{2\cdot 3\cdot 4}Bn^{c-3}$$
$$+ \frac{c(c-1)(c-2)(c-3)(c-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}Cn^{c-5}$$
$$+ \frac{c(c-1)(c-2)(c-3)(c-4)(c-5)(c-6)}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}Dn^{c-7} + \cdots$$

"etc., the exponent of the power continually being reduced by two until one arrives at *n* or *nn*." The coefficients "*A*, *B*, *C*, *D*, etc … are so established that each of the coefficients along with the others of its order add up to 1."[229] Together with those I have marked in green, these final coefficients in the sums of powers have been known as Bernoulli numbers B_0, B_1, B_2, etc., since 1755, at the suggestion of both de Moivre and Euler.[230]

So the formulae for the first few power series are as follows:

$$S_0 = \sum_{k=1}^{n} k^0 = \frac{1}{1}(1B_0n)$$

$$S_1 = \sum_{k=1}^{n} k^1 = \frac{1}{2}(1B_0n^2 + 2B_1n)$$

$$S_2 = \sum_{k=1}^{n} k^2 = \frac{1}{3}(1B_0n^3 + 3B_1n^2 + 3B_2n)$$

$$S_3 = \sum_{k=1}^{n} k^3 = \frac{1}{4}(1B_0n^4 + 4B_1n^3 + 6B_2n^2 + 4B_3n)$$

$$S_4 = \sum_{k=1}^{n} k^4 = \frac{1}{5}(1B_0n^5 + 5B_1n^4 + 10B_2n^3 + 10B_3n^2 + 5B_4n)$$

$$S_5 = \sum_{k=1}^{n} k^5 = \frac{1}{6}(1B_0n^6 + 6B_1n^5 + 15B_2n^4 + 20B_3n^3 + 15B_4n^2 + 6B_5n)$$

$$S_6 = \sum_{k=1}^{n} k^6 = \frac{1}{7}(1B_0n^7 + 7B_1n^6 + 21B_2n^5 + 35B_3n^4 + 35B_4n^3 + 21B_5n^2 + 7B_6n)$$

The coefficients of n^j for $j = m + 1$ to 1 in S_m are thus:

$$\frac{1}{m+1}\binom{m+1}{m+1-j}B_{m+1-j}$$

So the general formula for the sum of powers is:

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j n^{m+1-j}$$

where B_j is a Bernoulli number, defining each in terms of the previous ones thus:

$$B_j = \frac{1}{j+1}\sum_{i=0}^{j-1}(-1)^i\binom{j+1}{j-i-1}B_{j-i-1}$$

So binomial coefficients appear twice as coefficients when calculating the coefficients of powers of *n* in the sums of powers. However, there is some ambiguity in the recurrence equations that generate the sequence of Bernoulli numbers, which are rationals, not integers, unlike all other sequences in this section. Furthermore, they need some notational craftiness to explain and understand. I have found two approaches to this problem, leading to a further puzzle in the definition of these mysterious numbers.

Conway and Guy give one definition in *The Book of Numbers*, while Ken Ward gives another on his website on the Bernoulli numbers. To compare these, I'll modify the former's presentation with the latter's approach. Conway and Guy effectively use this recurrence equation, where *B* is a generic for the Bernoulli numbers:

$$(B-1)^{k+1} - B_{k+1} = 0 \qquad k \neq 0$$

For instance, setting k = -1 gives B_0 = 1. There is thus no need to set the initial value in the sequence, as is sometimes done in recurrence equations, for this is determined directly from the formula. Then, setting k = 1, to determine B_1, gives:

$$(B-1)^2 - B_2 = B^2 - 2B^1 + B^0 - B_2 = 0$$

Now, to explain the notational trick, interpret the powers of *B* as suffixes, as corresponding Bernoulli numbers, giving:

$$B_2 - 2B_1 + B_0 - B_2 = 0$$

From which we get B_1 = ½, which is the value of B_1 used in the sums of powers. However, we cannot set k = 0 in the recurrence equation, for this gives -1 = 0.

The other recurrence equation that generates the Bernoulli numbers is:

$$(B+1)^{k+1} - B_{k+1} = 0 \qquad k \neq 0$$

Once again, setting k = -1 gives B_0 = 1 and setting k = 0 leads to another contradiction, this time, 1 = 0. However, when k = 1, to determine B_1, gives, with the same notational trick:

$$B_2 + 2B_1 + B_0 - B_2 = 0$$

From which we get B_1 = -½, which is a valid value of B_1 in some circumstances. The ambiguity in the value of B_1 perhaps arises because it is the only odd-subscripted Bernoulli number that is not zero. So, we could add ½ and -½ to give zero, although it does not seem that doing so has any meaning.

Nevertheless, both recurrence equations give the same sequence of Bernoulli numbers from B_1 onwards. For instance, in the first case, we have:

$$3B_2 = 3B_1 - 1 = \frac{3}{2} - 1 = \frac{1}{2}$$

And in the second:

$$3B_2 = -(3B_1 + 1) = -\left(3\left(-\frac{1}{2}\right) + 1\right) = -\left(-\frac{1}{2}\right) = \frac{1}{2}$$

In both cases, B_2 = 1/6. Again, calculating B_3, we have:

$$4B_3 = 6B_2 - 4B_1 + 1 = \frac{6}{6} - \frac{4}{2} + 1 = 0$$

And:

$$4B_3 = -(6B_2 + 4B_1 + 1) = -\left(\frac{6}{6} - \frac{4}{2} + 1\right) = 0$$

To see the patterns in these sums, I created a table in Excel of the Bernoulli numbers to B_{11}:

	n+1	1	2	3	4	5	6	7	8	9	10	11	Σ	B_n
B_0	1	1											1	1
B_1	2	1											1	½
B_2	3	1½	-1										½	⅙
B_3	4	1	-2	1									0	0
B_4	5	0	-1⅔	2½	-1								-⅙	-1/30
B_5	6	-½	0	2½	-3	1							0	0
B_6	7	0	1⅙	0	-3½	3½	-1						⅙	1/42
B_7	8	⅔	0	-2⅓	0	4⅔	-4	1					0	0
B_8	9	0	-2	0	4⅕	0	-6	4½	-1				-3/10	-1/30
B_9	10	-1½	0	5	0	-7	0	7½	-5	1			0	0
B_{10}	11	0	5½	0	-11	0	11	0	-9⅙	5½	-1		⅚	5/66
B_{11}	12	5	0	-16½	0	22	0	-16½	0	11	-6	1	0	0

Here, then, are the first few Bernoulli numbers, with odd-subscripted numbers = 0 after B_1:[231]

Number	B_0	B_1	B_2	B_4	B_6	B_8	B_{10}	B_{12}	B_{14}	B_{16}	B_{18}	B_{20}
Value	1	±1/2	1/6	-1/30	1/42	-1/30	5/66	-691/2730	7/6	-3617/510	43867/798	-174611/330

Even though this sequence does not consist of integers, they still appear in the *On-line Encyclopedia of Integer Sequences* as:

OEIS	**Definition**
A027641	Numerators of Bernoulli numbers B_n, with B_1 = -½
A027642	Denominators of Bernoulli numbers B_n
A000367	Numerators of Bernoulli numbers B_{2n}
A002445	Denominators of Bernoulli numbers B_{2n}

These apparently haphazard numbers, which get larger and larger in absolute terms, are of such central importance in mathematics, that Ada Lovelace showed how they could be calculated with Charles Babbage's Analytical Engine, turning Bernoulli's formulae into tabular form, published at the end of her memoir to Menabrea's 'Sketch of the Analytical Engine' in 1843.[232] Not surprisingly, she did not do so without considerable effort, saying in a letter to Babbage, "I am in much dismay at having got into so amazing a quagmire & botheration with these *Numbers*."[233] I know only too well how she felt, having worked through these formulae so that I had, at least, a tentative understanding of them. The program is on the next page, with the omitted rubric 'Diagram for the computation by the Engine of the Numbers of Bernoulli', far more complex than the initial programs that ran on the first stored-program computers over a century later.

However, this was not the first program ever published. When Babbage gave a presentation of the Analytical Engine in Italy in 1840, he presented a procedure for solving a pair of simultaneous linear equations, which Luigi Menabrea, later to become prime minister of Italy, then published in French two years later.[234] In effect, this was the publication of the world's first program. Furthermore, even though Ada has been called the world's first programmer, she was clearly much assisted by Babbage himself.

						Data			Working Variables										Result Variables			
Number of Operation	Nature of Operation	Variables acted upon	Variables receiving results	Indication of change in the value on any Variable	Statement of Results	$^1V_1$	$^1V_2$	$^1V_3$	$^0V_4$	$^0V_5$	$^0V_6$	$^0V_7$	$^0V_8$	$^0V_9$	$^0V_{10}$	$^0V_{11}$	$^0V_{12}$	$^0V_{13}\ldots$	$^1V_{21}$	$^1V_{22}$	$^1V_{23}$	$^0V_{24}\ldots$
						○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
						0	0	0	0	0	0	0	0	0	0	0	0	0	B in a dec. fract.	B in a dec. fract.	B in a dec. fract.	0
						0	0	0	0	0	0	0	0	0	0	0	0	0				0
						1	2	4	0	0	0	0	0	0	0	0	0	0				0
						1	2	n											B_1	B_3	B_5	B_7
1	×	$^1V_2 \times {}^1V_3$	$^1V_4$, $^1V_5$, $^1V_6$	$^1V_2 = {}^1V_2$ $^1V_3 = {}^1V_3$	$= 2n$	......	2	n	$2n$	$2n$	$2n$											
2	−	$^1V_4 - {}^1V_1$	$^2V_4$	$^1V_4 = {}^2V_4$ $^1V_1 = {}^1V_1$	$= 2n - 1$	1	......	......	$2n - 1$													
3	+	$^1V_5 + {}^1V_1$	$^2V_5$	$^1V_5 = {}^2V_5$ $^1V_1 = {}^1V_1$	$= 2n + 1$	1	......	......	......	$2n + 1$												
4	÷	$^2V_5 \div {}^2V_4$	$^1V_{11}$	$^2V_5 = {}^0V_5$ $^2V_4 = {}^0V_4$	$= \frac{2n-1}{2n+1}$	......	......	......	0	0	......	......	......	......	......	$\frac{2n-1}{2n+1}$						
5	÷	$^1V_{11} \div {}^1V_2$	$^2V_{11}$	$^1V_{11} = {}^2V_{11}$ $^1V_2 = {}^1V_2$	$= \frac{1}{2} \cdot \frac{2n-1}{2n+1}$	......	2	......	......	......	......	......	......	......	......	$\frac{1}{2} \cdot \frac{2n-1}{2n+1}$						
6	−	$^0V_{13} - {}^2V_{11}$	$^1V_{13}$	$^2V_{11} = {}^0V_{11}$ $^0V_{13} = {}^1V_{13}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$	......	......	......	......	......	......	......	......	......	......	0	......	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$				
7	−	$^1V_3 - {}^1V_1$	$^1V_{10}$	$^1V_3 = {}^1V_3$ $^1V_1 = {}^1V_1$	$= n - 1 (= 3)$	1	......	n	......	......	......	......	......	......	$n - 1$							
8	+	$^1V_2 + {}^0V_7$	$^1V_7$	$^1V_2 = {}^1V_2$ $^0V_7 = {}^1V_7$	$= 2 + 0 = 2$	......	2	......	......	......	......	2										
9	÷	$^1V_6 \div {}^1V_7$	$^3V_{11}$	$^1V_6 = {}^1V_6$ $^0V_{11} = {}^3V_{11}$	$= \frac{2n}{2} = A_1$	......	......	......	......	......	$2n$	2	......	......	......	$\frac{2n}{2} = A_1$						
10	×	$^1V_{21} \times {}^3V_{11}$	$^1V_{12}$	$^1V_{21} = {}^1V_{21}$ $^3V_{11} = {}^3V_{11}$	$= B_1 \cdot \frac{2n}{2} = B_1 A_1$	......	......	......	......	......	......	......	......	......	......	$\frac{2n}{2} = A_1$	$B_1 \cdot \frac{2n}{2} = B_1 A_1$	......	B_1			
11	+	$^1V_{12} + {}^1V_{13}$	$^2V_{13}$	$^1V_{12} = {}^0V_{12}$ $^1V_{13} = {}^2V_{13}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}$	......	......	......	......	......	......	......	......	......	......	......	0	$\left\{-\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}\right\}$				
12	−	$^1V_{10} - {}^1V_1$	$^2V_{10}$	$^1V_{10} = {}^2V_{10}$ $^1V_1 = {}^1V_1$	$= n - 2 (= 2)$	1	......	......	......	......	......	......	......	......	$n - 2$							
13	−	$^1V_6 - {}^1V_1$	$^2V_6$	$^1V_6 = {}^2V_6$ $^1V_1 = {}^1V_1$	$= 2n - 1$	1	......	......	......	......	$2n - 1$											
14	+	$^1V_1 + {}^1V_7$	$^2V_7$	$^1V_1 = {}^1V_1$ $^1V_7 = {}^2V_7$	$= 2 + 1 = 3$	1	......	......	......	......	......	3										
15	÷	$^2V_6 \div {}^2V_7$	$^1V_8$	$^2V_6 = {}^2V_6$ $^2V_7 = {}^2V_7$	$= \frac{2n-1}{3}$	......	......	......	......	......	$2n - 1$	3	$\frac{2n-1}{3}$									
16	×	$^1V_8 \times {}^3V_{11}$	$^4V_{11}$	$^1V_8 = {}^0V_8$ $^3V_{11} = {}^4V_{11}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3}$	......	......	......	......	......	......	......	0	......	......	$\frac{2n}{2} \cdot \frac{2n-1}{3}$						
17	−	$^2V_6 - {}^1V_1$	$^3V_6$	$^2V_6 = {}^3V_6$ $^1V_1 = {}^1V_1$	$= 2n - 2$	1	......	......	......	......	$2n - 2$											
18	+	$^1V_1 + {}^2V_7$	$^3V_7$	$^2V_7 = {}^3V_7$ $^1V_1 = {}^1V_1$	$= 3 + 1 = 4$	1	......	......	......	......	......	4										
19	÷	$^3V_6 \div {}^3V_7$	$^1V_9$	$^3V_6 = {}^3V_6$ $^3V_7 = {}^3V_7$	$= \frac{2n-2}{4}$	......	......	......	......	......	$2n - 2$	4	......	$\frac{2n-2}{4}$	......							
20	×	$^1V_9 \times {}^4V_{11}$	$^5V_{11}$	$^1V_9 = {}^0V_9$ $^4V_{11} = {}^5V_{11}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4} = A_3$	......	......	......	......	......	......	......	......	0		$\left\{\frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4}\right\} = A_3$						
21	×	$^1V_{22} \times {}^5V_{11}$	$^0V_{12}$	$^1V_{22} = {}^1V_{22}$ $^0V_{12} = {}^2V_{12}$	$= B_3 \cdot \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4} = B_3 A_3$	......	......	......	......	......	......	......	......	......	......	0	$B_3 A_3$	......	......	B_3		
22	+	$^2V_{12} + {}^2V_{13}$	$^3V_{13}$	$^2V_{12} = {}^0V_{12}$ $^2V_{13} = {}^3V_{13}$	$= A_0 + B_1 A_1 + B_3 A_3$	......	......	......	......	......	......	......	......	......	......	......	0	$\{A_0 + B_1 A_1 + B_3 A_3\}$				
23	−	$^2V_{10} - {}^1V_1$	$^3V_{10}$	$^2V_{10} = {}^3V_{10}$ $^1V_1 = {}^1V_1$	$= n - 3 (= 1)$	1	......	......	......	......	......	......	......	......	$n - 3$							
					Here follows a repetition of Operations thirteen to twenty-three																	
24	+	$^4V_{13} + {}^0V_{24}$	$^1V_{24}$	$^4V_{13} = {}^0V_{13}$ $^0V_{24} = {}^1V_{24}$	$= B_7$	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	B_7
25	+	$^1V_1 + {}^1V_3$	$^1V_3$	$^1V_1 = {}^1V_1$ $^1V_3 = {}^1V_3$ $^5V_6 = {}^0V_6$ $^5V_7 = {}^0V_7$	$= n + 1 = 4 + 1 = 5$ by a Variable-card. by a Variable-card.	1	......	$n + 1$	......	......	0	0										

Generating functions

When cataloguing sequences of mostly natural numbers in this section, I have focused attention on presenting the difference or recurrence equations that generate them and expressions for their *n*th terms. However, there is another way of presenting these terms: as coefficients of infinite series, like those in complex analysis, but generally treated in a somewhat different manner.

These are the aptly named generating functions, which are often the polynomial expansions of closed-form expressions, which can greatly assist with the understanding of integer sequences. De Moivre was the first to use generating functions in 1730 in *Miscellanea analytica de seriebus et quadraturis*, finding the closed form of the generating function for the Fibonacci sequence,[235] leading to the solution to the general linear recurrence problem.[236]

Then, in 1741, Euler began studying generating functions in order to find recurrence equations for integer partitions, eventually proving the pentagonal number theorem in 1775. But it was not until 1812 when Pierre-Simon Laplace (1749–1827) coined the term *fonction génératrice* in *Théorie Analytique des Probabilités*,[237] the technique having been in use for eighty years in combinatory analysis and the theories of probabilities and numbers.[238]

When studying the intuitive way that Euler used generating functions, George Pólya wrote, “A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag.”[239] And in the words of Herbert S. Wilf in *Generatingfunctionology*, “A generating function is a clothesline on which we hang up a sequence of numbers for display.”[240]

As Donald E. Knuth said, “the use of generating functions opens up a whole new range of techniques, and broadly increases our capacity for problem solving.”[241] As he said in a book that he wrote with Ronald L. Graham and Oren Patashnik, “The most powerful way to deal with sequences of numbers … is to manipulate infinite series that 'generate' those sequences." They then showed an almost mechanical four-step way of using generating functions to solve recurrence equations, which I have focused attention on in this section on sequences.[242] Wilf gives another six-step method that generatingfunctionologists use.

However, while human intuition is often important in experimental mathematics, WolframAlpha has a number of functions to automatically create sequences and generating functions. This raises the key psychological issue about the relationships between intuition and rationality and plausible and demonstrative reasoning (by formal proof), which must wait until another time. As this book is more concerned with categorizing the underlying patterns and basic concepts in mathematics than on problem solving or proving theorems, this is what this subsection focuses attention on.

Wolfram *MathWorld* defines a generating function $f(x)$ as a formal power series:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

whose coefficients give the sequence $\{a_0, a_1, a_2, \ldots\}$, with the understanding that no value is assigned to the symbol x.[243]

However, there are other types of generating functions that are useful in various circumstances, such as exponential generating functions (EGF),[244] expressible as a variation of the power series expansion of the exponential function, which we look at later:

$$E(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!} = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + \cdots$$

The first definition of a generating function is then named as an ordinary generating function (OGF).[245] Indeed, there is no need to stop there. Comtet tells us that, like so much in mathematics, generating functions can be generalized to multiple sequences, the simplest being double sequences, useful in triangular sequences,[246] such as Pascal's triangle:

$$\Phi(x,y) = \sum_{n,k\geq 0} a_{n,k} x^n y^k \qquad \Psi(x,y) = \sum_{n,k\geq 0} a_{n,k} \frac{x^n}{n!} \frac{y^k}{k!} \qquad \Theta(x,y) = \sum_{n,k\geq 0} a_{n,k} \frac{x^n}{n!} y^k$$

So let us summarize the generating functions for the sequences we have looked at in this section.

In terms of the figurate numbers, the most basic of the ordinary generating functions generates a constant sequence of 1's, as the seed for all the others:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

The corresponding exponential generating function is simply:

$$e^x = \sum_{n=0}^{\infty} \frac{x^k}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This section mostly categorizes the OGFs, just mentioning the EGFs when the need arises. As there is sometimes some debate on whether a sequence begins at index 0 or 1, start with the former in the basic series, we should really multiply the initial generating function by x to set $a_0 = 0$. However, to keep things as clear and simple as possible, this factor is omitted in the generating functions listed in this subsection. With this proviso, the basic figurate numbers are generated from increasing powers of this function multiplied by a polynomial, which follows an obvious pattern illustrated in this table:

	Corner polys					Centred polys				
	Seed	2-D	3-D	4-D	5-D	Seed	2-D	3-D	4-D	5-D
Triangular	$\frac{1}{(1-x)^2}$	$\frac{1}{(1-x)^3}$	$\frac{1}{(1-x)^4}$	$\frac{1}{(1-x)^5}$	$\frac{1}{(1-x)^6}$	$\frac{x^2+x+1}{(1-x)^2}$	$\frac{x^2+x+1}{(1-x)^3}$	$\frac{x^2+x+1}{(1-x)^4}$	$\frac{x^2+x+1}{(1-x)^5}$	$\frac{x^2+x+1}{(1-x)^6}$
Square	$\frac{x+1}{(1-x)^2}$	$\frac{x+1}{(1-x)^3}$	$\frac{x+1}{(1-x)^4}$	$\frac{x+1}{(1-x)^5}$	$\frac{x+1}{(1-x)^6}$	$\frac{x^2+2x+1}{(1-x)^2}$	$\frac{x^2+2x+1}{(1-x)^3}$	$\frac{x^2+2x+1}{(1-x)^4}$	$\frac{x^2+2x+1}{(1-x)^5}$	$\frac{x^2+2x+1}{(1-x)^6}$
Pentagonal	$\frac{2x+1}{(1-x)^2}$	$\frac{2x+1}{(1-x)^3}$	$\frac{2x+1}{(1-x)^4}$	$\frac{2x+1}{(1-x)^5}$	$\frac{2x+1}{(1-x)^6}$	$\frac{x^2+3x+1}{(1-x)^2}$	$\frac{x^2+3x+1}{(1-x)^3}$	$\frac{x^2+3x+1}{(1-x)^4}$	$\frac{x^2+3x+1}{(1-x)^5}$	$\frac{x^2+3x+1}{(1-x)^6}$
Hexagonal	$\frac{3x+1}{(1-x)^2}$	$\frac{3x+1}{(1-x)^3}$	$\frac{3x+1}{(1-x)^4}$	$\frac{3x+1}{(1-x)^5}$	$\frac{3x+1}{(1-x)^6}$	$\frac{x^2+4x+1}{(1-x)^2}$	$\frac{x^2+4x+1}{(1-x)^3}$	$\frac{x^2+4x+1}{(1-x)^4}$	$\frac{x^2+4x+1}{(1-x)^5}$	$\frac{x^2+4x+1}{(1-x)^6}$

The general generating function for the Platonic numbers $Pl_V(x)$ is not clear to me at the moment, for Deza and Deza do not seem to include it, and the OEIS Wiki page gives three functions depending on the number of vertices. So here are the particular generating functions that they provide:

Platonic solid	**OEIS**	***f*(*x*)**	**Series**
Tetrahedron	A000292	$\frac{1}{(1-x)^4}$	$1+4x+10x^2+20x^3+35x^4+\cdots$
Octahedron	A005900	$\frac{x^2+2x+1}{(1-x)^4}$	$1+6x+19x^2+44x^3+85x^4+\cdots$
Cube	A000578	$\frac{x^2+4x+1}{(1-x)^4}$	$1+8x+27x^2+64x^3+125x^4+\cdots$
Icosahedron	A006564	$\frac{6x^2+8x+1}{(1-x)^4}$	$1+12x+48x^2+124x^3+255x^4+\cdots$
Dodecahedron	A006566	$\frac{10x^2+16x+1}{(1-x)^4}$	$1+20x+84x^2+220x^3+455x^4+\cdots$

The general generating function for the centred Platonic numbers $CPl_V(x)$ is:[247]

$$CPl_V(x)=\frac{(1+x)(1+2(k_V-1)x+x^2)}{(1-x)^4}$$

where $k_V=\{1,2,3,5,15\}$ for $V=\{4,6,8,12,20\}$, respectively. Here are the particular generating functions:

Centred	**OEIS**	***f*(*x*)**	**Series**
Tetrahedron	A005894	$\frac{(x+1)(x^2+1)}{(1-x)^4}$	$1+5x+15x^2+35x^3+69x^4+\cdots$
Octahedron	A001845	$\frac{(x+1)(x^2+2x+1)}{(1-x)^4}$	$1+7x+25x^2+63x^3+129x^4+\cdots$
Cube	A005898	$\frac{(x+1)(x^2+4x+1)}{(1-x)^4}$	$1+9x+35x^2+91x^3+189x^4+\cdots$
Icosahedron	A005902	$\frac{(x+1)(x^2+8x+1)}{(1-x)^4}$	$1+13x+55x^2+147x^3+309x^4+\cdots$
Dodecahedron	A005904	$\frac{(x+1)(x^2+28x+1)}{(1-x)^4}$	$1+33x+155x^2+427x^3+909x^4+\cdots$

Now, moving into the fourth dimension, here are the generating functions for the six regular polytopes in this dimension.

4-D polytope	**OEIS**	***f*(*x*)**	**Series**
5-cell	A000332	$\frac{1}{(1-x)^5}$	$1+5x+15x^2+35x^3+70x^4+\cdots$
16-cell	A014820	$\frac{(1+x)^3}{(1-x)^5}$	$1+8x+33x^2+96x^3+225x^4+\cdots$
Tesseract	A000583	$\frac{x^3+11x^2+11x+1}{(1-x)^5}$	$1+16x+81x^2+256x^3+625x^4+\cdots$

4-D polytope	OEIS	*f*(*x*)	Series
24-cell	A092181	$\frac{9x^3+43x^2+19x+1}{(1-x)^5}$	$1+24x+153x^2+544x^3+1425x^4+\cdots$
600-cell	A092182	$\frac{107x^3+357x^2+115x+1}{(1-x)^5}$	$1+120x+947x^2+3652x^3+9985x^4+\cdots$
120-cell	A092183	$\frac{543x^3+1993x^2+595x+1}{(1-x)^5}$	$1+600x+4983x^2+19468x^3+53505x^4+\cdots$

In higher dimensions, the generating functions for the sequences of powers, corresponding to hypercubes, arise from Worpitzky's identity,[248] which Julius Worpitzky (1835–1895) discovered in 1883:[249]

$$x^n = \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k-1}{n}$$

Thus the denominators are Eulerian polynomials, whose coefficients are Eulerian numbers *A*(*n*, *k*), defined on page 231.

n^m	OEIS	*f*(*x*)	Series
1	A000012	$\frac{1}{1-x}$	$1+x+x^2+x^3+x^4+\cdots$
n	A000027	$\frac{1}{(1-x)^2}$	$1+2x+3x^2+4x^3+5x^4+\cdots$
n^2	A000290	$\frac{x+1}{(1-x)^3}$	$1+4x+9x^2+16x^3+25x^4+\cdots$
n^3	A000578	$\frac{x^2+4x+1}{(1-x)^4}$	$1+8x+27x^2+64x^3+125x^4+\cdots$
n^4	A000583	$\frac{x^3+11x^2+11x+1}{(1-x)^5}$	$1+16x+81x^2+256x^3+625x^4+\cdots$
n^5	A000584	$\frac{x^4+26x^3+66x^2+26x+1}{(1-x)^6}$	$1+32x+243x^2+1024x^3+3125x^4+\cdots$
n^6	A001014	$\frac{x^5+57x^4+302x^3+302x^2+57x+1}{(1-x)^7}$	$1+64x+729x^2+4096x^3+15625x^4+\cdots$

The generating functions for the nexus numbers, which act as gnomic seeds for the sequences of powers, are thus:

d	OEIS	*f*(*x*)	Series
0	A000012	$\frac{1}{1-x}$	$1+x+x^2+x^3+x^4+\cdots$
1	A005408	$\frac{x+1}{(1-x)^2}$	$1+3x+5x^2+7x^3+9x^4+\cdots$
2	A003215	$\frac{x^2+4x+1}{(1-x)^3}$	$1+7x+19x^2+37x^3+61x^4+\cdots$
3	A005917	$\frac{x^3+11x^2+11x+1}{(1-x)^4}$	$1+15x+65x^2+175x^3+369x^4+\cdots$
4	A022521	$\frac{x^4+26x^3+66x^2+26x+1}{(1-x)^5}$	$1+31x+211x^2+781x^3+2101x^4+\cdots$
5	A022522	$\frac{x^5+57x^4+302x^3+302x^2+57x+1}{(1-x)^6}$	$1+63x+665x^2+3367x^3+11529x^4+\cdots$
6	A022523	$\frac{x^6+120x^5+1191x^4+2416x^3+1191x^2+120x+1}{(1-x)^7}$	$1+127x+2059x^2+14197x^3+61741x^4+\cdots$

Other examples of generating functions for polyhedral numbers are:

Polyhedron	OEIS	*f*(x)	Series
Truncated tetrahedron	A005906	$\frac{10x^2+12x+1}{(1-x)^4}$	$1+16x+68x^2+180x^3+375x^4+\cdots$
Stellated octahedron	A007588	$\frac{x^2+10x+1}{(1-x)^4}$	$1+14x+51x^2+124x^3+245x^4+\cdots$
Rhombic dodecahedron	A005917	$\frac{(x+1)(x^2+10x+1)}{(1-x)^4}$	$1+15x+65x^2+175x^3+369x^4+\cdots$

To find generating functions for Pascal's triangle, the ordinary one for the diagonals/columns, representing the pyramidal simplexes is:

$$\frac{x+1}{(1-x)^{k-1}}$$

where *k* is the column number. The corresponding EGF is:

$$(1+kx)\cdot e^x$$

However, it is also possible to find generating functions for the rows in Pascal's triangle, using double expressions, as Comtet showed:

$$\Phi(x,y)=\sum_{n,k\geq 0}\binom{n}{k}x^n y^k=\sum_{n\geq 0}x^n\left(\sum_{0\leq k\leq n}\binom{n}{k}y^k\right)=\sum_{n\geq 0}x^n(1+y)^n=\frac{1}{1-x(1+y)}$$
$$=1+x(y+1)+x^2(y+1)^2+x^3(y+1)^3+\cdots$$

Here, the coefficients of the expanded polynomials, as coefficients of the powers of *x*, give the rows in Pascal's triangle. The corresponding mixed OGF and EGF is:

$$\Theta(x,y)=e^{x(1+y)}=1+\frac{1}{1!}x(y+1)+\frac{1}{2!}x^2(y+1)^2+\frac{1}{3!}x^3(y+1)^3+\cdots$$

The corresponding double EGF ($\Psi(x,y)$) involves Bessel functions and, as it is rather complicated, is not considered very interesting.[250]

The generating functions for the multinomial triangles are derived directly from the polynomial $(1+x+x^2+\cdots x^{m-1})^n$, with specific values for *m*, giving each row in the triangle as *n* increases. However, the generating functions for the central multinomial coefficients are somewhat more complex. As the Catalan numbers are related to the first of these, Thomas Koshy shows how to use the differential and integral calculus to develop its generating function and that of the Catalan numbers themselves in his comprehensive book on the subject.[251] The first few are illustrated in this table, without proof:

n	OEIS	*f*(x)	Series
Binomial	A000984	$\frac{1}{\sqrt{1-4x}}$	$1+2x+6x^2+20x^3+70x^4+\cdots$
Trinomial	A002426	$\frac{1}{\sqrt{(1+x)(1-3x)}}$	$1+x+3x^2+7x^3+19x^4+\cdots$
Quadrinomial	A005721	See note.	$1+4x+44x^2+580x^3+8092x^4+\cdots$
Quinquenomial	A005191	$\sqrt{\frac{2\sqrt{5x^2-6x+1}-5x+2}{25x^3-10x^2-19x+4}}$	$1+x+5x^2+19x^3+85x^4+\cdots$

Note: Mark van Hoeij, Professor at Florida State University, gives this way of generating the generating function for the central quadrinomial coefficients:[252]

Let $Z(x)$ be a solution of

$$(-1+16x)\cdot(32x-27)^2\cdot Z^6+9(-9+64x)\cdot(32x-27)\cdot Z^4+81(80x-27)\cdot Z^2+729=0 \quad Z(0)=1$$

Compute a Puiseux series for $Z(x)$ at $x=0$, then $Z(x)$ in $C(\sqrt[3]{x})$. Remove all non-integer powers of *x*. The result is the generating function for A005721.

Understanding what this means is beyond my abilities. Nevertheless, it seems to demonstrate an example of a generating function that is created through a procedure rather than one that is presented in closed form.

There are many generating functions related to the Lucas, Fibonacci, and Pell numbers, revealing some generative patterns when viewed as a whole. Euler called such functions recurrent series, following DeMoivre.[253] As a foundation, when $(P, Q) = (k, -1)$ in Lucas's characteristic equation, the discriminant $\sqrt{k^2+4}$ plays a central role in the generation of the silver means, giving this generating function:

Name	OEIS	$f(x)$	Series
Discriminant	A087475	$\frac{4-7x+5x^2}{1-3x+3x^2-x^3}$	$4+5x+8x^2+13x^3+20x^4+29x^5+\cdots$

Each value of k generates a pair of sequences U_n and V_n with these convergent limits:

$$\lim_{n\to\infty}\frac{U_{n+1}}{U_n}=\lim_{n\to\infty}\frac{V_{n+1}}{V_n}=\frac{k+\sqrt{k^2+4}}{2}$$

In the case of U_n, each ratio of consecutive terms is a convergent of the continued fraction $[k;\overline{k}]$, with the denominators being the generated sequence and the numerators shifted over one to the left. Here are the generating functions for the first five pairs of Lucas sequences for the silver means.

Name	OEIS	k	Metal	Root	$f(x)$	Series
Fibonacci	A000045	1	Gold	$\frac{1+\sqrt{5}}{2}$	$\frac{x}{1-x-x^2}$	$x+x^2+2x^3+3x^4+5x^5+\cdots$
Lucas	A000032				$\frac{2-x}{1-x-x^2}$	$2+x+3x^2+4x^3+7x^4+11x^5+\cdots$
Pell	A000129	2	Silver	$1+\sqrt{2}$	$\frac{x}{1-2x-x^2}$	$x+2x^2+5x^3+12x^4+29x^5+\cdots$
Pell-Lucas	A002203				$\frac{2-2x}{1-2x-x^2}$	$2+2x+6x^2+14x^3+34x^4+82x^5+\cdots$
	A006190	3	Bronze	$\frac{3+\sqrt{13}}{2}$	$\frac{x}{1-3x-x^2}$	$x+3x^2+10x^3+33x^4+109x^5+\cdots$
	A006497				$\frac{2-3x}{1-3x-x^2}$	$2+3x+11x^2+36x^3+119x^4+393x^5+\cdots$
	A001076	4	Copper	$2+\sqrt{5}$	$\frac{x}{1-4x-x^2}$	$x+4x^2+17x^3+72x^4+305x^5+\cdots$
	A014448				$\frac{2-4x}{1-4x-x^2}$	$2+4x+18x^2+76x^3+322x^4+1364x^5+\cdots$
	—	5	Nickel	$\frac{5+\sqrt{29}}{2}$	$\frac{x}{1-5x-x^2}$	$x+5x^2+26x^3+135x^4+701x^5+\cdots$
	A087130				$\frac{2-5x}{1-5x-x^2}$	$2+5x+27x^2+140x^3+727x^4+3775x^5+\cdots$

In the cases of $\sqrt{2}$ and $\sqrt{5}$, A000129 and A001076 are also the denominators of their continued fraction convergents. However, as $a_0 = 1$ and 2 in their continued fraction representations, rather than $k = 2$ and 4, the numerators have slightly different generating functions. For the convergents p_i/q_i of continued fractions are given by this general recurrence equation, with $a_i = k$ for all i for the relevant Lucas sequences:[254]

$$\frac{p_i}{q_i}=\frac{a_i p_{i-1}+p_{i-2}}{a_i q_{i-1}+q_{i-2}} \qquad p_{-1}=1, q_{-1}=0, p_0=a_0, q_0=1$$

Indeed, all square roots of natural numbers can be represented as simple continued fractions, as we saw in Chapter 3, with $a_0=\lfloor\sqrt{n}\rfloor$. So, their convergents have sequences of numerators and denominators, with related generating functions, which we can compare to those for the convergent integer solutions to Pell's equation.

However, not all generating functions in the OEIS are presented in this way. What are called numerators and denominators of continued fraction convergents for $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{10}$ begin 1 and 0, respectively, as the zeroth terms, which are actually the values of p_{-1} and q_{-1}. As this does make sense in this context, I have adjusted the generating functions for these roots. The numerators and denominators of the continued fraction convergents for $\sqrt{6}$, $\sqrt{7}$, and $\sqrt{8}$, not in the printed version of Sloane and Plouffe, *The Encyclopedia of Integer Sequences*, from 1995, are correct. In all cases, *Mathematica* and the WolframAlpha statements for continued fractions generate the correct tables for the numerators and denominators.

Having resolved these inconsistencies, there are few obvious patterns in the generating functions for the convergents of these continued fractions, presumably because these lack discernible patterns themselves.

Root	OEIS	*f*(*x*)	Series
$\sqrt{2}$	A001333	$\frac{1+x}{1-2x-x^2}$	$1+3x+7x^2+17x^3+41x^4+99x^5+\cdots$
	A000129	$\frac{1}{1-2x-x^2}$	$1+2x+5x^2+12x^3+29x^4+70x^5+\cdots$
$\sqrt{3}$	A002531	$\frac{1+2x+x^2-x^3}{1-4x+x^2}$	$1+2x+5x^2+7x^3+19x^4+26x^5+\cdots$
	A002530	$\frac{1+x-x^2}{1-4x+x^2}$	$1+x+3x^2+4x^3+11x^4+15x^5+\cdots$
$\sqrt{5}$	A001077	$\frac{2+x}{1-4x-x^2}$	$2+9x+38x^2+161x^3+682x^4+2889x^5+\cdots$
	A001076	$\frac{1}{1-4x-x^2}$	$1+4x+17x^2+72x^3+305x^4+1292x^5+\cdots$
$\sqrt{6}$	A041006	$\frac{2+5x+x^2-x^3}{1-10x^2+x^4}$	$2+5x+22x^2+49x^3+218x^4+485x^5+\cdots$
	A041007	$\frac{1+2x-x^2}{1-10x^2+x^4}$	$1+2x+9x^2+20x^3+89x^4+198x^5+\cdots$
$\sqrt{7}$	A041008	$\frac{2+3x+5x^2+8x^3+5x^4-3x^5+2x^6-x^7}{1-16x^4+x^8}$	$2+3x+5x^2+8x^3+37x^4+45x^5+\cdots$
	A041009	$\frac{1+x+2x^2+3x^3-2x^4+x^5-x^6}{1-16x^4+x^8}$	$1+x+2x^2+3x^3+14x^4+17x^5+\cdots$
$\sqrt{8}$	A041010	$\frac{2+3x+2x^2-x^3}{1-6x^2+x^4}$	$2+3x+14x^2+17x^3+82x^4+99x^5+\cdots$
	A041011	$\frac{1+x-x^2}{1-6x^2+x^4}$	$1+x+5x^2+6x^3+29x^4+35x^5+\cdots$
$\sqrt{10}$	A005667	$\frac{3+x}{1-6x-x^2}$	$3+19x+37x^2+228x^3+1405x^4+8658x^5+\cdots$
	A005668	$\frac{1}{1-6x-x^2}$	$1+6x+37x^2+228x^3+1405x^4+8658x^5+\cdots$

In comparison, the generating functions for the integer solutions to Pell's equations $x^2-dy^2=1$ generate a pair of sequences whose ratios of corresponding terms converge on $\sqrt{d}$, much faster than for the convergents of the continued fractions. Here, the zeroth terms *are* (1, 0), with the coefficients of x being the same as for the series for the convergents of the continued fractions, with the exception of $\sqrt{7}$, giving these expressions:

Root	OEIS	*f*(*x*)	Series
$\sqrt{2}$	A001541	$\frac{1-3x}{1-6x+x^2}$	$1+3x+17x^2+99x^3+577x^4+3363x^5+\cdots$
	A001542	$\frac{2x}{1-6x+x^2}$	$2x+12x^2+70x^3+408x^4+2378x^5+\cdots$
$\sqrt{3}$	A001075	$\frac{1-2x}{1-4x+x^2}$	$1+2x+7x^2+26x^3+97x^4+362x^5+\cdots$

Root	OEIS	*f*(*x*)	Series
	A001353	$\frac{x}{1-4x+x^2}$	$x + 4x^2 + 15x^3 + 56x^4 + 209x^5 + \cdots$
$\sqrt{5}$	A023039	$\frac{1-9x}{1-18x+x^2}$	$1 + 9x + 161\,x^2 + 2889\,x^3 + 51841\,x^4 + 930249\,x^5 + \cdots$
	A060645	$\frac{4x}{1-18x+x^2}$	$4x + 72x^2 + 1292x^3 + 23184x^4 + 416020x^5 + \cdots$
$\sqrt{6}$	A001079	$\frac{1-5x}{1-10x+x^2}$	$1 + 5x + 49x^2 + 485x^3 + 4801x^4 + 47525x^5 + \cdots$
	A001078	$\frac{2x}{1-10x+x^2}$	$2x + 20x^2 + 198x^3 + 1960x^4 + 19402x^5 + \cdots$
$\sqrt{7}$	A001081	$\frac{1-8x}{1-16x+x^2}$	$1 + 8x + 127x^2 + 2024x^3 + 32257x^4 + 514088x^5 + \cdots$
	A001080	$\frac{3x}{1-16x+x^2}$	$3x + 48x^2 + 765x^3 + 12192x^4 + 194307x^5 + \cdots$
$\sqrt{8}$	A001541	$\frac{1-3x}{1-6x+x^2}$	$1 + 3x + 17x^2 + 99x^3 + 577x^4 + 3363x^5 + \cdots$
	A001109	$\frac{x}{1-6x+x^2}$	$x + 6x^2 + 35x^3 + 204x^4 + 1189x^5 + \cdots$
$\sqrt{10}$	A078986	$\frac{1-19x}{1-38x+x^2}$	$1 + 19x + 721x^2 + 27379x^3 + 1039681x^4 + 39480499x^5 + \cdots$
	A084070	$\frac{6x}{1-38x+x^2}$	$6x + 228x^2 + 8658x^3 + 328776x^4 + 12484830x^5 + \cdots$

Unlike the generating functions for the convergents of the continued fractions for square roots, the *x*- and *y*-value solutions to Pell's equation do follow a pattern, related to the generating function for Chebyshev polynomials,[255] named after Pafnuty Chebyshev (1821–1894).[256] The Chebyshev polynomials of the first and second kind are formed from these recurrence equations, closely related to those for the Lucas sequences:

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t) \qquad T_0(t) = 1, \quad T_1(t) = t$$
$$U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t) \qquad U_0(t) = 0, \quad U_1(t) = 2t$$

The general generating function for the *x*-value solutions to Pell's equation is then the generating function for the Chebyshev polynomials of the first kind:

$$\frac{1-tx}{1-2tx+x^2} = 1 + tx + (2t^2-1)x^2 + (4t^3-3t)x^3 + (8t^4-8t^2+1)x^4 + (16t^3-20t^3+5t)x^5 + \cdots$$

And the general generating function for the *y*-value solutions to Pell's equation is the generating function for the Chebyshev polynomials of the second kind multiplied by an additional factor *u*:

$$\frac{ux}{1-2tx+x^2} = ux + 2utx^2 + u(4t^2-1)x^3 + u(8t^3-4t)x^4 + u(16t^4-12t^2+1)x^5 + \cdots$$

The sequences of parameters *t* and *u* for the smallest positive integers *x* and *y* satisfying Pell's equation for nonsquare *d* are:

$\sqrt{d}$		2	3	5	6	7	8	10	11	12	13	14	15	17	18	19	20	21	22	23	24	26	27	28	29
A033313	*t*	3	2	9	5	8	3	19	10	7	649	15	4	33	17	170	9	55	197	24	5	51	26	127	9801
A033317	*u*	2	1	4	2	3	1	6	3	2	180	4	1	8	4	39	2	12	42	5	1	10	5	24	1820

There is no discernible pattern in these sequences, except that the ratios of their corresponding terms grow steadily as the initial values for the convergence on $\sqrt{d}$. Perhaps the most pertinent observation is that the values of *t* and *u* take a jump at $d = 13$ and 29, the roots in the discriminants of Lucas's characteristic equation for (*P*, *Q*) = (*k*, -1), which generate the silver means. This is unlikely to be a coincidence. So maybe there is an explanation.

Finally, here are the generating functions for $(P, Q) = (3, 2)$ in Lucas's characteristic equation, generating the Mersenne and Fermat numbers:

Name	OEIS	*f*(*x*)	Series
Mersenne	A000225	$\frac{x}{(1-x)(1-2x)}$	$x + 3x^2 + 7x^3 + 15x^4 + 37x^5 + \cdots$
Fermat	A000051	$\frac{2-3x}{(1-x)(1-2x)}$	$1 + 3x + 5x^2 + 9x^3 + 17x^4 + \cdots$

As the generating function for the Catalan numbers does not have the simple form of a polynomial with integer coefficients divided by a similar polynomial, like most others in this subsection, it is instructive to look at how it can be constructed. In the letters that Euler and Goldbach exchanged from September to December 1751, Euler showed how this generating function $A(x)$ could be created from his knowledge of the binomial formula:[257]

$$A(x) = 1 + 2x + 5x^2 + 14x^3 + 42x^4 + 132x^5 + \cdots = \frac{1 - 2x - \sqrt{1-4x}}{2x^2}$$

Goldbach responded by saying that $A(x)$ satisfies this quadratic equation:

$$1 + A(x) = A(x)^{\frac{1}{2}}$$

However, as this omits C_0, we need a slightly simpler generating function to include it, as Tom Davis showed in 2018.[258] First, we define the generating function in abstract terms thus:

$$f(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots$$

Now multiplying $f(x)$ by itself gives:

$$[f(x)]^2 = C_0C_0 + (C_1C_0 + C_0C_1)x + (C_2C_0 + C_1C_1 + C_0C_2)x^2 + \cdots$$

But the coefficients here are those in Segner's recurrence equation, with $C_0 = 1$, giving

$$f(x) = 1 + x[f(x)]^2$$

Solving this quadratic equation gives the closed-form expression for the generating function for the Catalan numbers (OEIS A000108):

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \cdots$$

Euler made the first significant use of generating functions in the history of this subject in his study of integer partitions, rather than the recurrence equations with which I introduced the subject on page 221. To present a few of these, I begin with those for $P(n, k)$, whose generative pattern is clear.

Name	OEIS	*f*(*x*)	Series
$P(n, 1)$	A000012	$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$
$P(n, 2)$	A004526	$\frac{1}{(1-x)(1-x^2)}$	$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + \cdots$
$P(n, 3)$	A001399	$\frac{1}{(1-x)(1-x^2)(1-x^3)}$	$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + \cdots$
$P(n, 4)$	A001400	$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$	$1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + \cdots$
$P(n, 5)$	A001401	$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$	$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 10x^6 + \cdots$
$P(n, 6)$	A001402	$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)}$	$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \cdots$

Here are some generating functions for the complete partition numbers and a few of their subsets we looked at earlier.

Name	OEIS	$f(x)$	Series
$p(n)$	A000041	$\prod_{k=1}^{\infty}\left(\frac{1}{1-x^k}\right)$	$1+x+2x^2+3x^3+5x^4+7x^5+11x^6+\cdots$
$q(n)$	A000009	$\prod_{k=1}^{\infty}(1+x^k)$	$1+x+x^2+2x^3+2x^4+3x^5+4x^6+\cdots$
Even $q(n)$	A067661		$1+x^3+x^4+2x^5+2x^6+3x^7+3x^8+\cdots$
Odd $q(n)$	A067659		$x+x^2+x^3+x^4+x^5+2x^6+2x^7+3x^8+\cdots$
Even $q(n)$*-odd* $q(n)$	A010815		$1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}\ldots$
Generalized Pentagonal Numbers	A001318	$\frac{1+x+x^2}{(1+x)^2(1-x)^3}$	$1+2x+5x^2+7x^3+12x^4+15x^5+22x^6+\cdots$
$G(n)$	A003114	$\prod_{k=1}^{\infty}\frac{1}{(1-x^{5k-1})(1-x^{5k-4})}$	$1+x+x^2+x^3+2x^4+2x^5+3x^6+\cdots$
$H(n)$	A003106	$\prod_{k=1}^{\infty}\frac{1}{(1-x^{5k-2})(1-x^{5k-3})}$	$1+x^2+x^3+x^4+x^5+2x^6+\cdots$

Although I approached the Stirling numbers from the perspective of recurrence equations on page 229, Stirling originally defined them in terms of what we call generating functions today, which greatly helps the understanding. The Stirling numbers of the first kind are the coefficients $s(n, k)$ in the expansion of the falling factorial

$$(x)_n = x^{\underline{n}} = x(x-1)(x-2)\ldots(x-n+1)$$

into powers of the variable x. For instance,

$$x^{\underline{4}} = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$$

So, reversing the coefficients, to match the fourth row in the Stirling triangle, $s(4, 1) = -6$, $s(4, 2) = 11$, $s(4, 3) = -6$, and $s(4, 4) = 1$. In turn, the unsigned Stirling numbers of the first kind are the coefficients $c(n, k)$ in the expansion of the rising factorial

$$x^{(n)} = x^{\overline{n}} = x(x+1)(x+2)\ldots(x+n-1)$$

into powers of the variable x. For instance,

$$x^{\overline{4}} = x(x+1)(x+2)(x+3) = x^4 + 6x^3 + 11x^2 + 6x$$

Creating a generating function for the Stirling numbers of the second kind and hence for the Bell numbers, is far from simple, as Khristo N. Boyadzhiev explains in a paper from 2018 that I found on the Web titled 'Close Encounters with the Stirling Numbers of the Second Kind'. So, I'll leave this puzzle for the moment.

There doesn't appear to be any generating function for the Lah numbers, as a whole, signed or unsigned. However, the exponential generating function for the kth column of the unsigned Lah numbers is:

$$\frac{1}{k!}\left(\frac{x}{1-x}\right)^k$$

Here are the exponential generating functions for the first five columns, together with the formulae for the *n*th term, from the general formula on page 232.

k	OEIS	*n*th term	EGF
1	A000142	$n!$	$\frac{x}{1-x}=\frac{x}{1!}+\frac{2x^2}{2!}+\frac{6x^3}{3!}+\frac{24x^4}{4!}+\frac{120x^5}{5!}+\cdots$
2	A001286	$n!\,(n-1)/2$	$\frac{x^2}{2(1-x)^2}=\frac{x^2}{2!}+\frac{6x^3}{3!}+\frac{36x^4}{4!}+\frac{240x^5}{5!}+\frac{1800x^6}{6!}+\cdots$
3	A001754	$n!\,C(n-1,2)/3!$	$\frac{x^3}{6(1-x)^3}=\frac{x^3}{3!}+\frac{12x^4}{4!}+\frac{120x^5}{5!}+\frac{1200x^6}{6!}+\frac{12600x^7}{6!}+\cdots$
4	A001755	$n!\,C(n-1,3)/4!$	$\frac{x^4}{24(1-x)^4}=\frac{x^4}{4!}+\frac{20x^5}{5!}+\frac{300x^6}{6!}+\frac{4200x^7}{7!}+\frac{58800x^8}{8!}+\cdots$
5	A001777	$n!\,C(n-1,4)/5!$	$\frac{x^5}{120(1-x)^5}=\frac{x^5}{5!}+\frac{30x^6}{6!}+\frac{630x^7}{7!}+\frac{11760x^8}{8!}+\frac{211680x^9}{9!}+\cdots$

The ordinary generating functions for the first three diagonals after the 1's are:

L(*n*, *n*-*j*)	OEIS	*n*th term	OGF
L(*n*, *n*-1)	A002378	$n(n-1)$	$\frac{2x}{(1-x)^3}=2x+6x^2+12x^3+20x^4+30x^5+\cdots$
L(*n*, *n*-2)	A083374	$n^2(n^2-1)/2$	$-\frac{6x(x+1)}{(x-1)^5}=6x+36x^2+120x^3+300x^4+630x^5+\cdots$
L(*n*, *n*-3)	A253285	$n(n+1)^2(n+2)^2(n+3)/6$	$-\frac{24}{(x-1)^4}-\frac{144}{(x-1)^5}-\frac{240}{(x-1)^6}-\frac{120}{(x-1)^7}$ $=24x+240x^2+1200x^3+4200x^4+11760x^5+\cdots$

The e.g.f.'s for the sum of the rows of the signed/unsigned Lah numbers (OEIS A293125 and A000262), with offset 0, are:

$$e^{\mp x/(1-x)}=1\mp x+\frac{3x^2}{2!}\mp\frac{13x^3}{3!}+\frac{73x^4}{4!}\mp\frac{501x^5}{5!}+\cdots$$

Finally, the generating functions for the first five diagonals in Euler's number triangle are:

OEIS	*f*(*x*)	Series
A000012	$\frac{1}{1-x}$	$1+x+x^2+x^3+x^4+\cdots$
A000295	$\frac{1}{(1-2x)(1-x)^2}$	$1+4x+11x^2+26x^3+57x^4+\cdots$
A000460	$\frac{1+x-4x^2}{(1-3x)(1-2x)^2(1-x)^3}$	$1+11x+66x^2+302x^3+1191x^4+\cdots$
A000498	$\frac{1+6x-43x^2+44x^3+52x^4-72x^5}{(1-4x)(1-3x)^2(1-2x)^3(1-x)^4}$	$1+26x+302x^2+2416x^3+15619x^4$ $+\cdots$
A000505	$\frac{1+22x-244x^2+442x^3+2575x^4-12012x^5+17828x^6-5664x^7-9552x^8+6912x^9}{(1-5x)(1-4x)^2(1-3x)^3(1-2x)^4(1-x)^5}$	$1+57x+1191x^2+15619x^3$ $+156190x^4+\cdots$

Spirals

One aspect of the Golden Ratio or Divine Proportion that I did not mention when exploring the Fibonacci sequence on page 202 is the Golden Spiral. For this topic raises a host of mathematical, causal, and aesthetic issues that need a distinct section, even another book, to deal with satisfactorily.

To set the Golden Spiral in context, I begin this section with a mathematical overview of some of the principal types of spiral, such as logarithmic, Archimedes, and Fermat's spirals. As spirals are a prime example of the growth of structure in the manifest Universe, it is not surprising that we see them in the world around us, in the plant and animal kingdoms and even in galaxies.

These structures raise the critical issue of how have they have come into being. What causes the sunflower, for instance, to grow as it does? Scientists have proposed many answers to this question. But they all fall short unless we admit the creative power of Life—emerging directly from the Divine Origin of the Universe—into our scientific reasoning.

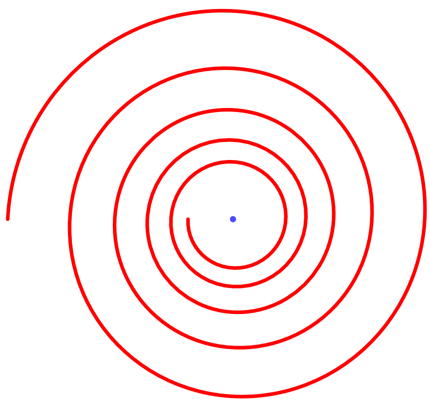
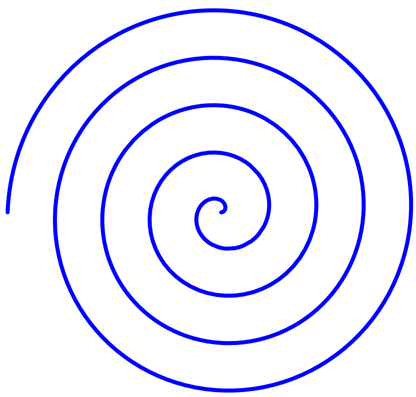
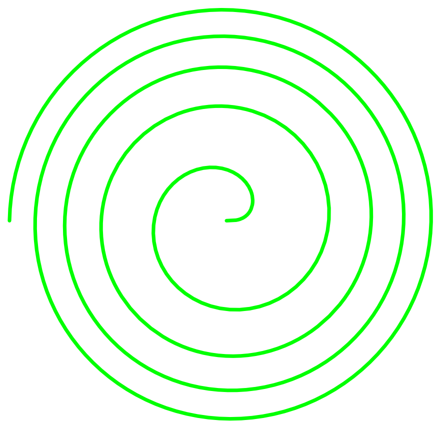
The cultural split between spirituality and science has also led to extreme aesthetic positions being taken in the arts. While some attempt to say that some ubiquitous mathematical structure underlies music, painting, and architecture, for instance, others go out of their way to debunk such attributions.

All these issues can be resolved when we see that mathematical structures reside in the Cosmic Psyche, inaccessible to our physical senses. By unifying mysticism and mathematics, the central theme of this book, we can then not only understand the underlying causality and aesthetics of art and nature, we can also apply the same principles to understand why we humans behave in the way that we do.

As there is as much confusion in this branch of mathematics as there is in the world of learning at large, let us shed some light on the topic of spirals to see what is revealed.

Mathematical perspective

As Fibonacci and other sequences represent growth patterns, it is not surprising that we find these patterns in plants and other structures in the material universe. What is of particular interest here is the world of spirals, indicating continuous growth at different rates, departing, for the moment, from step-wise growth. In this book, we need to consider only three types, where growth is geometric, arithmetic, or diminishing, as we see in the logarithmic, Archimedes', and Fermat's spirals, depicted here, with the general functions in polar form that generate them.

Logarithmic spiral	Archimedes' spiral	Fermat's spiral
		
$r(\theta) = a \cdot b^{\theta}$	$r(\theta) = a + b \cdot \theta$	$r(\theta) = a\sqrt{\theta}$

Sometimes, when plotting spirals, or, indeed, any polar equation, it is more convenient to express them parametrically thus:

$$x = r(\theta) \cos \theta$$
$$y = r(\theta) \sin \theta$$

For instance, by changing the signs of x and y, the spirals are reflected in the y- and x-axes, respectively. By changing both signs, reflective symmetries become a rotational symmetry, in this case by 180°. We look further at symmetries in Chapter 5 in the section on 'Abstract Algebra'.

The first mention of spirals in the mathematical literature was *On Spirals*, by Archimedes of Syracuse (c. 287–c. 212 BCE), written after the death of his friend Conon of Samos (c. 280–c. 220 BCE),[259] who Pappus of Alexandria (c. 290–c. 350) thought had first studied what is today called Archimedes' spiral. However, as Archimedes was in the habit of sharing his discourses with Conon before their publication, Thomas Heath is doubtful of this attribution.[260] Archimedes premised the definition of a spiral in this way, demonstrating the clarity of his writings:

> If a straight line one extremity of which remains fixed is made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line is revolving, a point moves at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane.[261]

Fermat introduced his spiral in 1636,[262] although it is not clear in which publication. In Wolfram *MathWorld*, Eric W. Weisstein defines both Archimedes and Fermat's spirals as special cases of what he calls Archimedean Spirals,[263] with this polar equation:

$$r = a\theta^{\frac{1}{n}}$$

where n is an integer, positive or negative. So Archimedes and Fermat's spirals are Archimedian spirals with $n = 1$ and 2, respectively, which is a bit confusing elsewhere in the literature, where Archimedes' spiral is sometimes referred to as an Archimedian spiral. I'll return to Fermat's spiral when we look at phyllotaxis.

In the meantime, let us look at logarithmic spirals, which René Descartes (1608–1647) began studying in 1638,[264] the year after the publication of *La géométrie*, as one of three examples of his method of seeking the truth in the sciences. Evangelista Torricelli (1596–1650) then independently showed in 1645 with infinitesimal methods, before the calculus was discovered, that the total length of a logarithmic spiral could be rectified, that is calculated exactly, a possibility that Descartes had rejected in *Geometry*.[265]

However, the logarithmic spiral is most closely associated with Jakob Bernoulli, who in the 1690s called it *spira mirabilis* 'miraculous spiral' because, although it is constantly changing, it always remains the same no matter how much it is scaled,[266] with the genuine property of self-similarity, known as *homothety* 'similar to itself' before fractals were discovered.

To understand what this means, I first convert the geometric polar equation I used to draw the above diagram[267] into an exponential polar equation:

$$r = ae^{k\theta}$$

where $k = \ln b$. So, if b < or > 1, k < or > 0, respectively. When $k = 0$, the logarithmic spiral degenerates into a circle. So, if we set the scaling factor a as 1, the logarithmic spiral spirals inwards and outwards to 0 and ∞ from 1, where $\theta = 0$. When k < or > 0, the spiral spirals inwards and outwards when θ > or < 0, respectively. This is unlike Archimedean spirals, which only spiral outwards from 0.

Now, to calculate the radial angle, we first need to determine the rate of change of the radius, which is:

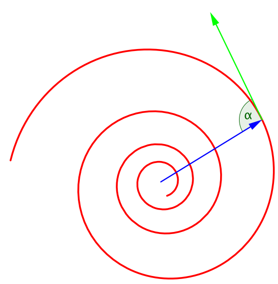
$$\frac{dr}{d\theta} = r' = ake^{k\theta} = kr$$

The angle α between the tangent and the radial line at point (r, θ) is then:[268]

$$\alpha = \tan^{-1}\frac{r}{r'} = \tan^{-1}\frac{1}{k} = \cot^{-1} k$$

Angle α is thus a constant, which is why the logarithmic spiral is also called equiangular. The logarithmic spiral can thus also be defined as:

$$r = ae^{\theta \cot \alpha}$$



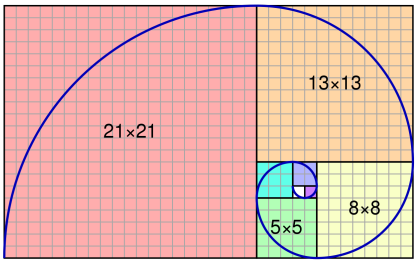
In this representation, when $\alpha = \frac{\pi}{2}$, $k = \cot\alpha = \frac{\cos\alpha}{\sin\alpha} = 0$, and the spiral becomes a circle. Thus, when k < or > 0, α is > or < $\frac{\pi}{2}$, respectively.

There is some confusion in the literature in this regard. The angle α is normally defined as the angle between the radial and tangential vectors, as illustrated in this diagram, adapted from one in the GeoGebra library.[269]

However, Wikipedia defines the constant angle as that of the tangent to the spiral and the corresponding polar circle, which is perpendicular to a ray from the centre of the spiral. This 'polar slope angle' β is thus $\beta = \frac{\pi}{2} - \alpha$ giving the polar slope $\tan \beta = \cot \alpha$, and the polar equation:

$$r = ae^{\theta \tan \beta}$$

Adding to the confusion, I have sometimes seen α referred to as the 'polar tangential angle'.



So, how does the Golden Spiral, based on the Fibonacci sequence, fit in here? Well, the Fibonacci spiral helps us to understand this from this diagram in Wikipedia. The dimensions of the rectangle are 34:21, which approximates to the aspect ratio of the Golden Rectangle, which is φ: 1. As an approximation to the Golden Spiral, quarter arcs are drawn in each square, diminishing by a factor that is approximately the reciprocal of φ, although "the true spiral cuts the squares at very small angles," Coxeter tells us.[270]

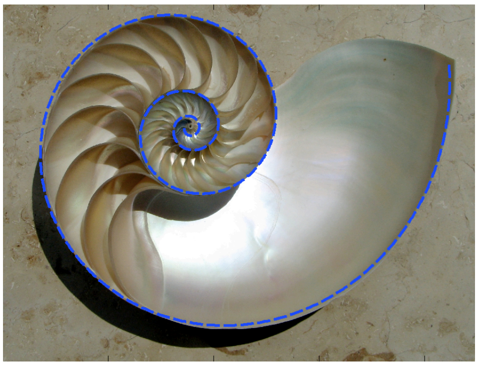
Reversing the perspective, the Golden Spiral grows by a factor of φ as θ increases by $\pi/2$. For every full rotation of 360 degrees or 2π radians, the Golden Spiral expands by a factor of $\varphi^4 = 6.85$. So, using the geometric form of the logarithmic spiral, *b*, as the rate of growth per radian, is given by:[271]

$$b = \varphi^{\frac{2}{\pi}} \approx 1.3585$$

The polar equation for the Golden Spiral is thus:

$$r = e^{\theta \ln (\varphi^{\frac{2}{\pi}})} \approx e^{0.30630} \approx e^{\theta \cot 72.97}$$

with the radial angle α in degrees.



What this means is that there are different ways to measure the rate of growth in a logarithmic, equiangular spiral, making it difficult sometimes to make comparisons. For instance, this diagram from Wikipedia shows how a logarithmic spiral with a growth factor of 1.1923 per radian matches the growth of a chambered nautilus (*Nautilus pompilius*) quite well, a lot less than 1.3585.[272] In contrast, Clement Falbo measured a collection of chambered nautiluses in the California Academy of Sciences in San Francisco in terms of growth per 90 degrees, easier, in practice, to measure. The measured ratios ranged from 1.24 to 1.43, and the average was 1.33, which he compares to 1.618, the Golden Ratio. In terms of growth rates per radian, the ranges were 1.15 to 1.26, with an average of 1.2. Also, the polar slope angle of the example in Wikipedia is approximately 10°, or 80°, as the more conventional radial slope angle.[273]

Despite these different ways of measuring logarithmic spirals in the chambered nautilus, it is quite clear that its growth rate falls far short of that of the golden spiral, contrary to many claims to the contrary, such as that of Priya Hemenway, who has placed the chambered nautilus on the front cover of her book *Divine Proportion*, declaring the ubiquity of the Golden Ratio in architecture, art, music, nature, science, and mysticism.[274]

As well as modelling the growth of forms of life in the animal kingdom, spirals can also be used to model creative growth in the hylosphere. The most vivid example is the Whirlpool Galaxy, which Charles Messier (1730–1817) discovered in 1773 when charting the skies looking for objects that might be confused with comets.[275] What is known today as M51 is 31 million light-years from Earth in the constellation Canes Venatici,[276] this photo having been taken in January 2005 with the Advanced Camera for Surveys aboard the NASA/ESA Hubble Space Telescope.[277]

In 1845, William Parsons (1800–1867), using the most powerful telescope in the world, noticed the spirals in the galaxy, the first of many to be discovered. At the time, Parsons was the Earl of Rosse, building a 72″ telescope on his estate in Ireland, known as the 'Leviathan of Parsonstown',[278] not unlike the way that Tycho Brahe, a Danish nobleman, had built a telescope in the late 1500s on the island of Hven, enabling Kepler to later discover that the planets circle the Sun in ellipses.

NASA tells us that the winding arms of the majestic spirals are star-formation factories, compressing hydrogen gas to create clusters of new stars.[279] And as the website for the Hubble Space Telescope tells us, "the Whirlpool's most striking feature is its two curving arms, a hallmark of so-called grand-design spiral galaxies,"[280] emanating from a black hole, which is thought to exist at the heart of the spiral.[281]

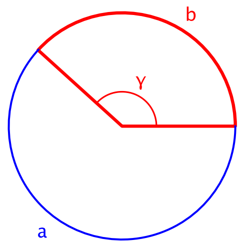
Returning to discrete models of growth, these can be most clearly applied to plant morphogenesis, in the way that leaves or florets grow, known as phyllotaxis 'the arrangement of leaves on a plant stem', from Greek *phullon* 'leaf' and *taxis* arrangement', a word that Charles Bonnet (1720–1793) coined in 1754.[282] As Roger V. Jean tells us in *Phyllotaxis: A Systemic Study in Plant Morphogenesis*, from 1994, "In various areas of botany, phyllotaxis is often considered to be the most striking phenomenon and the toughest subject, raising the most difficult questions." To address these as clearly as possible, he set out in this monograph "to present a universal theory of phyllotaxis",[283] musing, in particular in Part III 'Origins of phylogenetic patterns', on the central problem of causality.

To provide an overview of the subject, Jean wrote a paper in 1997 with Irving Adler and Denis Barabé on 'A History of the Study of Phyllotaxis'. They write, "We have divided the history of the study of phyllotaxis into three periods: (1) the Ancient Period (up to the fourteenth century); (2) the Modern Period (from the fifteenth century to 1970); and (3) the Contemporary Period (from 1970 onwards)." Although the third is the shortest period, "it contains at least half of the most meaningful developments in the history of phyllotaxis research."[284] Nevertheless, there are still many unanswered questions. The principal reason for these unanswered questions is that we cannot understand phyllotaxis in its Cosmic Context, or any other subject for that matter, without understanding why the pace of scientific discovery is accelerating at unprecedented exponential rates of acceleration.

So, to understand the way that plants grow in our gardens, we first need to understand ourselves, in how the Divine power of Life and the Logos guide our creativity within the Cosmic Psyche that we all share, that vast part of the Universe that is inaccessible to our physical senses, including the whole of mathematics. That, in essence, is what I am endeavouring to do in this book on *Unifying Mysticism and Mathematics*. However, as rational, systemic introspection lies far beyond what is considered acceptable in scientific circles today, all I can do for the moment is summarize the situation as it exists in my external world, trusting that one day any outstanding questions could be answered.

As I read the situation, being new to the subject, there are unanswered questions in both causality and mathematics. Regarding these questions, Jean regards Arthur Harry Church (1865–1937) to be the first to study them in a systemic manner, developing a mathematical 'equipotential theory' to explain phyllotaxis.[285] However, of course, nothing new can ever by created from mechanical processes. So, during the twentieth century, scientists have been seeking alternative explanations for those phenomena that defy the second law of thermodynamics in physics. I look at some of these after I have outlined the mathematics, as I understand it at the present time.

The mathematical foundations of the study of phyllotaxis were laid down in the 1830s before the printing of Leonardo Fibonacci's *Liber Abbaci* in 1867 and Lucas's paper of 1878, when he attributed the Fibonacci sequence, much seen in phyllotaxis, to Fibonacci. Karl Friedrich Schimper (1803–1867),[286] Alexander Braun (1805–1877),[287] and Louis Bravais (1801–1843) and Auguste Bravais (1811–1863), a botanist and crystallographer, respectively,[288] wrote the seminal papers.



Most significantly, the Bravais brothers realized that the Divine Proportion plays a central role in phyllotaxis. To see this, rather than looking at the way the Golden Section divides the segment of a line, we can apply it to the circumference of a circle, forming the Golden Angle, as the angle subtended by the smaller arc of length *b*, where:

$$\frac{a+b}{a} = \frac{a}{b} = \varphi$$

This means that the fraction *f* of the circumference subtended by the golden angle is

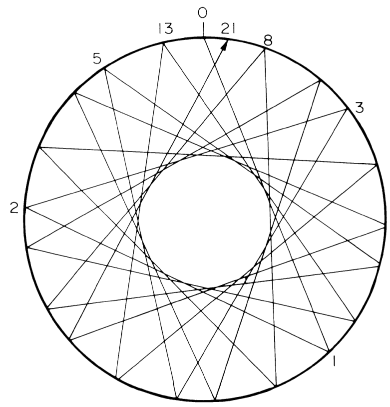
$$f = \frac{b}{a+b} = \frac{1}{1+\varphi} = \frac{1}{\varphi^2} \approx 0.381966$$

So the Golden Angle, which Schimper called the 'Ideal Angle', which I'll denote with gamma (γ), as it doesn't seem to have a standard symbol, is either, as calculated by the Bravais brothers:[289]

$$\gamma = \frac{360}{\varphi^2} = 180(3-\sqrt{5}) = 137.507764° = 137°\,30'\,28.936''$$

or

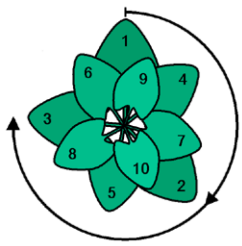
$$\gamma = \frac{2\pi}{\varphi^2} = \pi(3-\sqrt{5}) = 2.399963 \text{ radians}$$



That is, the Golden Angle is approximately 137½ degrees, known as the divergence angle.[290] Under the guidance of H. S. M. Coxeter in 1983, Robert Dixon, a mathematical artist, provided an excellent explanation for the central importance of the Golden Angle in the growth of plants. Starting at position 0, he drew a further 21 points on the circumference of a circle, each spaced at the Golden Angle from the previous one. As you can see, consecutive pairs of Fibonacci numbers get ever closer to the starting point, without ever reaching it. Because φ is the most irrational number, as we see in its continued fraction representation in Chapter 3, the Golden Angle provides the optimal arrangement, making the most efficient use of the plant's limited space, as leaves and florets successively emerge.

Nick Seewald, a statistician, similarly summarizes the benefits of this arrangement in his web pages on 'The Myth of the Golden Ratio':

> Assuming that the sun and rain come from above, orthogonal (perpendicular) to the plane of the leaf, the divergence angle must be such as to minimize blockage of lower leaves by higher leaves. Therefore, any sort of periodic leaf arrangement must be avoided, if possible, as this will result in such blockage. So, the most optimal arrangement is obtained if we divide the circle formed by the plant … by an irrational number—the more irrational the better.[291]



The diagram on the left from Wikipedia provides another illustration of the way that leaves or florets are spaced γ degrees from each other as a plant grows.[292] This is depicted in an actual plant as crisscrossing spirals in an *Aloe* *polyphylla*.[293]

Turning now to plants in the Asteraceae or Compositae family, such as daisies and sunflowers, their flower heads or capitula consist of a dense flat cluster of small flowers or florets sequentially placed at the golden angle from each other on what Schimper and Braun called the 'genetic spiral'. However, what is more conspicuous is a spiral pattern consisting of a pair of families of parastichies, from Greek *para*-'adjacent' and *stikhos* 'row, rank', a word that Schimper coined.[294]

Now, what is fascinating is that the number of sets of intersecting spirals, as parastichy pairs, are adjacent Fibonacci numbers, an effect that Gerrit van Iterson (1878–1972) noticed in 1907, publishing his discoveries in a Ph.D. thesis on phyllotaxis.[295] As Livio tells us, "Most commonly there are thirty-four spirals going one way and fifty-five the other, but sunflowers with ratios of numbers of spirals of 89/55, 144/89, and even … 233/144 have been seen."[296]

But which spiral is most appropriate to use as a model for the sunflower? Well, Jean and Dixon, in their illustrations of the fundamental principles, used a logarithmic spiral, with a constant plastochrone ratio, as the ratio of distances to the apex of the capitulum of two successively numbered primordia. H. E. Huntley made a similar assumption in *The Divine Proportion*, in a chapter titled 'Spira Mirabilis'.[297] However, in 1979, in a paper titled 'A Better Way to Construct the Sunflower Head', Helmut Vogel proposed that Fermat's spiral should be used as the genetic spiral, as the one that optimizes the packing of the seeds, very close to the optimal hexagonal packing.[298] The n seeds are positioned in polar coordinates at these points:

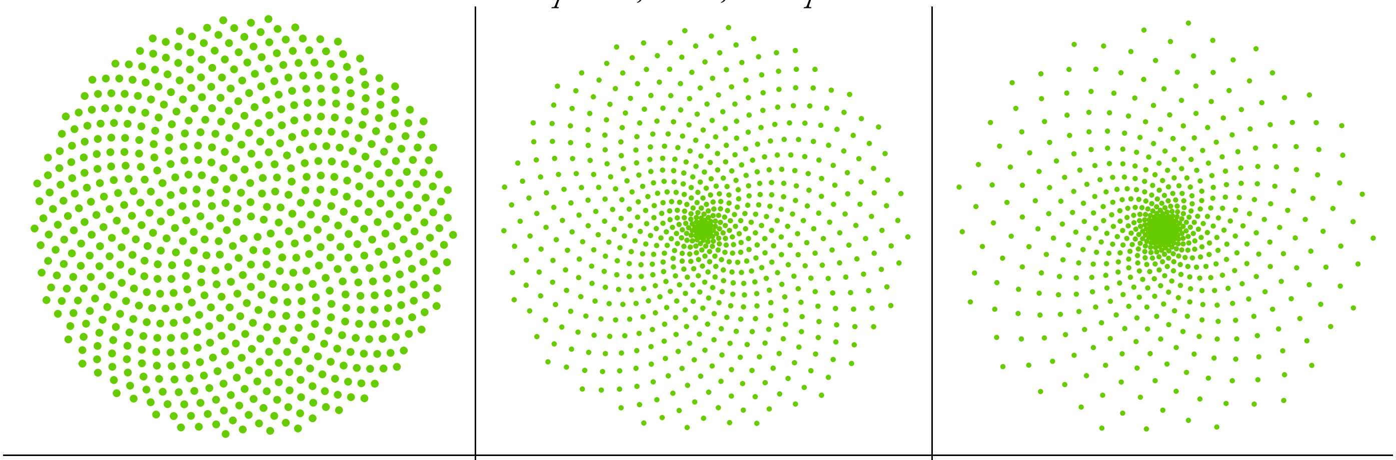
$$(c\sqrt{n}; n\gamma)$$

where $c = \sqrt{\gamma}$ and γ is the Golden Angle. So does Fermat's spiral or a logarithmic spiral best serve as the genetic spiral underlying the flower head of sunflowers? Well, as I have not found any such comparison in the literature and as these mathematical models are so fascinating, I spent a couple of weeks in the late summer of 2019 exploring the respective benefits of these two perspectives. For, as you can see from the above picture of a 'perfect' specimen of a *Helianthus annuus*, the common sunflower,[299] the seeds actually spread out as they move away from the centre, as in the logarithmic spiral, the opposite effect of Fermat's spiral.

Nevertheless, as it is Vogel's radial factor of $\sqrt{k\gamma}$ that is used in the basic model of the sunflower in GeoGebra's library,[300] this is what I first used in exploring these spirals, as p-parastichies, with my limited graphical and mathematical abilities. One thing that I have discovered is that the number of these intersecting spirals is not dependent on the radial factor for the positions of the seeds. For comparison, I drew three figures with these polar coordinates

$$((k\gamma)^a; k\gamma)$$

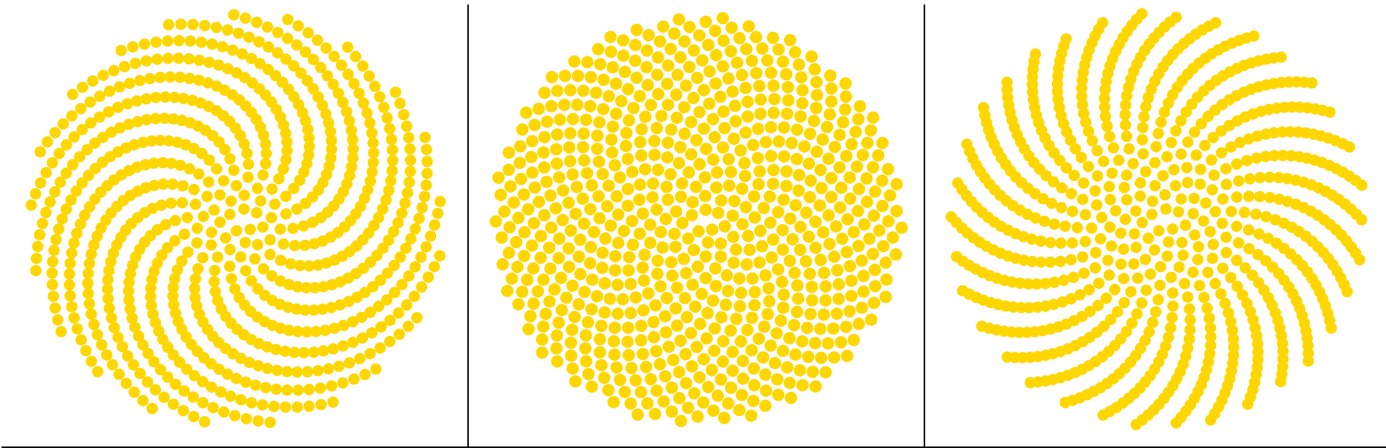
where a = ½, 1, and 2, and k = 0 to 700. The result is these three diagrams:



$(\sqrt{n\gamma}; n\gamma)$	$(n\gamma; n\gamma)$	$((n\gamma)^2; n\gamma)$

In the first two arrangements, the second of which places the seeds in an Archimedes' spiral, the ratio of the clockwise and anticlockwise spirals is 55/34. However, in the power series on the right, the ratio is 34/21, for some reason.

On the other hand, the packing of the seeds is incredibly sensitive to the angle in these polar coordinates: $(\sqrt{k\alpha}; k\alpha)$. The next diagrams show three models, with α converted to degrees in the table, also illustrated in Przemyslaw Prusinkiewicz and Aristid Lindenmayer's *The Algorithmic Beauty of Plants* and Jean's *Phyllotaxis*.[301]



$\alpha = 137.3°$	$\alpha = 137.5078°$	$\alpha = 137.6°$

Inspired by a Mathologer video that Burkard Polster published on his YouTube channel in 2016,[302] I then investigated the intersecting spirals in the mathematical model of the sunflower, as a whole, a perspective I haven't seen anywhere else. For apart from its potential to map the growth of a sunflower, I find the way that all these spirals relate to each other quite amazing.

As I have discovered, if the seeds are sequentially numbered *s*, starting with zero, the *k*th parastichy in the P_n family passes through these points:

$$s \equiv k \pmod{P_n}$$

where $P_n = F_{n+2}$. For instance, when $n = 2$, $P_2 = 2$, and two spirals pass through the even- and odd-numbered points, not unlike the two spirals in the Whirlpool Galaxy. As the diagram on page 258 indicates, the larger the value of P_n, the closer the points numbered with the same modulus move together, enabling the spirals to be seen, even when not explicitly drawn. There was just one snag if I were to draw a diagram to display all these interlocking spirals. None of the books and papers I consulted mentioned the general polar equation for these spirals, most significantly the angle between each consecutive point.

In the end, I found the answer to this puzzle on a website with domain name mathcurve.com, with the rubric *Encyclopédie des formes remarquables: courbes, surfaces, fractals, polyèdres*, set up by pupils at Lycée Fénelon in Paris in 1993, under the guidance of Robert Ferréo, presumably their teacher.[303] They have used a logarithmic spiral as the genetic one, with the *s* points given by these coordinates[304]

$$M_s = (c^s; s\gamma)$$

located on the genetic spiral:

$$r = c^{\theta/\gamma}$$

The *k*th logarithmic spiral for the P_n family of parastichies then has this general polar equation:

$$r = c^{k+P_n\frac{\theta-k\gamma}{P_n\gamma-2q_n\pi}}$$

where $0 \leq k < P_n$ and q_n is the integer closest to

$$\frac{P_n\gamma}{2\pi} = \frac{P_n}{\varphi^2}$$

which is F_{n-2}. The angle γ_n between two consecutive points of a parastichy is:

$$\gamma_n = P_n\gamma - 2q_n\pi$$

For P_1, the formula reduces to the simple genetic one above. And when $k = 0$, we have:

$$r = b^{\frac{\theta \cdot P_n}{\gamma_n}}$$

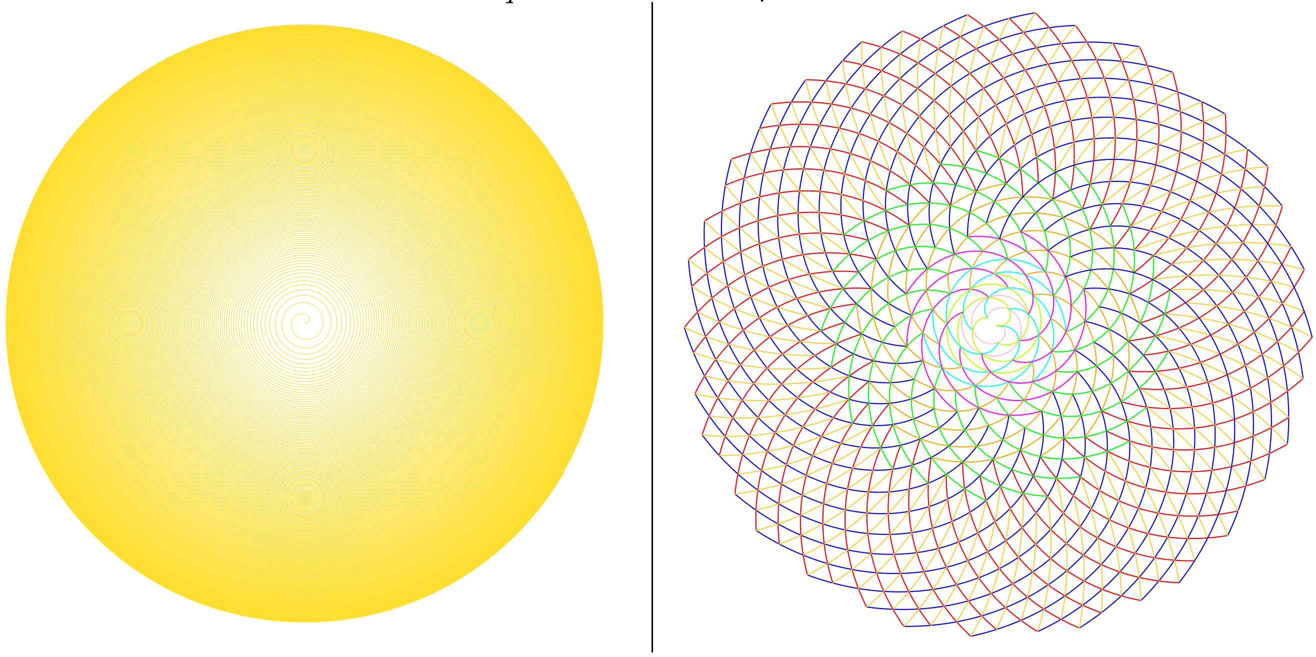
This table gives the angles for the first few families, with the minimum number of points R_n in each spiral that are passed through to exceed just one rotation of 360° or 2π radians.

n	1	2	3	4	5	6	7	8	9	10
P_n	1	2	3	5	8	13	21	34	55	89
q_n	0	1	1	2	3	5	8	13	21	34
γ_n	137.5°	-85.0°	52.5°	-32.5°	20.1°	-12.4°	7.7°	-4.7°	2.9°	-1.8°
R_n	3	5	7	12	18	30	47	77	123	200

The corresponding general formula for parastichies based on Fermat's spiral is:

$$r = \sqrt{\gamma(k + \frac{P_n(\theta - k\gamma)}{\gamma_n})}$$

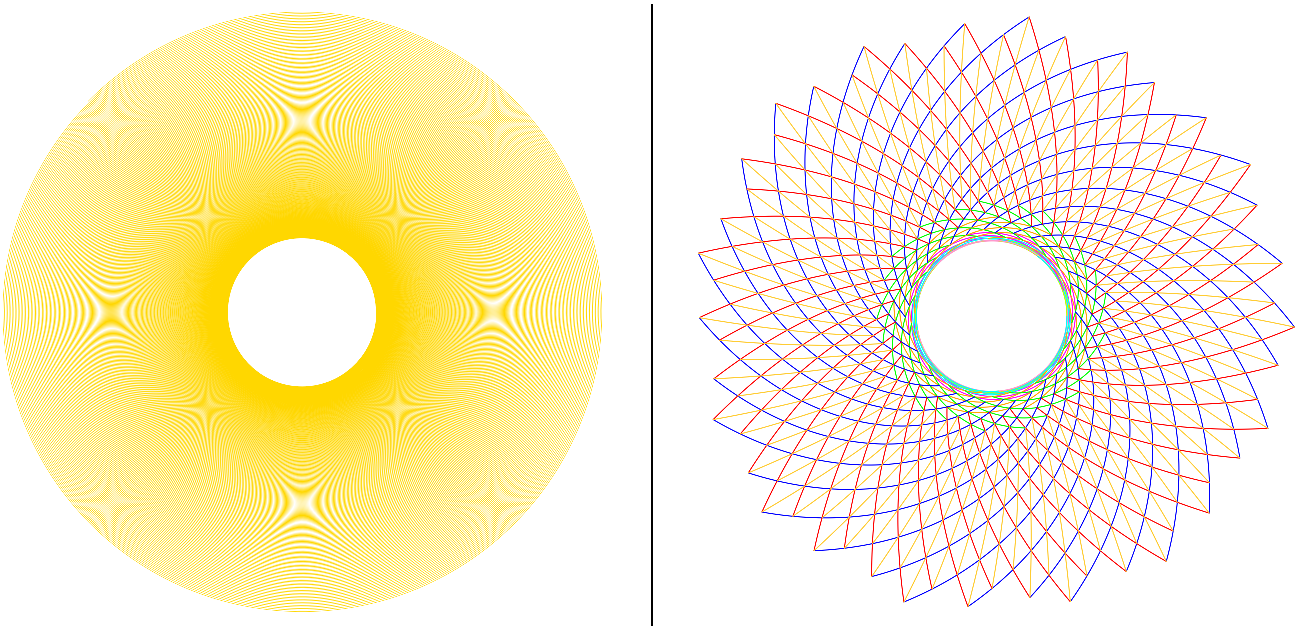
The diagram on the left below shows the genetic spiral, as the 1-parastichy, passing through all 701 points sequentially, with the details visible electronically by zooming in the diagram in pdf file. That on the right shows the nine families of P_n-parastichies after the first.



As all families of spirals pass through all points, to keep this diagram as simple as possible, I've only shown one spiral intersecting two others, stopping short when the number of spirals meeting at a point exceeds three. There are two situations here. For instance, I have terminated the set of 'circular' green spirals passing through the intersections of the red and blue ones where the 'radial' spirals coloured sunglow begin. This table shows the relationships of all sets of spirals in the diagram:

Colour	# Spirals	Radial aspect	Circular aspect
Sunglow	89	Red & blue	
Red	55	Blue & green	
Blue	34	Green & orange	Sunglow & Red
Green	21	Orange & magenta	Red & blue
Orange	13	Magenta & cyan	Blue & green
Magenta	8	Cyan & lime	Green & orange
Cyan	5	Lime & pink	Orange & magenta
Lime	3		Magenta & cyan
Pink	2		Cyan & lime

These two diagrams show the corresponding models when the genetic spiral is logarithmic:



As there are 701 points in the diagram, from $r = 1$, to get them all in, I have used a very small value for $c = 1.005$, giving a growth rate of:

$$1.005^{1/\gamma} = 1.005^{0.41667} = 1.00208$$

As an exponential function, the genetic spiral is thus:

$$r = e^{\theta \ln 1.00208} = e^{0.002078\theta} = e^{\theta \cot 1.5687}$$

In degrees, the radial tangential angle is 89.88°, very nearly circular, a very far remove from that of the Golden Spiral.

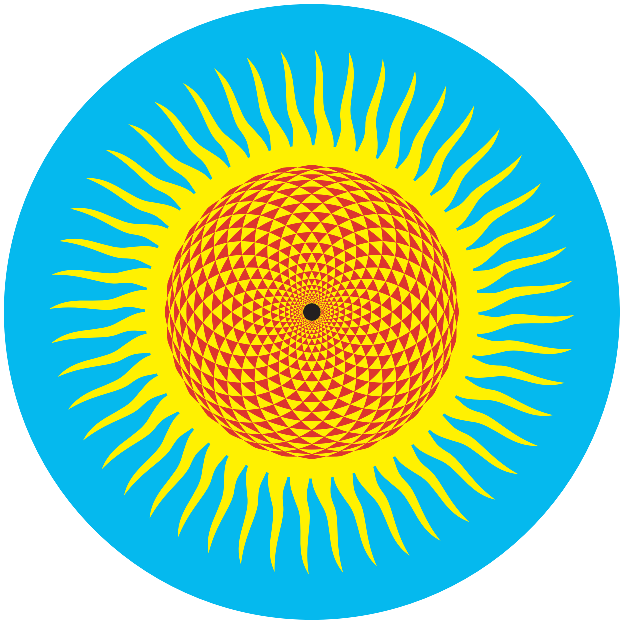
So, which model best corresponds to the florets or seeds in the sunflower? The logarithmic spiral best maps the outer parastichies, which are most evident. However, it is hopelessly inadequate to map the inner florets in the flowerhead. Archimedes' spiral, lying between the two, or a power spiral, might be a better balance between inner and outer. Indeed, Jean points out that several polar equations could serve as the generative spiral, with only the logarithmic one having a plastochrone ratio constant.[305] Furthermore, there is far more variety in phyllotactic effects than in this comparatively simple model, as Coxeter pointed out in his chapter on 'The Golden Section and Phyllotaxis' in *Introduction to Geometry*. As he said, "Thus we must face the fact that phyllotaxis is really not a universal *law* but only a fascinating prevalent *tendency*."[306]

To reflect the complexity of actual plants, Jean has developed a much more elaborate mathematical model, based on what he calls the Bravais-Bravais theorem, which defines different divergent angles, derived from the Golden Angle, including phyllotactic effects that appear cylindrically as lattices, such as those in pineapples.[307] However, I have taken the mathematics as far as I can for the moment. If this book were ever published with the assistance of specialist mathematicians, this exposition could, no doubt, be much improved.

Causal implications

In the meantime, after this brief overview of the mathematical structure that most closely matches the growth of a sunflower, what is causing the sunflower to evolve as it does? Well, this question is not unlike the one that Kepler faced in the first decade of the 1600s, as he strove to make sense of Tycho's measurements of the orbits of the planets around the Sun. He was doing so within a cultural environment that held to Aristotle's separation of physics and astronomy, as being exclusively concerned with causality and mathematics, respectively. To heal this split, he confidently placed God in the centre of the Sun, obsoleting the epicycles of Ptolemy and Copernicus's geocentric and heliocentric models of the solar system, coming close to the inverse square law that Newton later discovered in *Principia* with his concept of gravitation.

Similarly, to explain what causes the sunflower to grow in the way that it does, I place God at its centre. I do so, not by studying phyllotaxis, per se, but by looking within with Self-reflective Intelligence, wondering what is causing my creativity and that of all other humans. For what is causing my own creativity in my psyche is no different from that guiding the growth of any other structure in the Universe, emanating from and through the Cosmic Psyche. What we call 'man-made' designs and forms are just a part of nature, as Pierre Teilhard could see. Indeed, Jean points out that similarities with phyllotactic patterns are also found in art, giving the example of a mosaic in the Museo Nazionale Romano,[308] similar to this one I found on the Web, which seems to be a floor tile based on a mosaic in the Getty Museum.[309] In turn, this pattern is very similar to the centre of the symbol I have been using since the 1990s for the Sun of Consciousness,[310] as the Coherent Light emerging from a black hole, enabling us to view the Cosmos holographically.



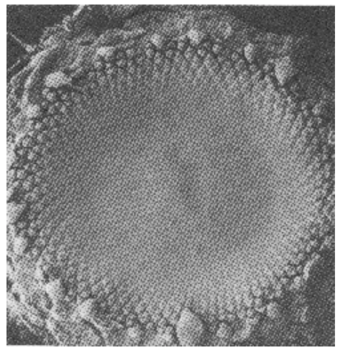
As I describe on page 24 in Chapter 1 on 'Business Modelling', I have come to this realization because synergistic data, arising from the Datum of the Universe, is causal and hence energetic. Indeed, as data is ubiquitous, unlike the Divine Proportion, there is no other possible source of causality than data patterns, viewing the Cosmos as an infinitely dimensional network of hierarchical relationships emerging directly from the Source, as the Divine Origin of the Universe.

To explain this, I use Heraclitus' mystical meaning of *Logos*, as the Immanent Divine Intelligence governing the Cosmos. Just as the Logos has shown me how to intelligently integrate all knowledge into a coherent whole—with Integral Relational Logic providing the Cosmic Context, Gnostic Foundation, and coordinating framework—the Logos has the intelligence to 'know' how to optimize the packing of seeds of sunflowers in spiral arrangements. For, as mathematics is inaccessible to our physical senses, its influence must reside in the Cosmic Psyche, the 99% of the Universe that does not consist of matter. Such notions lie at the very heart of unifying mysticism and mathematics in order to reveal Love, Peace, Wholeness, and the Truth, the central theme of this book.

Yet, like Kepler, I live in a schizoid culture, one that separates science and spirituality, denying the existence of Life, the Logos, and the brilliant Light of Consciousness that enables our innate Intelligence to understand what is happening to our species at the present time. Faced with the dogma of the second law of thermodynamics, the best that conventional scientists can do to explain growth processes is through what called Humberto Maturana and Francisco Varela called self-organizing, -producing, or -creating structures in 1972. In technical terms, they called this process *autopoiesis*, from the Greek *poien* 'to make, do, produce, create', which is also the root of *poetry*. To them, autopoietic machines are homeostatic machines, with one peculiarity:

> An autopoietic machine is a machine organized (defined as a unity) as a network of processes of production (transformation and destruction) of components which: (i) through their interactions and transformations continuously regenerate and realize the network of processes (relations) that produced them; and (ii) constitute it (the machine) as a concrete unity in space in which they (the components) exist by specifying the topological domain of its realization as such a network.[311]

The physicists Stéphane Douady and Yves Couder used this self-organizing model in 1991 to explain how parastichies can be produced in the laboratory, "due to the system's trend to avoid rational (periodic) organization, thus leading to a convergence towards the golden mean".[312] Drops of a magnetic fluid were dropped into a dish full of silicone oil in a magnetic field stronger near the dish's edge than at the centre, as they demonstrate in a YouTube video.[313] As Livio succinctly explains, "Physical systems usually settle into states that minimize energy. The suggestion is therefore that phyllotaxis represents a state of minimal energy for a system of mutually repelling buds."[314]



However, it is uncertain whether florets actually emerge one by one from the centre of the capitulum fully formed, pushing the ones that have already emerged out to the perimeter, which is sometimes suggested in the literature. Indeed, here is a scanning electron micrograph of a very young capitulum of *Helianthus annuus*, about 2.5 mm in diameter, "showing the process of floret initiation proceeding toward the centre on the generative front with a remarkable degree of symmetry".[315] For me, this picture shows that the sunflower emerges as a whole, presumably through cell differentiation, not unlike the development of the human embryo. This is essentially how this book is evolving. I visualized it as a whole in the summer of 2018, without the details, which are emerging directly from the Divine Origin of the Universe as I write.

So, as Jean points out, the root meaning of *phyllotaxis* is rather restrictive, applying far beyond so-called living systems, beyond just the arrangement of leaves in botany, inexplicable within the framework of reductionist science, dividing natural phenomena into separate parts.[316] So, while seeking a mechanistic explanation for phyllotactic effects, rooted in the supposed physicochemical basis of the universe, Jean has been searching for an alternative explanatory approach. The one he favours is the autoevolutionism of António Lima-de-Faria, emeritus professor of cytogenetics at Lund University, who wrote a book titled *Evolution without Selection: Form and Function by Autoevolution* in 1988, presumably after retiring. This polemical book is popular in Russia, but almost unknown in the USA, questioning, as it does, a fundamental tenet of Neo-Darwinism.[317]

Lima-de-Faria has the distinction of having an entry in RationalWiki, which has the purpose to fight what it regards as 'pseudoscience'. He proposed that evolution is 'ordered' by physicochemical processes—not by natural selection—as a form of orthogenesis,[318] "the hypothesis that evolution has an innate tendency to evolve in a unilinear fashion due to some internal or external force or mechanism."[319] So scientifically establishing Life as the ultimate cause of creativity, confirming what many sense within them, still has a mountain to climb.

It is here that we need to recognize that scientific research is as much a social activity as an objective, rational process, as Thomas S. Kuhn pointed out in *The Structure of Scientific Revolutions* in 1962. Then, in 1970, Imre Lakatos introduced the notion of an unchangeable 'hard core' that scientific research programmes should adhere to.[320] As billionaires and large companies fund much of technological research,[321] anyone questioning this hard core is likely to be ostracized, even losing their positions in academia, making materialistic, mechanistic science today even more dogmatic than Medieval religious beliefs.

Yes, many individuals and institutions are today questioning some of the core beliefs of science in order to explain everyday effects that are anomalous within the prevailing materialistic, mechanistic paradigm, with some even seeking to unify science and spirituality. Yet, almost no one is ready to awaken to Total Revolution, recognizing that holding on one-sidedly to the status quo at these times of unprecedented rates of change is insane, as Vimala Thakar pointed out in 1984 in *Spirituality and Social*

Action: A Holistic Approach. In particular, we cannot understand what is happening to us all as a species by following a traditional work ethic, trapped, as cogs, within the economic machine, believing that money provides a sense of security in life.

As the banks will disappear when *Homo sapiens* becomes extinct in the near future, all I can do under these circumstances is continue to live in solitary Wholeness, as an outsider to society, paradoxically recognizing that none of us is ever separate from any other being for an instant, exploring the psychogenesis of mathematics, the evolutionary growth process that underlies morphogenesis and all other genetic processes. For to model the growth of form in all its myriad manifestations, we can do no better than generalize the modelling methods of information systems architects in business, which underlie the Internet, as I describe in Chapters 1 and 2 of this book.

What is of particular interest in this regard is that Alan Turing's attempts to study morphogenesis in 1952 with simultaneous first- and second-order differential equations has led to an evolutionary dead end,

as Aristid Lindenmayer (1925–1989) pointed out in his seminal papers in 1968 of what are today known as L-systems.[322] Similarly, topological catastrophe theory, which René Thom (1923–2002) introduced in 1975 in *Structural Stability and Morphogenesis: An Outline of a General Theory of Models*, is too complex to adequately model bifurcations in dynamical systems, as I explained in 2016 in my book *Through Evolution's Accumulation Point: Towards Its Glorious Culmination*. To develop almost natural looking representations of plants, in particular, as Lindenmayer's collaborator Przemyslaw Prusinkiewicz does with computer graphics, like the above roses,[323] requires finite mathematics, much simpler, generating forms and structures of limitless complexity.

But we need to remember that L-systems, short for Lindenmayer systems, are a language, as symbols representing the structure of concepts in the mind, whose emergence from our Divine Source we need to cognitively experience to understand morphogenesis. So, as fascinating as this language is, to fully understand causality in humans, nature, and the Universe, it is vitally important to map the mathematical structures in the Cosmic Psyche, before they are expressed in language.

Anthropomorphic and aesthetic considerations

Having looked briefly at the confusion around which mathematical models work best to describe spiralling effects in nature and the impossibility of materialistic, mechanistic science to explain the causality of phyllotaxis, we now need to turn to what Martin Gardner (1914–2010) called the 'Cult of the Golden Ratio' in his mockery of those associating φ aesthetically with art and nature, even attributing the Divine Proportion to quantitative relationships in the 'ideal' male human body.

Gardner said that erroneously finding φ ubiquitously in nature and art has all the earmarks of pseudo-math, as an extension of pseudoscience,[324] a term that Wikipedia editors often use to debunk attempts to explain any anomalous phenomena that don't fit into the materialistic and mechanistic paradigm. We live in a culture dominated by a science that denies the existence of Spirit and Life, which make life worth living, explaining who we humans are—in contrast to conforming machines—and why we behave as we do.

The claim is that scientists are being rational by dismissing Life emanating from the Datum of Universe from science, for the Absolute is regarded as being beyond reason. Yet, as I describe on pages 72 and 73 in the section titled 'Transcending the categories' in the chapter on 'Integral Relational Logic', in October 1983 I was able to use David Bohm's simple method of bringing order to quantum physics to form the concept of the Absolute in exactly the same way as I form the concept of any being in the relativistic world of form. Then, following 25 years of profound spiritual practice, God became a sound scientific concept in the Altai Mountains in Siberia, the original home of the shamans, and I became completely free of the limiting delusions that govern Western civilization.

So, since then, I have been living in a quite different world from most of those around me, writing a dozen books explaining what it truly means to be an intelligent human being in comparison to machines with so-called artificial intelligence in the context of evolution, as a whole, and the psychodynamics of society. It is with such Self-awareness that I am writing this book on *Unifying Mysticism and Mathematics* in order to reveal Love, Peace, Wholeness, and the Truth.

Regarding the anthropomorphic and aesthetic implications of the golden ratio, we first look at the root of *ratio*, *rational*, and *reason*, for by studying what Bohm called the 'archaeology of language' we learn much about the evolutionary influences on our lives. These words all derive from Latin *ratiō* 'calculation, reckoning', from *ratus*, past participle of *rērī* 'to reckon, think; consider, suppose, judge', from PIE base **ar-* 'to fit together', also root of *harmony* and *order*. The word *reason* entered English before 1200 from Old French to mean 'explanation, ability to think'.

Associating reckoning or counting with thinking shows to what extent Western science has become imprisoned in the tyranny of number, which George Boole began to liberate us from in the mid 1800s, as we see in the next chapter on 'Universal algebra'. For mathematical logic was to lead information systems architects in business to treat qualitative and quantitative relationships in exactly the same manner. Yet, old habits die hard.

For instance, in business, it is said, "If you cannot measure, you cannot manage," probably inspired by Lord Kelvin's view of physical science, "To measure is to know," and "When you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind."[325]

In contrast, Arthur Koestler said in *The Ghost in the Machine* that scientists' obsession with quantitative measurement is the fourth of four pillars of unwisdom in science and economics, which I have redefined and expanded, as described on page xiv in the Prologue. And there is no greater absurdity than that of money, as a quantitative measure of what we value in life. Most significantly, because we have become separated from the Immortal Ground of Being that we all share, we have reified money, as a supposed unit of measure, turning a unit, like metres and grams, into a commodity, which can be bought and sold in the financial markets.

Yet, as Bohm pointed out in a profound passage on how to heal the fragmentated mind in Wholeness,[326] this is not what the ancient Greeks understood by measure. The word *measure* derives from Greek *metron* 'measure, rule, standard', also the root of *metrios* 'moderate, within measure' and *metriotes* 'moderation, modesty', from PIE base **med-* 'to take appropriate measures', also root of *medical*, *remedy*, *moderate*, and *module*.

So, to the ancient Greeks, what scientists call measurement today was a secondary activity, as the "outer display or appearance of a deeper 'inner measure', which played an essential role in everything". As Bohm pointed out,

> When something went beyond its proper measure, this meant not merely that it was not conforming to some external standard of what was right but, much more, that it was inwardly out of harmony, so that it was bound to lose its integrity and break up into fragments. … So, physically, socially, and mentally, awareness of the inner measure of things was seen as the essential key to a healthy, happy, and harmonious life.[327]

This inner sense of measure is thus key to our reasoning, to keeping the totality of everything in its proper proportion. For the Greek word for *proportion* is *analogia*, from *ana-* 'upon, according to' and *logos* 'ratio, reckoning', the root of *analogy*, of course. In English, *proportion* derives from Old French *proporcion* 'measure, proportion', from Latin *proportionem* (nominative *proportio*) 'comparative relation, analogy', from the phrase *pro portione* 'according to the relation' (of parts to each other), from *pro* 'for' and ablative of **partiō* 'division', related to *pars* 'a part, piece, portion, share', from *partire* 'to share out, distribute, divide'.

Euclid's *Elements* well illustrates the distinction between *mensuration* and *enumeration*, at the core of combinatorics, a central theme of this chapter. The word *measure* is frequently used, but there is no counting in units, such as metres, seconds, grams, or dollars. Euclid is more concerned with relationships and proportions in geometrical figures, adding some theorems about primes, coprimes, and perfect numbers, most famously proving by the mathematical technique of induction that there are an infinite number of primes,[328] as we see in the previous chapter. So to measure $\sqrt{2}$, which cannot be expressed rationally, as the ratio of integers, the Greeks "used a length equal to the hypotenuse of a right triangle whose sides were one unit in length."[329]

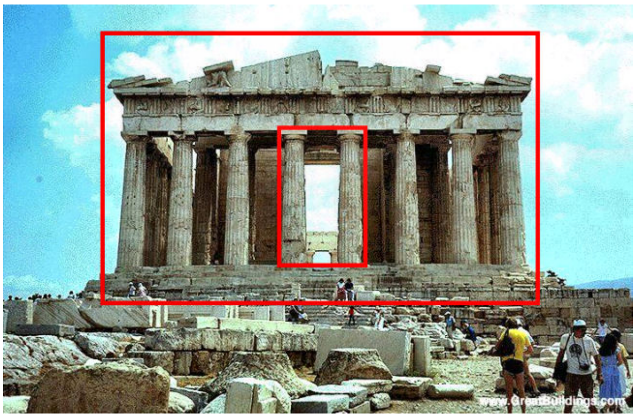
In contrast, in the East, as Bohm points out, what is immeasurable (that which cannot be named, described, or understood through any form of reason) is regarded as the primary reality. For while Sanskrit *matra* 'measure' has the same PIE base, this is also the root of *māyā*, usually translated as 'illusion'. To cocreate a harmonious and healthy society, healing the split between East and West and ending the war between science and religion, we thus need to unify the immeasurable and measurable within the deepest recesses of the Cosmic Psyche, as this book on *Unifying Mysticism and Mathematics* is endeavouring to demonstrate.

Not that this is easy, as Bohm points out: "to develop new insight into fragmentation and wholeness requires a creative work even more difficult than that needed to make fundamental new discoveries in science, or great and original works of art". As he says, it is not enough to imitate Einstein's ideas or apply them in new ways. Rather, one who learns from Einstein does something original, assimilating what is valid in Einstein's work and yet goes beyond this work in qualitatively new ways. And that requires us "to learn afresh, to observe, and to discover for ourselves the meaning of wholeness."[330] For, as Bohm said in 1985, if we do not question the assumptions and beliefs of the cultures we live in then humanity is not a viable species.

In the context of this short overview of the etymology of *ratio*, *proportion*, and *measure*, we can now look briefly at how we might resolve the conflict between sceptical scientists and those seeking to transcend the materialism and mechanism of science through their experience and spiritual insights. Like all conflicts between people, this disagreement arises from a lack of understanding of the essential nature of Wholeness, encapsulated in the fundamental law of the Universe, which I define on page 73: *Wholeness is the union of all opposites*. On the next page in that chapter, I show how the Principle of Unity or Hidden Harmony can be expressed in mathematical notation as the Cosmic Equation, the simple, elegant equation that can explain everything, which Einstein and Hawking sought in their futile endeavours to solve the ultimate problem of human learning within physics.

In particular, we can trace the widespread confusion about the role of the Divine Proportion in nature and art back to 1855, when the German psychologist Adolf Zeising (1810–1876) published a book titled: *A New Theory of the proportions of the human body, developed from a basic morphological law which stayed hitherto unknown, and which permeates the whole nature and art, accompanied by a complete summary of the prevailing systems*.[331] Then, in the posthumously published *Der Goldner Schnitt* in 1884, he claimed that the dimensions of the Parthenon in Athens, for which Phidias provided the sculptures, display the aspect ratio of the Golden Rectangle.[332]

Thus, the myth of the Golden Ratio was born, as people increasingly wanted to see the Divine Proportion in art and nature where it does not exist. Nick Seewald provides a well-balanced overview of the myth of phi on his website, written in 2010, when he was an undergraduate.[333] He began by quoting from Mario Livio's excellent book *The Golden Ratio*, which says, "It is probably fair to say that the Golden Ratio has inspired thinkers of all disciplines like no other number in the history of mathematics," a notion encapsulated in the two subtitles of this book: *The Story of Phi, the Extra ordinary Number of Nature, Art and Beauty* or *The Story of Phi, the World's Most Astonishing Number*.



Yet, the proportions of the Parthenon are not in the Golden Ratio, as this image from an Internet post in 2005 titled 'Laputan Logic – The Cult of the Golden Ratio' illustrates.[334] When people see the Divine Proportion in the Parthenon, this depends on the bounds of what they measure. I have also seen that some adjust the aspect ratio of the source to prove their point, corrupting scientific inquiry.

We also should not forget that the ancient Greeks only understood the Golden Ratio in terms of Euclid's definition on page 202. As Julia Calderone reminds us in an article titled 'The golden ratio and Fibonacci numbers don't prove beauty' in *Business Insider*, "A given set of numbers is said to be in a golden ratio if the following occurs:"[335]

$$\frac{a}{b} = \frac{a+b}{a} \quad a > b$$

The Golden Ratio would thus seem to be a rational number if integers *a* and *b* could be found to satisfy this equation. However, as we see on page 203, Kepler discovered that there are no such integers, other than consecutive terms in the Fibonacci sequence, whose ratio comes ever closer to 1.618034 as the terms get larger and larger. So, it is most unlikely that the Greeks could have consciously used any irrational number in their architecture. Indeed, as we see on page 274, the golden mean was not one of the ten means that the Pythagoreans studied in their theory of proportion and means, particularly related to harmony in music, a subject that also Kepler addressed in his wonderful book *The Harmony of the World*, as we see on page 102 in the previous chapter.

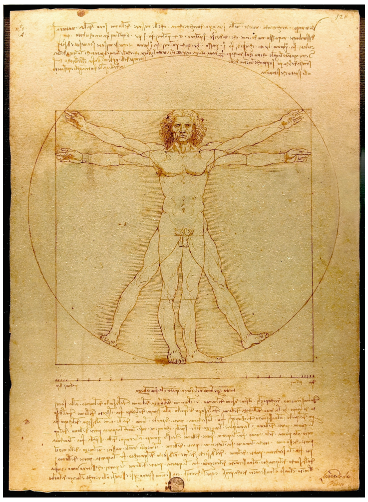
This preference for rational numbers prevailed even into the 1600s, when irrational numbers, which Kepler called ineffable, would be "avoided as much as possible, and certainly so when the architect was laying out a building's principal proportions", the art historian George L. Hersey (1927–2007) tells us in *Architecture and Geometry in the Age of the Baroque*. As the Modenese architect Guarino Guarini (1624–

1683) wrote in the posthumously published *Architettura civile*, effable proportions, as easy-to-measure shapes, were deemed to be superior to ineffable ones.[336] Guarini was much influenced by the ancients, developing a 'Universal Mathematics' as an augmentation of Euclid, "transcending all material concern to be analogous to a *divina scientia*", and accepting the traditions of Vitruvius in architecture.[337]

In *De architectura* (*The Ten Books on Architecture*) from around 15 BCE, Marcus Vitruvius Pollio (80–70 BCE–20–10 BCE) showed that the proportions of the Greek temples, including the Parthenon, were analogous to the proportions of the human body. As he said in Book III, Chapter I 'On Symmetry: In Temples and in the Human Body',

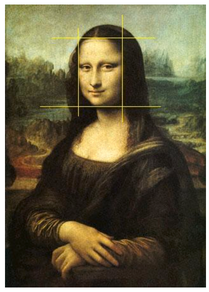
> Proportion is a correspondence among the measures of the members of an entire work, and of the whole to a certain part selected as standard. From this result the principles of symmetry. … Without symmetry and proportion there can be no principles in the design of any temple; that is, if there is no precise relation between its members, as in the case of those of a well-shaped man.[338]

To Vetruvio, a man is four cubits or six feet, as six and four palms, respectively, which, in turn, are four fingers. He also gave other proportions, saying, for instance, "from the chin to the top of the forehead and the lowest roots of the hair is a tenth part of the whole height." Continuing, he said, "The other members, too, have their own symmetrical proportions, and it was by employing them that the famous painters and sculptors of antiquity attained to great and endless renown."



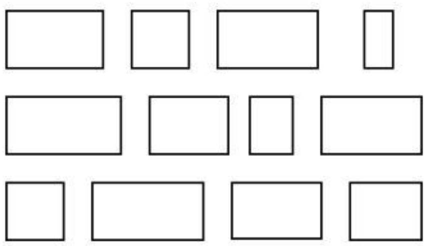
Then, around 1490, Leonardo da Vinci, similarly inspired by the proportions of the human body in his studies of anatomy, architecture, and engineering, drew this famous picture, known as Vitruvian Man.[339] The text above and below the drawing in mirror writing is very much as Vetruvio wrote it.[340] As the art historian Ludwig Heinrich Heydenreich (1903–1978) tells us, Leonardo envisaged "Vitruvian Man as a *cosmografia del minor mondo* (cosmography of the microcosm), believing the workings of the human body to be an analogy, in microcosm, for the workings of the universe". For,

> In an era that often compared the process of divine creation to the activity of an artist, Leonardo reversed the analogy, using art as his own means to approximate the mysteries of creation, asserting that, through the science of painting, "the mind of the painter is transformed into a copy of the divine mind, since it operates freely in creating many kinds of animals, plants, fruits, landscapes, countrysides, ruins, and awe-inspiring places.[341]



There is no mention of the Golden Ratio here, despite the fact that Leonardo illustrated Pacioli's *De Divina Proportione* over fifteen years later, as mentioned on page 202. But this has not stopped people saying that Leonardo's famous painting of the *Mona Lisa* is based on the Golden Rectangle, which is clearly false, as we see in this picture, also from Laputan Logic.

After Zeising, the next person to perpetuate the myth of the golden ratio was Gustav Theodor Fechner (1801–1887), the founder of experimental aesthetics in psychology, who set out in the 1860s to prove Zeising's 'pet theory', publishing the results of his experiments in 1876 in *Vorschule der Aesthetik* (*Introduction to Aesthetics*).[342] For instance, he conducted an experiment in which numerous observers were asked to select which of a number of rectangles, like this example, were the most aesthetically pleasing to them. Livio tell us that "76 percent of all choices centred on the three rectangles having the ratios 1.75, 1.62, and 1.50, with the peak at the Golden Rectangle (1.62)." I am not sure what this proves, as beauty is not to be found within any

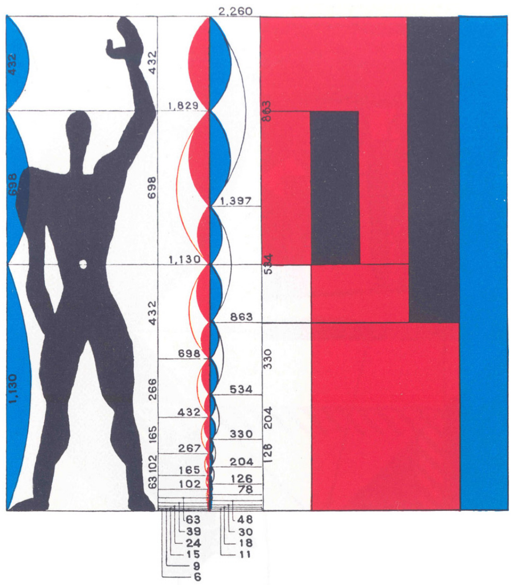


one form in isolation. Nevertheless, George Markowsky repeated the experiment in 1992, showing in a much-quoted paper titled 'Misconceptions about the Golden Ratio' that Zeising and Fechner's claims were flawed, dispelling many other misconceptions relating to the Golden Ratio.[343]

One architect who explicitly attempted to use the Fibonacci sequence to find a universal solution to the problems of human proportion was Charles-Édouard Jeanneret (1887–1965), known as Le Corbusier, after his maternal grandfather. Specifically, he sought a system of measurement that would be naturally based on the proportions of the human body, reconciling the metric system with feet and inches, the system invented by the Anglo-Saxons. Le Corbusier favoured anthropic units, for a metre "is nothing but a length of metal at the bottom of a well at the Pavillion du Breteuil near Paris",[344] and while *inch* derives from Latin *uncia* 'a twelfth', an inch is usually regarded as the width of a thumb.

He called this anthropocentric system 'Le Modulor', from French *moduler* 'to modulate', a measure that creates modules and modulates.[345] The modulor is thus "a measuring tool based on the human body and mathematics", depicted in this diagram, which Le Corbusier called the 'trade mark'.[346] It displays a man with upraised arm divided into three intervals, which are approximately in the ratio of the Golden section.

Le Corbusier initially set the height of the 'ideal' man at 1.75 metres, but realized that this was too French. On the other hand, "in English detective stories, the good-looking men, such as policemen, are always six feet tall."[347] Apparently "the female body was belatedly considered and rejected as a source of proportional harmony."[348]



Rounding six feet to the nearest millimetre, the modulor man is 1.829 metres tall. Dividing this by φ gives the height of the navel as 1130 mm, which Le Corbusier doubled to give the height of the tips of the fingers in the upraised arm as 2260. Thus, the ratios of the three principal sections of the modulor man are 1130/698 = 1.6157 and 699/432 = 1.6189, apparently using these values rather than the actual differences of 699 and 431, for they give ratios closer to 1.6180.

To develop his measuring tool, Le Corbusier then used 1829 and 2260 as the base for two Fibonacci sequences, called the red and the blue, dividing and multiplying them by φ and rounding to the nearest millimetre.[349] Thus, in most cases, each term is the sum of the two previous ones, the simplest of all second-order recurrence equations. The red and blue sequences wind around each other like a double helix, which "appears beside the sculptured image of the modulor man in many Le Corbusier buildings".[350]

Now, as each interval in a Fibonacci sequence, as the previous term, is approximately equal to the sum of all previous terms, Le Corbusier saw his measuring tool as "a flawless fabric formed of stitches of every dimension, from the smallest to the very largest, a texture of perfect homogeneity". The fabric itself was formed by plotting the red and blue sequences against each other, forming a lattice of rectangles that could be used in any design or construction work, represented at the right of the 'trade mark'.[351]

There are two central weaknesses of Le Corbusier's measuring tool, ingenious as it is. First, it is "an entirely analytical method of proportion", as P. H. Scholfield pointed out in 1958.[352] Secondly, as a consequence, it is not based on one's inner sense of proportion, but based arbitrarily on the external proportions of the human body, arranged to fit into the Divine Proportion, even though there is no reason why they should do so.

Then, in the second half of the twentieth century people's obsessive fascination with the Divine Proportion became so out of proportion that Stephen Strogatz was moved to write an opinion piece on 'Proportion Control' in the *New York Times* in 2012.[353] However, it is not easy to separate the myth from the maths, freeing the Golden Section from much hype, as he urged us to do. Most significantly, as the myths contain much ancient wisdom, lost to conventional science, it is important that we do not throw the spiritual baby out with the murky bath water.

For the Golden Ratio is actually quite ordinary, arising as a limiting ratio in Lucas sequences, along with an infinite number of others, as Clement Falbo pointed out in 2005 in 'The Golden Ratio: A Contrary Viewpoint'.[354] So, it doesn't appear to be as special as π and *e*, for instance. However, φ is special in one respect. The Golden Ratio has the distinction of being the most irrational number, as we see in its continued fraction representation in Chapter 3, a property that gives it central importance in the morphogenesis of plants, as we see on page 258.

To look for a number that is ubiquitous in nature, the only other one I would suggest is δ, as the little-known bifurcation velocity constant, which Mitchell J. Feigenbaum says lies at the heart of what he calls 'universality theory'.[355] For the reciprocal of this number—as the constant ratio of a geometric series, which we look at on page 279—helps us to model the entire history of evolution since the most recent big bang, enabling us to understand why the world is degenerating into chaos at the present time, as I explain in my 2016 book *Through Evolution's Accumulation Point: Towards Its Glorious Culmination*.

Infinite series

While Euclid and Archimedes studied the geometric series a little, it was not until the fourteenth century that mathematicians really became fascinated by infinite series. However, they could not get very far because, while they had imagination and precision of thought, they did not have the necessary algebraic and geometric facility.[356] It was not until the 1600s and 1700s, at the birth of modern mathematics, that the study of such series really took off, today playing a central role in analysis.

Eric Weisstein, on his *MathWorld* Wolfram website, defines *series* as "an infinite ordered set of terms combined together by the addition operator," saying, "the term 'infinite series' is sometimes used to emphasize the fact that series contain an infinite number of terms."[357] In mathematical terms, an infinite series is defined as the limit of the partial sums of the sequence of terms, which could be integers or real or complex numbers:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

As a_i could be positive or negative, unlike the terms in the finite sequences of natural numbers we looked in the first section, to determine whether a series converges regardless of sign, mathematicians conduct a ratio test[358] by calculating:

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

If ρ < or > 1, the series converges absolutely or diverges to infinity. If $\rho = 1$, it is not possible to use the ratio test to determine convergence or divergence. In some such cases, the series can either converge or diverge, defining the latter as that which does not converge, even those series that converge to oscillating values, which I would consider in a different category.

Before we look at the symbolic expressions of infinite series, it is important to remember that they first exist in the Cosmic Psyche in nonphysical form. As such, they are universal, becoming manifest in mathematical symbolism, as in this book, and in the material world through the action of the creative power of Life emanating directly from the Divine Origin of the Universe. As infinite series play a central role in mathematical analysis, having some understanding of this branch of mathematics is central to understanding how the Universe is designed and hence what is currently causing scientists and technologists to drive the pace of evolutionary change at unprecedented exponential rates of acceleration, an understanding that arises through unconditioned self-inquiry.

Infinite sums of reciprocals of finite sequences

For me, the natural place to start exploring infinite series is with the sums of the reciprocals of numbers generated from recurrence equations, such as figurate and Catalan numbers, which we explored in the first section of this chapter. In this, I am in good company. Jakob Bernoulli was similarly interested in the reciprocals of figurate numbers,[359] as we look at later.

Although such series don't generally follow a pattern, a Wikipedia page titled 'List of sums of reciprocals' does give two properties of such series that are significant. Does the series diverge or converge and, if the latter, is the limit value rational or irrational, and, if the latter, is that value algebraic or transcendental? This subset of infinite series thus has one common feature. If R_k is the kth number in the sequence generated from a recurrence equation, then we need to evaluate:

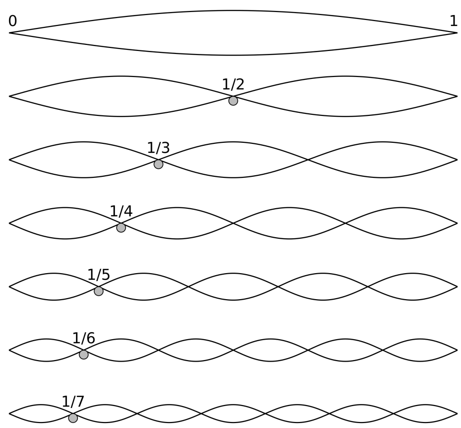
$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{R_k}=\sum_{k=1}^{\infty}\frac{1}{R_k}$$

So, one approach to evaluating the sums of reciprocals is first to determine if they converge and then to seek a closed-form expression for the first N terms, whose limit we can then determine. For instance, applying the ratio test, $\rho = 1$ for the harmonic series and reciprocals of the figurate numbers. But the former diverges, while the latter converge. On the other hand, $\rho < 1$ for the reciprocals of the Fibonacci and Catalan numbers, for instance, so these converge absolutely. The difference seems to be because the recurrence equations of the arithmetic progressions and figurate numbers are additive, while those of the others are multiplicative, like geometric series, which we look at in a moment.

Let us look at a few examples of this subset of infinite series to illustrate the variability.

Harmonic series

The most fundamental of the sums of reciprocals is that of the natural numbers, called a harmonic series, from Greek *ȧrmoniā* 'union, agreement, concord of sounds, harmony, proportion', from *ȧrmos* 'joint', from PIE-base *-*ar* 'to fit together', also root of *art*, *order*, and *ratio*. We can see the relationship between the reciprocals of the natural numbers and harmonics in music from this diagram, which shows the periods of a plucked string, for instance, as the inverse of its frequency, as the Pythagoreans discovered.



In mathematics, the harmonic series is defined as:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

Just as each term in the sequence of natural numbers is the arithmetic mean of the two numbers either side of it, the reciprocal of the harmonic mean in the harmonic series is the arithmetic mean of the reciprocals of the two numbers *a* and *b* either side of it, as this general formula with two terms shows:

$$\frac{1}{H} = \frac{\frac{1}{a} + \frac{1}{b}}{2} = \frac{a+b}{2ab}$$

giving

$$H = \frac{2ab}{a+b}$$

To illustrate, as *a* and *b* are themselves reciprocals in the harmonic series, the harmonic mean of ⅓ and ⅕ is:

$$\frac{2 \cdot \frac{1}{3} \cdot \frac{1}{5}}{\frac{1}{3} + \frac{1}{5}} = \frac{\frac{2}{15}}{\frac{3+5}{15}} = \frac{2}{8} = \frac{1}{4}$$

Now as the arithmetic mean *A* is $(a + b)/2$, the product of the arithmetic and harmonic means is the square of the geometric mean (*G*), as this formula shows:

$$A \cdot H = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = G^2$$

Pythagoras studied the arithmetic, geometric, and harmonic means with reference to the theory of music and arithmetic, in his general studies of the theory of proportion. The harmonic mean was initially called 'subcontrary', which the Pythagoreans Archytas and Hippasus later changed to what it is today.[360]

Regarding arithmetic series or progressions, in general, their reciprocals can also be regarded as harmonic series or progressions, as Euler pointed out.[361] For each term is the harmonic mean of the two numbers either side of it. Taking the odd numbers as an example, the harmonic mean of ⅓ and 1/7 is:

$$\frac{2 \cdot \frac{1}{3} \cdot \frac{1}{7}}{\frac{1}{3} + \frac{1}{7}} = \frac{\frac{2}{21}}{\frac{3+7}{21}} = \frac{2}{10} = \frac{1}{5}$$

The partial sums of the harmonic series are defined as harmonic numbers H_n, the first few terms being

n	1	2	3	4	5	6	7	8	9	10	11	12
H_n	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$	$\frac{7381}{2520}$	$\frac{83711}{27720}$	$\frac{86021}{27720}$
	1	1.5	1.83	2.08	2.28	2.45	2.59	2.72	2.83	2.93	3.02	3.1

The numerators and denominators of the harmonic numbers are given in the OEIS as A001008 and A002805, respectively. The harmonic numbers are obviously steadily increasing. But do they ever converge on a finite value as the increments get smaller and smaller? This is a question that the eminent French philosopher Nicole Oresme (c. 1323–1382) addressed in the middle of the 1300s. He proved that the harmonic series converges with a simple proof still taught in schools today.[362] From the harmonic series, he formed a smaller one in this manner:

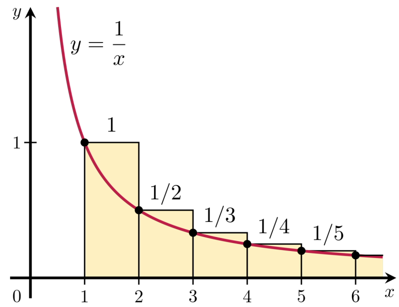
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots$$

As each group of 2^k terms in parentheses sums to ½ and as there are an infinite number of them, the harmonic series must diverge, 'a barely known masterpiece of medieval mathematics'.[363] This was little known until centuries later because Oresme's manuscript was lost.[364] In the event, Pietro Mengoli (1626–

1686), an 'unappreciated mathematician',[365] and Johann Bernoulli proved the divergence of the harmonic series in 1647 and 1687, respectively,[366] Bernoulli not knowing that Oresme and Mengoli had done so already.[367] His brother Jakob published a proof in his 1689 work *Tractatus de Seriebus Infinitis* (Treatise on Infinite Series),[368] falsely claiming that his brother was the first to do so.[369] He closes with these words:[370]

Even as the finite encloses an infinite series
And in the unlimited limits appear,
So the soul of immensity dwells in minutia
And in the narrowest limits no limit in here.
What joy to discern the minute in infinity!
The vast to perceive in the small, what divinity!

It seems from these lyrical words that the Bernoulli brothers and their contemporaries were somewhat bemused by the convergence of the harmonic series, as some still are today. In a short essay titled 'The Bernoullis and the Harmonic Series', William Dunham encapsulated the situation with these words: "Seasoned mathematicians tend to forget how surprising this phenomenon appears to the uninitiated student—that, by adding ever more negligible terms, we nonetheless reach a sum greater than any preassigned quantity."[371]



We can shed some further light on this situation with perhaps the most elegant formal proof of the divergence of the harmonic series, known as an 'integral test'. To illustrate, the area of all the rectangles in this diagram totals to the sum of the harmonic series, stretching to infinity. In comparison, the area under the curve, which is clearly smaller, is given by the integral:

$$\int_{x=1}^{\infty} \frac{1}{x} = \ln x|_1^{\infty} = \infty - 0 = \infty$$

Now, the rate at which $\ln x$ grows is given by its derivative, which is $1/x$, which gets smaller and smaller, but does not reach an infinitesimal value until x itself is infinite, unlike convergent infinite series, which we look in a moment. Indeed, in his famous paper on 'Harmonic Progressions' from 1734, Euler suggested that this fact provides a proof of the divergence of the harmonic series. In the same paper, Euler wondered about the total size of the 'triangular' pieces of each rectangle above the curve, as the finite difference of two expressions tending to infinity. He calculated it as 0.577218, acknowledging it as an approximate result, for the last decimal place should be 6.

This mathematical constant, which seems to appear frequently in the study of infinite series, was initially known as Euler's Constant. Then, in 1790, Lorenzo Mascheroni (1750–1800) calculated it to 32 decimal places. although only the first 19 places were correct. Nevertheless, as Mascheroni showed a deep understanding of Euler's calculus,[372] Euler's Constant is today called the Euler-Mascheroni Constant, denoted by γ, where

$$\gamma = 0.5772156649\dots.$$

This constant, whose decimal expansion is given in OEIS A001620, is so named because it is closely related to the digamma and gamma functions,[373] as a generalization of factorials, which we'll look at later.

What this integral test clearly shows is that the growth rate of the harmonic series, closely related to $\ln x$, is extremely slow, but nevertheless never stopping in finite time. Indeed, the growth rate is so slow that the partial sum of the harmonic series does not pass 10 until 12367 terms and in 1968 John W. Wrench Jr calculated that it would not pass 100 until the number of terms passed fifteen tredecillion ($15*10^{43}$), which OEIS A082912 gives exactly as

15,092,688,622,113,788,323,693,563,264,538,101,449,859,497

Indeed, this result raises the question whether a series could grow even more slowly than the harmonic series. Indeed, there is such a series! In 1737, Euler proved that the sum of the reciprocals of the prime numbers is also divergent, at a rate closely related to $\ln \ln x$.[374] In 1874, Franz Mertens (1840–1927) then calculated the asymptotic form of the harmonic series for the sum of reciprocal primes, corresponding to the Euler–Mascheroni constant, today known as Mertens Constant,[375] whose value is 0.2614972128… (OEIS A077761). Wikipedia calls this mathematical constant the Meissel–Mertens Constant, after Ernst Meissel (1826–1895), who made an uncertain contribution in 1866.[376]

There is no need to stop there. Divergent growth rates for $\ln \ln \ln x$, $\ln \ln \ln \ln x$, etc. are possible, as can be shown with the integral test for convergence, but they must diverge very, very slowly.[377] On the other hand, a subset of the reciprocals of the primes is so sparse that it actually converges. In particular, in 1919, Viggo Brun (1885–1978) proved that the sum of the reciprocals of the twin primes is convergent, even though it is conjectured that there are an infinite number of them. Twin primes are those that differ by two. However, what this sum converges to (Brun's Constant)[378] can only be determined heuristically to eight decimal places as of 2018, OEIS A065421 tells us: 1.90216058. It seems that earlier attempts at greater accuracy are considered doubtful.

Sums of reciprocals of figurate numbers

The harmonic series is a special case of the sums of the reciprocals of the powers of the natural numbers, with $k = 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

When $k > 1$, these are examples of reciprocals of powers, which are figurate numbers, whose sums historically proved to be rather elusive, leading to some challenging mathematical problems, still not entirely resolved today, which we look at later.

In the meantime, let us look at the triangular numbers, the most basic of the figurate numbers, where we need to find the limit of the partial sum:

$$\sum_{n=1}^{N} \frac{2}{n(n+1)}$$

where the summand is the reciprocal of the *n*th triangular number. Using partial fractions, this can be expressed as:

$$2\sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N-1} - \frac{1}{N} + \frac{1}{N} - \frac{1}{N+1} = 2\left(1 - \frac{1}{N+1}\right)$$

As you can see, this is a telescoping series, where terms from each consecutive pair cancel each other out. Thus, as N tends to infinity, the final term $1/(N+1)$ goes to zero and so the sum of the reciprocals of the triangular numbers is 2, as Mengoli discovered, although this credit is usually given to the better-known Christiaan Huygens (1629–1695).[379]

Similarly, the limit of the sums of the reciprocals of the tetrahedral and pentatope numbers can be calculated using partial fractions and telescoping series, giving the limit values 1½ and 1⅓, respectively. The same technique can be used to calculate the sums of the reciprocals of the *k*-simplexes, also known as binomial coefficients $C(n + k - 1, k)$, as we see in Pascal's triangle on page 198. However, this is rather tedious. Andrew M. Rockett seems to provide an inductive proof that the limit of the sums of the reciprocals of the *k*-simplexes is $k/(k - 1)$,[380] although I have not followed the reasoning in detail.

We can extend the reciprocals of the triangular and square numbers into polygonal numbers, as another basic set of figurate numbers, which have been little explored until this century.[381] Here are a few examples of the reciprocal numbers that I have found on the Web,[382] including the OEIS, which provides all the formulae and decimal expansions.

Polygonal number	Sum of reciprocals	Numerical value	OEIS
Pentagonal	$3\ln 3 - \frac{\sqrt{3}\pi}{3}$	1.4820375018…	A244641
Hexagonal[383]	$2\ln 2$	1.3862943611…	A016627
Heptagonal[384]	$\frac{1}{15}\pi\sqrt{25-10\sqrt{5}} + \frac{2}{3}\ln 5 + \frac{1+\sqrt{5}}{3}\ln\left(\frac{1}{2}\sqrt{10-2\sqrt{5}}\right) + \frac{1-\sqrt{5}}{3}\ln\left(\frac{1}{2}\sqrt{10+2\sqrt{5}}\right)$	1.3227792531…	A244639
Octagonal[385]	$\frac{3\ln 3}{4} + \frac{\sqrt{3}\pi}{12}$	1.2774090576…	A244645
Nonagonal	$\frac{1}{5}\left(2\ln 14 + 4\cos\frac{\pi}{7}\cdot\ln\left(\cos\frac{3\pi}{14}\right) + \sin\frac{\pi}{14}\cdot\ln\left(\sin\frac{\pi}{7}\right) - \sin\frac{3\pi}{14}\cdot\ln\left(\cos\frac{\pi}{14}\right) + \pi\tan\frac{3\pi}{14}\right)$	1.2433209262…	A244646
Decagonal[386]	$\ln 2 + \frac{\pi}{6}$	1.2167459562…	A244647

Given the complexity of these formulae, especially those related to odd-sided polygons, it is not surprising that the reciprocals of figurate numbers have not been explored further. Besides, as all figurate numbers stretch out more than triangular numbers, whose reciprocals sum to 2, the sums of the reciprocals of all these other examples must lie between 1 and 2, not particularly interesting in the context of this book, other in the case of the reciprocals of powers.

Sums of reciprocals of Fibonacci and Lucas numbers

While Mengoli, the Bernoulli brothers, and Euler made some progress in the seventeenth and eighteenth centuries with the sums of the reciprocals of the figurate numbers, it was not until the end of the nineteenth that mathematicians began to wonder about the sum of the reciprocals of the Fibonacci numbers. And, as far as I can tell, a general solution to the sums of the reciprocals of all Lucas sequences was not found until the 1980s. Alwyn F. Horadam in Australia[387] and the brothers Jonathan M. Borwein and Peter B. Borwein in Canada,[388] for instance, seem to have independently found solutions in 1986 and 1987, respectively.

Horadam provides a brief history of the topic, telling us that it was early recognized that to solve this problem a different approach was needed for the odd- and even-indexed Fibonacci numbers. First, Lucas sought to express the reciprocals of the even-indexed Fibonacci numbers in 1878 using Lambert series, named after Johann Heinrich Lambert (1728–1777), who defined this series in 1771 as:

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}$$

Then, in 1883, Catalan attempted to solve the reciprocals of the odd-indexed Fibonacci numbers using Jacobian elliptic functions, which Edmund Landau (1877–1938) elaborated on in 1899 in terms of theta functions. Also in 1899, Charles-Ange Laisant (1841–1920) seems to be the first to desire to evaluate:

$$\sum_{n=1}^{\infty} \frac{1}{F_n}$$

noting that this series evidently converges.[389]

As I am not familiar with these more advanced branches of mathematics, it is not easy for me to follow the reasoning of how mathematicians evaluate these reciprocals. Eric W. Weisstein provides an outline of the calculations of the sums of the reciprocals of the Fibonacci and Lucas numbers, telling us that they have both been given names, indicating their significance as examples of mathematical constants: Reciprocal Fibonacci Constant[390] and Reciprocal Lucas Constant.[391]

However, these constants are examples of the more general set of sums of the reciprocals of Lucas sequences, defined in the first section. We can first note that the ratio test shows that $\rho < 1$ for most of these series, as this table indicates:

Lucas sequence	ρ	Related to	Numerical value
Fibonacci/Lucas	$\frac{1}{\varphi}=\frac{2}{1+\sqrt{5}}$	Golden ratio	0.6180339887…
Pell/Pell-Lucas	$1/(1+\sqrt{2})$	Silver ratio	0.4142135624…
	$2/(3+\sqrt{13})$	Bronze mean	0.3027756377…
	$1/(2+\sqrt{5})$	Copper mean	0.2360679775…
	$2/(5+\sqrt{29})$	Nickel mean	0.1925824036…
Mersenne/Fermat-Lucas	1		1

To calculate the sums of the reciprocals of the Lucas sequences mentioned in the first section, I have used WolframAlpha to calculate the partial sums of the first 100 terms,[392] which converge quickly, rather than a general algorithm. As you can see, the decimal expansions of only the three sequences that have been given names are included in the OEIS.

Sum of reciprocals of Lucas sequences	nth term	Numerical value	OEIS
Fibonacci (Reciprocal Fibonacci Constant)	$\sqrt{5}/\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$	3.3598856662…	A079586
Lucas (Reciprocal Lucas Constant)	$1/\left(\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$	1.9628581732…	A093540
Pell	$2\sqrt{2}/\left((1+\sqrt{2})^n-(1-\sqrt{2})^n\right)$	1.8422030498…	—
Pell-Lucas	$1/\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right)$	0.7883239758…	—
‘Bronze’	$\sqrt{13}/\left(\left(\frac{3+\sqrt{13}}{2}\right)^n-\left(\frac{3-\sqrt{13}}{2}\right)^n\right)$	1.4767947263…	—
	$1/\left(\left(\frac{3+\sqrt{13}}{2}\right)^n+\left(\frac{3-\sqrt{13}}{2}\right)^n\right)$	0.4640730686…	—
‘Copper’	$2\sqrt{5}/\left((2+\sqrt{5})^n-(2-\sqrt{5})^n\right)$	1.3270042779…	—
	$1/\left((2+\sqrt{5})^n+(2-\sqrt{5})^n\right)$	0.3227787301…	—
‘Nickel’	$\sqrt{29}/\left(\left(\frac{5+\sqrt{29}}{2}\right)^n-\left(\frac{5-\sqrt{29}}{2}\right)^n\right)$	1.2476357312…	—
	$1/\left(\left(\frac{5+\sqrt{29}}{2}\right)^n+\left(\frac{5-\sqrt{29}}{2}\right)^n\right)$	0.2458834938…	—
Mersenne (Erdős-Borwein Constant)	$1/(2^n-1)$	1.6066951524…	A065442
Fermat-Lucas	$1/(2^n+1)$	0.7644997803…	—

Sum of reciprocals of Catalan numbers

Even though the Catalan numbers have a multitude of different interpretations, the sum of their reciprocals is comparatively simple. To evaluate it, we need to determine:

$$\sum_{n=0}^{\infty}\frac{1}{C_n}=\sum_{n=0}^{\infty}\frac{n+1}{\binom{2n}{n}}=\sum_{n=0}^{\infty}\frac{n!\,(n+1)!}{(2n)!}$$

Also, from the recurrence equation for the Catalan numbers, the ratio test gives:

$$\frac{1}{\rho}=\lim_{n\to\infty}\frac{C_{n+1}}{C_n}=\lim_{n\to\infty}\frac{4n+2}{n+2}=4$$

So the sum of the reciprocals of the Catalan numbers clearly converges. Rather surprisingly, it was not until 2014 that mathematicians found what the sum converged to, as far as I can tell. Using a generating function and some rather nifty calculus and algebra, Thomas Koshy and Zhenguang Gao found:[393]

$$\sum_{n=0}^{\infty}\frac{1}{C_n} = 2 + \frac{4\sqrt{3}}{27}\pi = 2.80613305077\ldots$$

This is A268813 in the OEIS, 1 more than A121839, the decimal expansion from $n = 1$. Koshy and Gao even calculated the sum of the reciprocals of the alternating Catalan numbers finding:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{C_n} = \frac{14}{25} - \frac{24\sqrt{5}}{125}\ln\varphi = 0.3534037083\ldots$$

where φ is the golden ratio $(1+\sqrt{5})/2$. The OEIS does not give this decimal expansion.

Sum of reciprocals of partition numbers

The OEIS's Wiki page on the Partition function gives:[394]

$$\sum_{n=0}^{\infty}\frac{1}{p(n)} = ?$$

Nevertheless, OEIS A078506 gives the decimal expansion of the sum of the inverses of the unrestricted partition function as 2.51059748389…. It seems from this that mathematicians have not found an algorithm for this sum, even using the most powerful tools relating to infinite series.

Geometric series

Just as an arithmetic series is formed from an arithmetic progression by adding a constant to an initial value, geometric series are formed from a geometric progression, which is a sequence of numbers formed by multiplying the last term in a sequence by a fixed, non-zero number, called the 'common ratio'.[395] Such a geometric sequence is thus that formed from this recurrence equation:

$$a_{n+1} = ra_n \qquad ra_0 = a \neq 0$$

where r is the common ratio. The general form of the geometric series is thus defined as

$$S_n = \sum_{k=0}^{n} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^n$$

To derive a closed-form expression for this geometric series, multiply each term by r to form:

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n+1}$$

Subtracting one sum from the other, most of the terms cancel, giving:

$$(1-r)S_n = 1 - ar^{n+1}$$

giving

$$S_n = \sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}$$

This expression has a long history, Euclid proving it in his own manner, as Thomas Heath explains.[396]

When $r > 1$, the geometric series clearly diverges to infinity. However, when $|r| < 1$, and setting $a = 1$, the series converges to this classic form:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots$$

As you can see, this equation has the form of the most fundamental generating function on page 244, but where r or x is no longer just a place-holder. With infinite series, in general, the variable is just that, it has a value, which could affect how the series behaves. This particular geometric series appears so frequently in mathematics that blackpenredpen calls this formula his 'best friend' on his YouTube channel.

As an aside and in preparation for looking at the way infinite series can be expressed as infinite products, Mu Prime Math shows on his YouTube channel such a relationship, which I have not seen elsewhere in the literature:[397]

$$\sum_{k=0}^{\infty} x^k = \prod_{k=0}^{\infty}(1 + x^{2^k}) = \frac{1}{1-x}$$

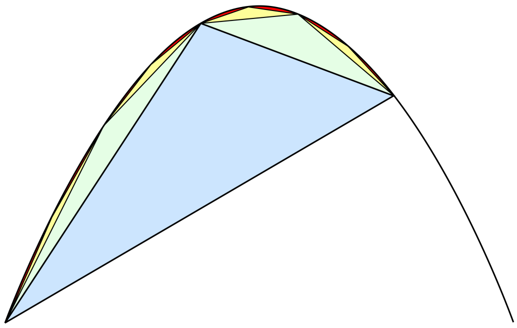
When $|r| < 1$, it is sometimes convenient to view the common ratio as the reciprocal of a number greater than one, such as $|q| > 1$. For instance, when $q = 2$, we have:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} = 2$$

We have a situation here that is quite different from the harmonic series, which is convergent even though successive terms tend towards zero, greatly puzzling the uninitiated, as we see on page 275. For even though the reciprocals of the natural numbers in a geometric series similarly diminish to zero, the series itself is convergent, again causing much puzzlement. For here we have another series—as the sum of an infinite number of finite terms, but which has a finite value, a mystery that we need to unravel if we are to understand what is happening to humanity at the present time.

Zeno of Elia (c. 490–430 BCE) was particularly perplexed by such a situation. As David M. Burton puts it in *The History of Mathematics*, "Zeno pointed out the logical absurdities arising from the concept of 'infinite divisibility' of time and space."[398] This led Zeno to propose four clever paradoxes, the most famous of which is that of Achilles and the tortoise, which Aristotle described thus: "This claims that the slowest runner will never be caught by the fastest runner, because the one behind has first to reach the point from which the one in front started, and so the slower one is bound always to be in front."[399]

However, what Aristotle called the dichotomy paradox did not trouble Archimedes, when he came to find the limit of a geometric series, using the method of exhaustion, which Eudoxus of Cnidus[400] formalized from an original idea of Antiphon the Sophist,[401] and which Archimedes also used to estimate the value of π. In *The Quadrature of the Parabola*, Archimedes sought to find the area of a segment of a parabola cut off by a chord, as in this diagram from Wikipedia.[402] After Propositions 1–17, which found the area by mechanical means, Propositions 18–24 gave a geometric solution to the problem.[403]



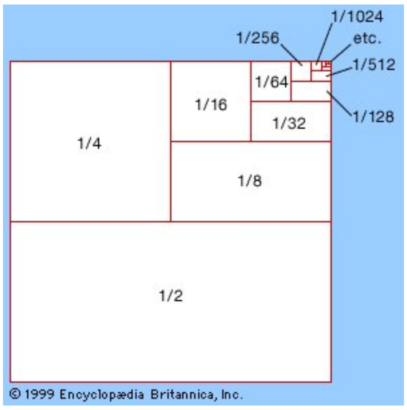
Archimedes first drew a triangle of area T to the point on the parabola where the tangent is parallel to the chord, knowing that a vertical dropped from this point would divide the horizontal base in half. He then drew further triangles, marked in green, based on the blue triangle, knowing from his knowledge of the properties of parabolas that the height of each green triangle is a quarter of the blue one. Thus, the area of the two green triangles is an eighth of the blue one. But he did not stop there. He successively drew further triangles in a similar manner, knowing that by the method of exhaustion they would eventually fill the entire space. In modern terms, we know that the area K of the segment is $4T/3$, as this expression shows:

$$K = T + 2\left(\frac{T}{8}\right) + 4\left(\frac{T}{8^2}\right) + 8\left(\frac{T}{8^3}\right) + \cdots 2^n\left(\frac{T}{8^n}\right) + \cdots = \left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots \frac{1}{4^n} + \cdots\right)T = \frac{4T}{3}$$

However, "Archimedes did not refer to the sum of the infinite series, for infinite processes were frowned on in his day; instead he proved a double *reductio ad absurdum* that K can be neither more nor less than $4T/3$."[404]

Little progress was made in understanding infinite series for a millennium and a half. Having proved that the harmonic series converges, Oresme then addressed the problem of convergent infinite series, apparently describing the general form of a geometric series when he wrote:

> … when the infinite series is of the nature that to a given magnitude there are added 'proportional parts to infinity' and the ratio a/b determining the proportional parts is less than one, the series has a finite sum. But when $a > b$, 'the total would be infinite'; that is, the series would be divergent.[405]



However, Oresme did explore some particular convergent geometric series, including that when $r = ½$, using a geometrical solution, not unlike this in Encyclopaedia Britannica. To find the sum of this series, Oresme first divided a square of area 1 in half, and then successively divided one of the halves that remained in half, filling the square, totalling 1. Adding this to the area of the original square gives the sum of 2 for this geometric series, as we see above.

Oresme went even further. Boyer tells us that Oresme calculated this expression, where each term in the Archimedes geometric series is multiplied by a multiple of 3:[406]

$$\frac{1\cdot 3}{4}+\frac{2\cdot 3}{16}+\frac{3\cdot 3}{64}+\cdots+\frac{n\cdot 3}{4^n}+\cdots=\frac{4}{3}$$

He did so by first evaluating:

$$B=\sum_{k=1}^{\infty}\frac{k}{2^k}=\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots+\frac{n}{2^n}+\cdots$$

with an ingenious geometric solution.[407] In effect, Oresme found the finite sum of an infinite series of infinite series in this way:

1/2 +	1/4 +	1/8 +	1/16 +	…	= 1
	1/4 +	1/8 +	1/16 +	…	= 1/2
		1/8 +	1/16 +	…	= 1/4
			1/16 +	…	= 1/8
				⋮	
1/2 +	2/4 +	3/8 +	4/16 +	…	= 2

Then, three hundred years later, Mengoli discovered an even more astonishing property of infinite series. Having proved that the harmonic series diverges in his own way, he proved that the alternating harmonic series converges to a finite value.[408] This is not obvious, because by the ratio test, as for the harmonic series itself,

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$$

To prove that the alternating harmonic series converges, Chris Odden has a YouTube video showing that the partial sums of the even and odd terms in the series converge on the same finite value, from both sides, with the even and odd increasing and decreasing, respectively.[409] But this does not tell us what this limit is.

Rather surprisingly, we can use the geometric series to help here, as 3Blue1Brown explains on his own YouTube channel.[410] We are seeking the sum S of

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots + \frac{(-1)^n}{n} + \cdots$$

Now sometimes it is easier in mathematics to solve a more general problem than a particular one, as George Pólya points out in *How To Solve It*. In this case, let us form $f(x)$ as:

$$f(x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \cdots + \frac{(-1)^n x^n}{n} + \cdots$$

Then differentiating, we have a geometric series with $r = -x$:

$$f'(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \cdots = \frac{1}{1+x}$$

Now integrating $1/(1+x)$ between 0 and 1, as the antiderivative, gives:

$$S = \int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)|_0^1 = \ln(2) - \ln(1) = \ln(2) = 0.6931471806 \ldots$$

Newton later found the finite sum of the alternating harmonic series, calculating its value to 16 decimal places (OEIS A002162). However, it is important to note here that the alternating harmonic series is conditionally convergent, not absolutely, leading to a quite remarkable property, known as the Riemann Series Theorem. This states that a conditionally convergent series may be made to converge to any desired value or to diverge by a suitable rearrangement of terms,[411] a situation that is closely related to the Euler–Mascheroni Constant γ.[412] For instance:

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \cdots = \frac{1}{2}\ln 2$$ [413]

and

$$\left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \left(\frac{1}{9} + \frac{1}{11} - \frac{1}{6}\right) + \cdots = \frac{3}{2}\ln 2$$ [414]

As a somewhat more complex example, Mathologer shows how we can rearrange the alternating harmonic series to converge to π or any other arbitrary number. First, collect all the positive and negative terms together to form two infinite series. Then, to set ∞ - ∞ to π, first find the partial sum of the positive numbers that just exceeds 3.14159. Then, add the first of the negative values, to go below π. To home in on π, add as many positive and negative terms as necessary to lead the partial sums closer and closer to π to as many decimal digits as desired.[415]

If we now set $r = -1$ in the basic geometric series, we obtain this alternating series, sometimes called a Grandi's series, after Guido Grandi (1671–1742),[416] which is causing mathematicians problems, even today:

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

We saw in Chapter 3 that Bernard Bolzano looked at this series in the late 1840s in *Paradoxes of the Infinite* when addressing the tricky nature of infinity. He told us that in 1830, someone identified as *M. R. S.* had 'proved' this sum to be ½, which Bolzano declared to be a fallacy.[417]

Similarly, in 2014, a physics professor at my old university in Nottingham 'proved' on a Numberphile YouTube video that this series converges to ½ using erroneous mathematical reasoning, as the first step to proving Ramanujan's 1913 assertion that under some circumstances the infinite sum of the natural numbers is -1/12,[418] which we look at page 307. As this video has been watched over seven million times, it has caused no end of confusion, which Mathologer splendidly clarified in a YouTube video in 2018, with 1.7 million views as of June 2020.[419]

As mentioned on page 273, mathematicians divide all infinite series into those that converge on a finite value and those that do not, calling the latter divergent. However, there are different types of divergence. Some infinite series, like the natural numbers, diverge to infinity. Yet, in the case of this alternating series, the values of the partial sums oscillate between 0 and 1, finite numbers. So, is there any sense in which this series converges? Indeed, there is, as Mathologer explains.

If we first find its partial sums, as:

$$1\;0\;1\;0\;1\;0\;1\;0\;1\;0\;\ldots$$

and then take the average of the first *n* terms, we obtain these values:

$$1\;\frac{1}{2}\;\frac{2}{3}\;\frac{1}{2}\;\frac{3}{5}\;\frac{1}{2}\;\frac{4}{7}\;\frac{1}{2}\;\frac{5}{9}\;\frac{1}{2}\;\ldots$$

The averages for even-indexed numbers are ½ and for those for the odd ones, in position $2k - 1$, are $k/(2k\text{-}1)$ for $n \geq 1$, which also tends to ½. So, in some sense, we can say that this series does converge to ½. It is 'Cesaro convergent', named after Ernesto Cesàro (1859–1906).

But how about the alternating series of the natural numbers:

$$1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + \cdots$$

This is the reciprocal of the alternating harmonic series, which we have seen converges to ln(2). So, we might assume that this series diverges, as the opposite of convergence. But maybe there is also a sense in which it converges, which is, indeed, the case. As Mathologer explains, we again first form the partial sums, which eventually oscillate between +∞ and -∞, unlike those in the previous example, which oscillate between finite values.

$$1\;\text{-}1\;2\;\text{-}2\;3\;\text{-}3\;4\;\text{-}4\;5\;\text{-}5\;\ldots$$

So, it is perhaps not surprising that the averages of their partial sums do not converge:

$$1\;0\;\frac{2}{3}\;0\;\frac{3}{5}\;0\;\frac{4}{7}\;0\;\frac{5}{9}\;0\;\ldots$$

However, if we take the average of these averages, we obtain:

$$1\;\frac{1}{2}\;\frac{5}{9}\;\frac{5}{12}\;\frac{34}{75}\;\frac{34}{90}\;\frac{298}{735}\;\frac{298}{840}\;\frac{1069}{2835}\;\frac{1069}{3150}\;\ldots$$

which slowly tends to ¼, as Numberphile stated by dubious reasoning, although what the formula for the *n*th term is is not easy to determine. So, once again, we have a non-convergent series that does converge in some sense.

There is one other variation of a geometric series that merits mention. We have seen that Oresme created a convergent geometric series with the numerators as the multiple of a constant. In *Tractatus de Seriebus Infinitis*, Jakob Bernoulli generalized this series with an arithmetic series in the numerator:[420]

$$\frac{a}{b} + \frac{a+c}{bd} + \frac{a+2c}{bd^2} + \frac{a+3c}{bd^3} + \cdots$$

Bernoulli evaluated this series by decomposing it into an infinite geometric series of geometric series, not unlike Oresme, but using algebraic reasoning rather than geometric. In this way, he found that its sum is:

$$\frac{ad^2 - ad + cd}{bd^2 - 2bd + b} = \frac{d(ad - a + c)}{b(d^2 - 2d + 1)}$$

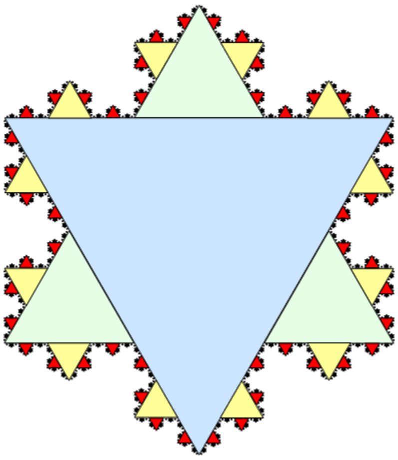
For instance, when $a = 1$, $b = 3$, $c = 5$, and $d = 7$, we have:[421]

$$\frac{1}{3} + \frac{6}{21} + \frac{11}{147} + \frac{16}{1029} + \frac{21}{7203} + \frac{26}{50421} + \cdots = \frac{77}{108}$$

Bernoulli's 36-page treatise marks the culmination of mathematicians' endeavours to master infinite series during the seventeenth century. So, it is rather strange that there is no translation into English of this significant work in the history of mathematical ideas, as far as I can tell.

But before we look at the developments that came after Bernoulli, let us briefly look at a few of the ubiquitous applications of the geometric series. One use is in fractional-reserve banking, which allows banks to lend far more money than they have in reserve, effectively creating money out of nothing, as I explain in my book *The Theory of Everything* from 2014. As this is an insane way of conducting our business affairs, enabling wars to be funded, there is no need to dwell on this application further.

Rather, if we are ever to live in love, peace, and harmony with each other, we need to make the most radical change to the work ethic since the invention of money some four thousand years ago. By being free of our mechanistic cultural conditioning, we can take the abstractions of mathematicians, outlined in Chapter 5 on 'Universal Algebra', to the utmost level of generality, viewing the manifest Cosmos as a meaningful, holographic information system with the property of self-similarity, emerging directly from the meaningless Absolute, as the Datum of the Universe.



To give a simple illustration, Wikipedia shows us how we can use a geometric series to determine the area of a fractal like the Koch snowflake, named after Helge von Koch (1870–1924). Each added triangle is one ninth of the size of the initiating triangle and, after the green triangles, their number increases by a factor of 4, giving the total area as:

$$A = 1 + 3\left(\frac{1}{9}\right) + 3 \cdot 4\left(\frac{1}{9}\right)^2 + 3 \cdot 4^2\left(\frac{1}{9}\right)^3 + \cdots$$

Ignoring the first term, this is a geometric series whose first term is a = 3(1/9) = 1/3 and whose constant ratio is r = 4/9. The total area of the Koch snowflake is thus:

$$A = 1 + \frac{a}{1-r} = 1 + \frac{1/3}{1-4/9} = \frac{8}{5}$$

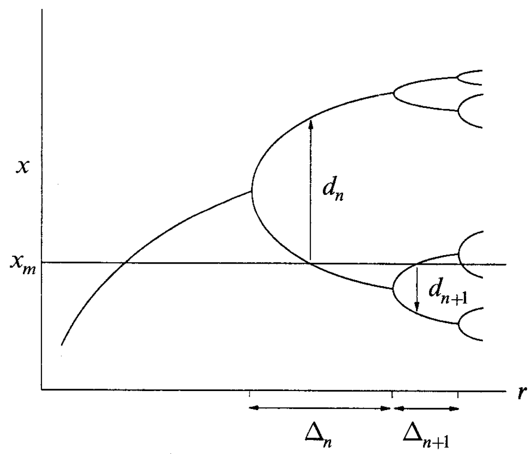
In contrast, the perimeter of the Koch snowflake tends towards infinity. For at each iteration, the number of edges increases by a factor of 4, while the length of each edge decreases by 1/3, giving r = 4/3 in the geometric series, which thus converges. So the infinite perimeter of the Koch snowflake encloses a finite area, another surprising result.

As Integral Relational Logic shows that the underlying structure of the Cosmos is a multidimensional network of hierarchical relationships, simply represented as a holographic, self-similar mathematical graph, we can map the whole of evolution since the most recent big bang as a coherent whole. Then, as my 2016 book *Through Evolution's Accumulation Point: Towards Its Glorious Culmination* explains, we can use nonlinear systems dynamics to develop a comprehensive evolutionary model of the whole of evolution since the most recent big bang, describing why society is degenerating into chaos at the present time.

The most appropriate mathematical tool for this study is the logistic map,[422] the discrete form of the logistic function, which Pierre-François Verhulst (1804–1849) introduced in 1844 to study the potential growth of the population of the newly formed nation of Belgium.[423] To represent growth under constraint, Robert M. May wrote a seminal paper on this first-order, nonlinear difference equation in 1976,[424] when studying a hypothetical population of fish living in a pond, whose growth, by its nature, is limited.[425] The canonical form of the logistic map is,

$$x_{n+1} = rx_n(1 - x_n)$$

where r denotes the rate of growth in some sense, lying in the range [0,4], and the term 1 - x_n keeps the growth within bounds, since as x_n rises, 1 - x_n falls.[426]



May, who later became Chief Scientific Adviser to the UK Government and President of the Royal Society of London, was staggered by the results. Without going further into the mathematical details, he discovered that the iterations of the recursive logistic map first converge on single values. Then they begin to bifurcate, as this diagram from Steven H. Strogatz's popular *Nonlinear Dynamics and Chaos* illustrates.[427]

In 1978, Mitchell J. Feigenbaum then published an even more amazing result. The periods between successive bifurcations, denoted by the deltas, diminish by a constant factor, known today as the reciprocal of the Feigenbaum bifurcation velocity constant δ,[428] which is given by this formula (OEIS 006890):

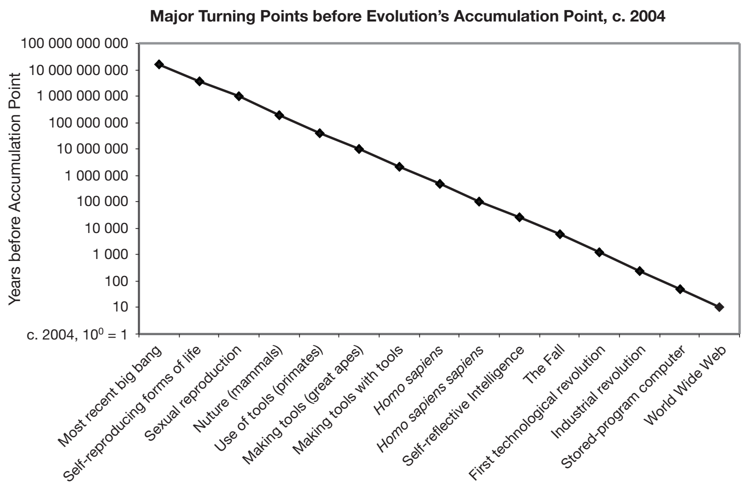
$$\lim_{n\to\infty} \frac{\Delta_n}{\Delta_{n+1}} = 4.6692016091 \ldots$$

This mathematical constant is not found only in the logistic map. It also lies at the heart of a wide-range of functions, a characteristic that Feigenbaum called 'universality theory'. As he said, "This definite number must appear as a natural rate in oscillators, populations, fluids, and all systems exhibiting a period-doubling route to turbulence! … So long as a system possesses certain qualitative properties that enable it to undergo this route to complexity, its quantitative properties are determined."[429]

However, bifurcations do not continue indefinitely, for they get shorter and shorter by a factor approaching 1/4.6692 in a geometric series, giving its approximate sum as:

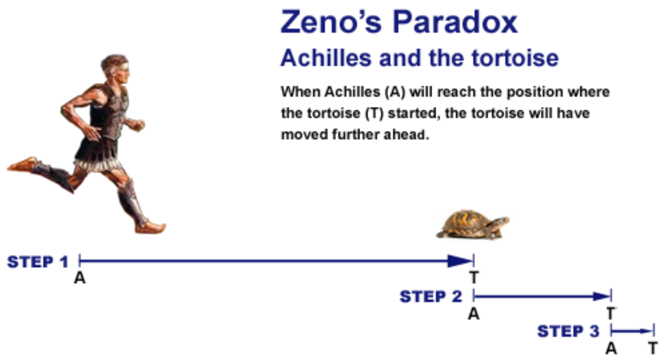
$$1 + \frac{1}{4.6692} + \frac{1}{4.6692^2} + \frac{1}{4.6692^3} + \cdots = \frac{1}{1 - \frac{1}{4.6692}} \approx 1.2725$$

As the whole of evolution is a bifurcating system, we can use this geometric series to calculate what its limiting point is, known as evolution's Accumulation Point, as illustrated in this diagram:



As the most recent big bang happened about fourteen billion years ago, we need to set *a* in the general geometric series in order to approximate the sum of all the periods between evolution's major turning points to the length of time since then. This we can do by setting *a* = 14,000,000,000/1.2725, which is about eleven billion years, the length of time between the most recent big bang and the emergence of the first self-reproducing forms of life on Earth. Of course, this calculation does not enable us to calculate when evolution's Accumulation Point happens to any accuracy.

All that is necessary to do this is to note that the periods between the last three points in the above chart, whose dates we know reasonably accurately, diminish by the reciprocal of δ. In this way, we can estimate that evolution's Accumulation Point happened about 2004, give or take a couple of years, explaining why the world is degenerating into increasing chaos at the moment. Nick Hoggard, a software developer, who presented this evolutionary model at the continental meeting of the Scientific and Medical Network in Sweden in 2000, likened this model to a dripping tap, where the increasing rate of drips corresponds to evolution's major turning points. As evolution has now passed through its Accumulation Point, the evolutionary tap is now turned full on, with no more discrete turning points to be discerned.



However, this does not mean that evolution has come to an end, any more than Achilles and the tortoise stop running or crawling when they come to the end of their race. We are effectively now living at a time beyond the infinite, the most momentous event in evolutionary history, requiring us to make the most radical change to the way we live our lives, if we are able to awarely adapt to our rapidly changing environment.

Taylor and Maclaurin series

Before we look at the exponential function and its related series, as the archetypal growth series in mathematics, it is helpful to look at the so-called Taylor and Maclaurin series, often taught to high-school students when they begin to study calculus. However, there is some confusion in the literature about nomenclature, which I have only unravelled when researching this topic.

It is easiest to begin with Colin Maclaurin (1698–1746), who was something of a prodigy, becoming a professor of mathematics at nineteen,[430] who wrote his treatise nearly thirty years after that of Brook Taylor (1685–1713). Maclaurin wrote *A Treatise of Fluxions* in 1742, "partly in response to the criticism of the foundations of the theory of fluxions voiced by George Berkeley eight years earlier",[431] when Berkeley said that fluxions are 'ghosts of departed quantities', mentioned in Chapter 3.

In his treatise, Maclaurin wondered whether it would be possible to express an infinitely differentiable, continuous function in terms of a polynomial with suitable coefficients *A*, *B*, *C*, *D*, and so on, for integrating and differentiating terms in a polynomial, as powers of *x*, lie at the foundation of the calculus:

$$f(x) = A + Bx + Cx^2 + Dx^3 + \cdots$$

By differentiating this polynomial over and over again and setting $x = 0$, thereby successively leaving only the first term each time, Maclaurin found values for the coefficients thus:[432]

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

For instance, here are the familiar Maclaurin series for examples of complementary trigonometric and hyperbolic functions:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1}$$

Maclaurin acknowledged that his series is 'only' a special case of the more general Taylor series, not knowing that James Stirling had published this particular series more than a dozen years earlier than Maclaurin in *Methodus Differentialis*.[433] In turn, Taylor, secretary of the Royal Society for a time, published his series in *Methodus Incrementorum Directa et Inversa* in 1715, not knowing that James Gregory (1638–1675) had formulated such a series in *Geometriae pars universalis* (*The Universal Parts of Geometry*) in 1668, coming close to discovering the calculus, unknown to Newton.[434]

Be that as it may, here are two expressions for the Taylor series, formed in a different manner from that of Maclaurin:

$$f(x+a) = f(a) + f'(a)x + f''(a)\frac{x^2}{2!} + f'''(a)\frac{x^3}{3!} + \cdots + f^{(n)}(a)\frac{x^n}{n!} + \cdots$$ [435]

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$ [436]

As you can see, setting $a = 0$ gives the Maclaurin series. However, in general, the Taylor series is not directly expressed as a polynomial. For instance, Wolfram *MathWorld* gives:

$$\sin x = \sin a + \cos a(x-a) - \frac{1}{2}\sin a(x-a)^2 - \frac{1}{6}\cos a(x-a)^3 + \cdots$$

Setting $a = 0$ immediately gives the Maclaurin series for $\sin x$ as a polynomial. But presumably setting a to any other value would also do so by expanding terms like $(x-a)^n$, such as $(x-\frac{\pi}{2})^n$ or even $(x-\frac{\pi}{4})^n$, for each n, where $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Yet the more commonly used Maclaurin series is often referred to as a Taylor series. For instance, in their influential textbook *What is Mathematics?*, Courant and Robbins derive the Maclaurin series along the lines above, but call it a Taylor series.[437] The general Taylor series is not mentioned in this book, presumably because it is less useful, and neither is the name of Colin Maclaurin.

Exponential and logarithmic functions

Having seen that infinitely differential functions can be expressed as an infinite series, we now need to look at the two functions that concern us most in our studies of our rapidly changing world: the exponential function, which models accelerating growth and rates of change, and its dual, the logarithmic function.

We saw in Chapter 3 that exponentiation can be viewed as repeated multiplication, just as multiplication is repeated addition. But this interpretation only works when the exponentiation factor is a natural number. It does not apply when this factor is a real or complex number, as briefly illustrated with the expression i^i in the previous chapter. We need exponential series, as polynomials, to best understand the exponential function, as Euler demonstrated in the eighteenth century.

But first, we once again need to go back to first principles, with logarithms rather than exponentials.[438] For historically, a logarithmic series appeared before an exponential one, with mathematicians in the 1600s coming close to the mysterious constant e without understanding what it is.[439] Let us therefore look at the relationship between the arithmetic and geometric progressions, illustrated in this table:

0	1	2	3	4	5	6	7	8
1	2	4	8	16	32	64	128	256

The German monk Michael Stifel (1487–1567) published such a table in *Arithmetica Integra* in 1544, pointing out that the sum of two terms in the upper arithmetic progression has a connection with the corresponding product of two terms in the lower geometric progression.[440] For instance,

$$8 \times 32 = 2^3 \times 2^5 = 2^{3+5} = 2^8 = 256$$

This relationship probably gave John Napier (1550–1617), Baron Merchiston of Scotland, the idea of developing a procedure that would substitute the operations of addition and subtraction for those of multiplication and division, making calculations very much easier. Accordingly, Napier set out to pair the terms of a geometric series with those of an arithmetic one, spending twenty years calculating what he was to call logarithms 'reckoning number', from Greek *logos* 'reckoning, ratio' and *arithmos* 'number'. He published his results in 1614 in a small Latin volume of 147 pages—90 of which were tables—with the title *Mirifici Logarithmorum Canonis Descriptio* (*A Description of the Marvellous Rule of Logarithms*). His son Robert then posthumously published Napier's account of how the tables were constructed, written earlier, in *Mirifici Logarithmorum Canonis Constructio* (*The Construction of the Wonderful Canon of Logarithms*).[441]

There is no need to go into Napier's method of calculating logarithms in detail because it doesn't enhance conceptual understanding, not the least because he had no idea of the concept of base for logarithms, as we know it today. We only need to note that Napier used a formula not unlike this, effectively using a base of $1/e$ for his calculations:[442]

$$\frac{1}{e} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$$

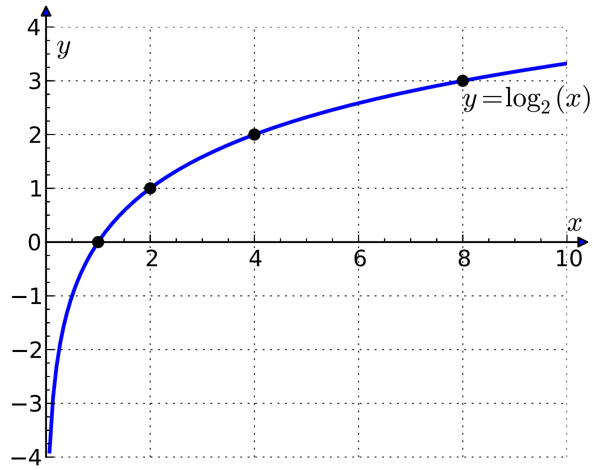
Napier's concept of logarithms quickly caught on, not the least with Kepler, who had performed thousands of tedious calculations without the use of logarithms in discovering the first two laws of planetary motion, published in *New Astronomy* in 1609, ignored by Galileo but not Newton. In 1620, Kepler wrote a laudatory oration to the Baron of Merchiston, not knowing that he had died,[443] and then, in his own industrious manner, seeking to go to the heart of the matter, he set out in the winter of 1621–22 to write his own book on logarithms, published in 1623. Kepler was thus able to complete the task for which he had been appointed as *Imperial Mathematicus* to the Holy Roman Emperor in Prague in 1601, following the death of Tycho Brahe: to publish Tycho's one thousand measurements of the stars and those of planetary motion. Following a dispute with Tycho's relatives, these were eventually published in 1627 as *Tabulæ Rudolphiæ* 'Rudolphine Tables', in honour of Rudolf II.[444]

In the meantime, in England, Henry Briggs (1561–1631) embarked on the tedious task of preparing the first set of common, or Briggsian, logarithms using 10 as a base, published in 1624 as *Arithmetical Logarithmica*, which the Dutchman Adrian Vlacq (1600–1666) expanded in a second edition in 1628, calculated to 14 and 10 decimal places. Also, in 1622, William Oughtred (1574–1660) invented the slide rule, as a mechanical device based on the additive power of logarithms.[445] Thus were established the basic tools I needed as a mathematician at school and university in the 1950s and 60s, not now much used following the invention of the pocket calculator, personal computer, and smart phones .

However, while logarithms were of great practical use, it was to take some time before they were fully understood as a function, rather than as a means of simplifying multiplication. Today, it is obvious to us that logarithms are the dual of exponents. For, if $x = b^t$, then we know that $t = \log_b x$. That is, t is log to the base b of x. But it wasn't until the 1680s that this relationship began to appear in the consciousness of mathematicians and hence in their writings. And it was not until 1748, when Euler published *Introductio in analysin infinitorum* (*Introduction to the Analysis of the Infinite*), laying down the foundations of modern mathematical analysis, that a full understanding was developed.[446]

There is no need to go into this long learning process, one of many examples illustrating the way evolution progresses in the noosphere. All that is relevant for this book is to highlight a couple of points that could help us become more aware of what is happening to humanity at the present time.,

First, the following diagram plots the logarithmic function for base 2, corresponding to Stifel's table on page 287, denoted by the bullets on the curve. For instance, we can see that log 1 is 0, which is true of all logarithms, no matter what their base. Also, $\log_2 2 = 1$, which is an example of the general principle



$\log_b b = 1$. The graph then shows how the next few terms on the x-axis, which are in an arithmetic series, become a logarithmic scale on the y-axis.

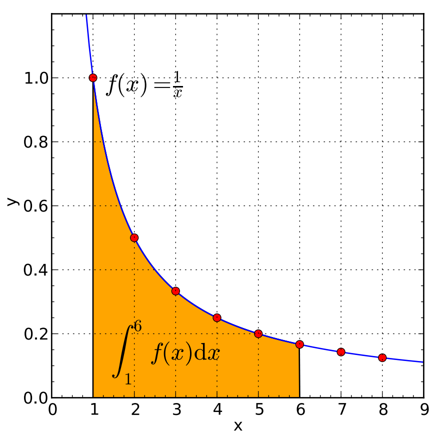
This is particularly useful when plotting geometric and exponential phenomena, for their graphs rapidly disappear off the page using arithmetic scales. On the other hand, when using a semilogarithmic chart, in which exponents follow a geometric progression, exponential growth can be depicted as a straight line, nevertheless still stretching out to infinity. Or, in the case of a diminishing geometric series, when $r < 1$ in the expression for S_n on page 279, the straight line crosses the x-axis, when $y = 0$, at a finite, limiting point. As evolution as a whole can be represented as a diminishing geometric series, such a plot is especially useful in seeing humanity's place in the overall scheme of things.

We now need to ask what is so special about logarithms to the base e, which is approximately 2.71828. Surely it would be much simpler to use an integer as a base. For instance, if we express the x-axis in the plot of $\log_2 x$ in binary, then 2, 4, and 8 become 10, 100, 1000, as they are in $\log_{10} x$, or indeed any other number system such as base 3, 7, or 16, with hexadecimal digits used in computers.

The key publication in this regard was *Logarithmotechnia* by Nicholas Mercator (c. 1619–1687), who called logarithms to the base e 'natural logarithms' in 1668.[447] Mercator, not to be confused with Gerardus Mercator (1512–1594) of map projection fame, did so from the equation $xy = 1$ for the rectangular hyperbola, which led him to find this Taylor series, although Gregorius Saint-Vincent (1584–1667) had already found a similar expression in 1647 in a monumental, but flawed work titled *Opus geometricum quadraturae circuli sectionum coni* (*Geometric work on the quadrature of the circle of conic sections*):[448]

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

As we can see, by setting $x = 1$, we have the value of the alternating harmonic series, which Mengoli discovered in 1647, as described on page 282. Today, the general series is known as the Mercator Series,[449] but sometimes as the Mercator-Newton Series,[450] because Newton had found it in October 1666, when hiding from the plague in London. Newton was upset in 1669 because someone else had published this result before him. So he wrote a short tract titled *De Analyst per Æquationes Numero Terminorum Infinitas* (*On the Analysis by Equations Unlimited in the Number of Their Terms*), which probably led Isaac Barrow (1630–1677) to recommend Newton as the Lucasian professor of mathematics at Cambridge University.[451]



One way of seeing why e is the base of the natural logarithm is with the fundamental theorem of the calculus, in which Leibniz and Newton showed that the slope of a curve at a particular point, obtained through differentiation, is the inverse of integration, as the area under a curve, as we see in Chapter 3. For instance, the area shaded orange in this Wikipedia diagram of the hyperbolic function is log 6 to base e, generally denoted as ln, as we have seen. In general,

$$\ln a = \int_1^a \frac{1}{x}\,dx$$

So, if $a = e$, the area under the curve is 1, a geometric representation of e, as Mathologer points out on his YouTube channel.[452] A somewhat more complex way to illustrate the naturalness of e as the base of logarithms arises from reversing the integral. Of course, if we differentiate $\ln x$, we get back to $1/x$. But what is the differential of $\log_b x$? Well, to convert logs from one base to another, we use this formula:

$$\log_b x = \frac{\log_k x}{\log_k b}$$

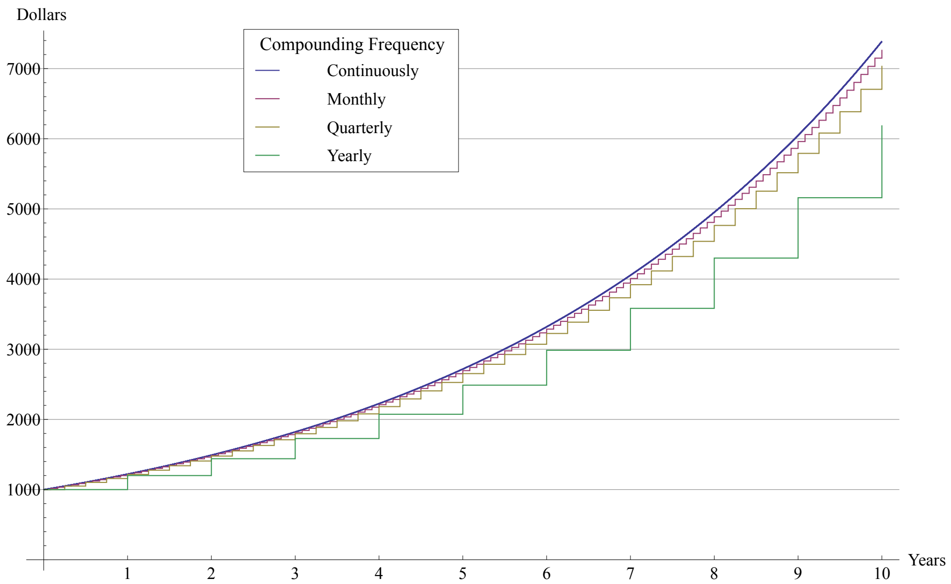
So, differentiating this function, using Leibniz's notation for the differential:

$$\frac{d}{dx}\log_b x = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b}\cdot\frac{d}{dx}(\ln x) = \frac{1}{x\ln b}$$

Now when $x = 1$, $\log_b x = 0$, and the slope of the logarithm function as it crosses the x-axis is $1/\ln(b)$. This number is greater or less than one depending on whether b is less or greater than e. For instance, ln 2 and ln 10 are 0.693147 and 2.302585, respectively. But when $b = e$, the slope at this critical point is 1. Wikipedia suggests that this key characteristic of e is what makes logarithms to this fundamental mathematical constant natural.[453] This explanation is rather elusive, not nearly as obvious as the concept of π.

Despite mathematicians using natural logarithms to base e for much of the second half of the seventeenth century, it was to take until the midddle of the next century before Euler eventually showed how to calculate its value. In the meantime, in 1683, Jakob Bernoulli found the integer bounds of e when studying compound interest,[454] having little understanding of the broader and deeper aspects of what he was doing, characteristic of so much human learning since the dawn of history.

The following chart, from Wikipedia,[455] illustrates the central issue, where the compound interest is 20%. In general, if we begin with a principal of P, which is compounded at an annual rate of interest r, then at the end of t years it is $P\cdot(1+r)^t$, which is a geometric progression from year to year.



However, what happens if the interest is paid more frequently? In general, if interest is paid n times per year at the rate r/n, then the principal P_t at t years is:

$$P_t = P\cdot\left(1+\frac{r}{n}\right)^{nt}$$

But then Bernoulli wondered what is the value of:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

where the rate of interest is effectively 100% and interest is paid continuously? Well, he used the binomial theorem to calculate that the range of possible values is between 2 and 3:[456]

$$\left(1+\frac{1}{n}\right)^n < 1+\frac{1}{1}+\frac{1}{1\cdot 2}+\cdots+\frac{1}{1\cdot 2\ldots n} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^{n-1}} < 3$$

In the event, it was not until 1748 that Euler was able to develop a formula for this expression that enables it to be calculated to any level of accuracy whatsoever. He did so in *Introductio in analysin infinitorum* (*Introduction to Analysis of the Infinite*), regarded by many as the greatest mathematics book ever written, although Euler had difficulty in finding a publisher for it, having spent most of the 1740s writing it.[457]

One reason why this book has been so influential is that it starts from first principles, defining a few basic concepts before building structures of ever-increasing complexity in a natural evolutionary manner. For instance, Chapter 1 'On Functions in General' begins with definitions of constant and variable quantities before giving this definition of function:

A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.[458]

Although this definition seems to equate *function* with *formula*, "nonetheless this analytic definition was a vast improvement over the ill-considered geometric notion of 'curve'," introducing a profound change in the mathematical landscape. Rather, *function* is defined more today as "to each x there corresponds a unique y in the range," a formulation that Euler was later to approach.[459]

Constants, variables, and functions today play a similar role in software development to that of mathematics. For instance, as a function $f(x)$ is a variable quantity in modern notation, it can be both the input to a function, as $g(f(x))$, and the output. But does this mean that computer functions could create new functions that have never existed before without human intervention?

As this book is at pains to point out, the answer is a resounding NO! Functions are essentially mechanistic, operating in the horizontal dimension of time. So, if we are to become free of our mechanistic behaviour and realize our fullest potential as superintelligent humans, then we need to admit the creative power of Life into science, emerging directly from the Absolute, as the Divine Datum of the Universe, that which is given.

Even though Euler did not admit Life into his reasoning, as doing so is countercultural, he nevertheless made enormous progress in providing mathematicians with the tools they need to study growth and rates of change. He did so by showing that logarithms are functions, complementary duals of exponential functions, in conformity with the Principle of Unity, the fundamental law of the Universe. However, these tools are mainly applied in physics and engineering, for instance, not generally applied in evolutionary and psychological studies, which are necessary if we are to understand what it truly means to be a human being.

Starting as usual with simple, basic concepts, Chapter VI 'On Exponentials and Logarithms' begins with the basic exponential function a^z, where a is a constant and the exponent z is a variable, standing for 'all determined numbers'.[460] When z is a natural number, a^z is a member of a geometric progression or sequence, regarding exponentiation as repeated multiplication. However, when z is a real or complex number, such notation breaks down. To clarify this situation, mathematicians sometimes denote e^z as exp(z),[461] although Euler did not do so. We could therefore denote the general exponential function as aexp(z).

So, as logarithms are functions, while a^z gives y, for instance, $z = \log(y)$,[462] not unlike the way that differentiation and integration are duals of each other. So, just as logarithms convert multiplication into addition, like this:

$$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$$

exponentials convert addition into multiplication, like this:[463]

$$a^{x+y} = a^x \cdot a^y$$

Then, in Chapter VII 'Exponentials and Logarithms Expressed through Series', Euler set out to do just that. He began by defining $a^\omega = 1 + \varphi$, where ω and φ are infinitesimal quantities, a device that Euler often used before the formal development of limits in the next century, as we see in Chapter 3. He then let $\varphi = k\omega$, giving $a^\omega = 1 + k\omega$. Thus, $\omega = \log(1 + k\omega)$, where a is the base of the logarithm and k is a finite number that depends on the value of the base a in some, as yet, unknown way.

To determine an infinite series for a^ω, Euler then introduced the variable j, as a finite exponential power, to give $a^{j\omega} = (1 + k\omega)^j$. Then, setting $j = z/\omega$, we have $\omega = z/j$ and so:[464]

$$a^z = \left(1 + \frac{kz}{j}\right)^j$$

This is an expression not unlike that which Bernoulli had explored over half a century earlier. Then Euler expanded the series using Newton's generalized binomial series,[465] gathering all the coefficients in j to $k^n z^n/n!$ together, noting that they all tend to 1 as n tends to infinity. By thus eliminating j from the expansion, Euler arrived at this Maclaurin series, changing z to the more usual x:

$$a^x = 1 + kx + \frac{k^2}{2!}x^2 + \frac{k^3}{3!}x^3 + \frac{k^4}{4!}x^4 + \cdots$$

Now, choosing a as the particular base for which $k = 1$ and also setting $x = 1$, we can see that

$$a = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

which Euler calculated as $a = 2.71828182845904523536028\ldots$ (OEIS A001113). As he said, "when this base is chosen, the logarithms are called natural or hyperbolic. The latter name is used since the quadrature of a hyperbola can be expressed through these logarithms. For the sake of brevity for this number 2.718281828459… we will use the symbol *e*."[466] So, after nearly a century and half of struggle, Euler had found the value of the base of the natural logarithm, which is today known as 'Euler's number', although *e* stands for neither *exponential* nor his name.

Setting $a = e$ and $k = 1$ in the infinite series for a^x, we have this general expression for the exponential function, itself:

$$\exp(x) = e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = \sum_{n=0}^{\infty}\frac{x^k}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

We can see immediately from this infinite series why the exponential function archetypically measures growth and rates of change, determined by differentiating the function. For, by doing so, we obtain:

$$f'(x) = 0 + \frac{1}{1!} + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The differential of the exponential function is thus the exponential function itself. Furthermore, acceleration is also denoted by the exponential function, for $f''(x) = e^x$, as are all successive derivatives. The rate at which acceleration changes and accelerates is also exponential, and so on *ad infinitum*! In other words, the exponential function, as an expression of accumulative processes, such as evolution, never slows down if there are no constraints for it do so, such as population growth, as I explain in my book *Through Evolution's Accumulation Point*.

Now, as Taylor and Maclaurin showed that infinitely differentiable functions can be expressed as infinite power series, there is a fascinating relationship between exponential and trigonometric and hyperbolic functions. To prove this, we need to introduce the strange notion of an imaginary exponent, which is a far remove from exponentiation as repeated multiplication. To do so, we first set $x = iz$ in the expansion of the exponential function, giving:

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{3!} + \frac{(iz)^5}{3!} + \cdots = 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{3!} + \frac{iz^5}{3!} +$$

for $i^{4n} = 1$, $i^{(4n+1)} = i$, $i^{(4n+2)} = -1$, and $i^{(4n+3)} = -i$, for $n \geq 0$. Gathering the real and imaginary parts together, we thus have:

$$e^{iz} = 1 - \frac{z^2}{2!} + \frac{z^4}{3!} - \cdots + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

Now, the real and imaginary parts of this expansion are those for the cosine and sine functions, respectively, a discovery usually attributed to Newton.[467] However, modern research indicates that Madhava of Sangamagrama in Kerala, India, was the first to find Maclaurin series for these functions in 1400, over two and a half centuries before Newton did so in Europe around 1670.[468] Representing z as an angle θ, gives:

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

and

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

We can thus see that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

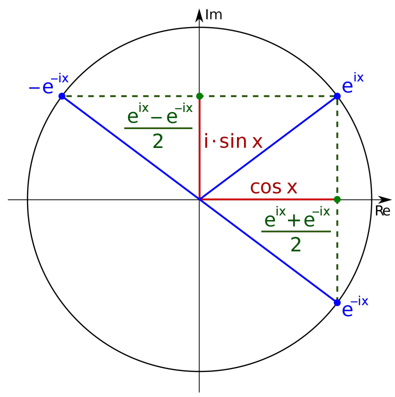
known today as Euler's formula, which shows the relationship between analysis and trigonometry,[469] although this is not how Euler proved it in Chapter VIII 'On Transcendental Quantities Which Arise from the Circle' of *Introductio*, as the Eulerian scholar Ed Sandifer explains in his Web column *How Euler Did It*.[470] Rather, using his familiar technique of infinitesimals and infinity, Euler first found these two expressions for cos and sin and derived his formula from them:[471]

$$\cos\theta = \Re\left(e^{i\theta}\right) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin\theta = \Im\left(e^{i\theta}\right) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

This diagram from Wikipedia shows these relationships using the unit circle in the complex plane. Here, the polar coordinates of a point on this circle are expressed as $e^{i\theta}$, where θ is the angle between the real axis and the vector ending on the unit circle.



Now setting θ to π or π/2 in Euler's formula gives the most amazing formula in mathematics, known as Euler's identity:

$$e^{i\pi} = -1$$

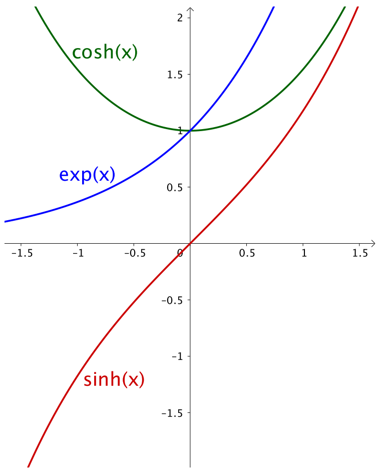
This is sometimes rearranged as:

$$e^{i\pi} + 1 = 0$$

giving a relationship between five fundamental constants in mathematics in the simplest possible terms. We can see from the diagram that $e^{i\pi}$ is the point -1 on the unit circle at its leftmost point, 180° from the base real axis.

However, Euler never explicitly set θ to π in his identity, although he had twice come close to his famous formula earlier in his studies, as had Roger Cotes (1662–1716), Johann Bernoulli, and DeMoivre. Indeed, it is not clear from the literature who first explicitly expressed this formula in writing, Sandifer tells us.[472]

If we now set θ to ix in the formulae for cos and sin, we find a relationship between the exponential and hyperbolic functions:



$$\cos(ix) = \frac{e^{-x} + e^{x}}{2} = \cosh(x)$$

$$\sin(ix) = \frac{e^{-x} - e^{x}}{2i} = i\left(\frac{e^{x} - e^{-x}}{2}\right) = \sinh(x)$$

In other words, all the alternating signs in the trigonometric expansions become positive in the hyperbolic ones and we have:

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} = x + \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

giving, as this diagram shows:

$$e^{x} = \cosh(x) + \sinh(x)$$

Madhava did not find infinite series expansions for just sine and cosine around 1400. He also found the power series for the inverse of tangent,[473] officially written 'arctan',[474] to avoid the ambiguity of 'tan^{-1}', still widely used today, even though it is deprecated. He did so over two and half centuries before Gregory found this formula in 1668 or 1671, known today as 'Gregory's series':[475]

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Differentiating gives:

$$1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1 + x^2}$$

So, integrating this series gives us back arctan, a formula well familiar to calculus students:

$$\int_0^x \frac{1}{1 + t^2} dt = \arctan(x)$$

Having found a series expansion for arctan, by setting it to suitable angles in radians, Madhava was able to calculate the value of π to several decimal places. In particular, as arctan(1) = π/4, he found:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

Leibniz also found this formula in 1674, "one of the most beautiful mathematical discoveries of the seventeenth century".[476] Hence it was long known as 'Leibniz's formula for π', not acknowledging Madhava and Gregory's precedence.

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \ddots}}}}}}$$

William Brouncker (c. 1620–1684), the first president of the Royal Society, found this amazing continued fraction for π/4, the reciprocal of the fraction that John Wallis (1616–1703) gave in

Arithmetica Infinitorum in 1656.[477]

In this same work, Wallis, who introduced ∞ as a symbol for infinity in 1655, probably from the Roman numeral for 100 million,[478] presented this infinite product for π/2, known today as the 'Wallis product':[479]

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \dots$$

This was not the first infinite product for π/2. In 1593, François Viète (1540–1603) had found another product, most suitably presented as the reciprocal:[480]

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \dots$$

This product can most simply be derived from this trigonometric identity, which presumably predates Euler's product for sinθ/θ, which we look at in the next subsection:

$$\frac{\sin\theta}{\theta} = \prod_{n=1}^{\infty} \cos\frac{\theta}{2^n} = \cos\frac{\theta}{2} \cdot \cos\frac{\theta}{4} \cdot \cos\frac{\theta}{8} \cdot \dots$$

Viète's formula can be derived from this general product by setting θ =π/2, with repeated application of the half-angle formula

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$$

from which we get, as a first iteration, $\cos{}^{\pi}/_{4} = {}^{1}/_{\sqrt{2}} = {}^{\sqrt{2}}/_{2}$.

As far as I am aware, Viète's formula is the first to represent π as an infinite product. However, there are many others, related to infinite series, which we look at in the next subection.

Riemann zeta function and series

We now come to the infinite series that mathematicians during the seventeenth century found to be the most intractable: what is the sum of the reciprocal of a sequence of powers, such as:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots + \frac{1}{k^2} + \cdots$$

Mengoli was the first to ponder this problem, in 1644[481] or 1650 in *Novæ quadraturæ arithmetic*, but was unable to solve it.[482] In 1655, Wallis then commented on the problem, computing its value to three decimal places.[483] He was followed by Jakob Bernoulli, who wrote in 1689 in *Tractatus de Seriebus Infinitis* that this problem "is more difficult than one would expect … If anyone finds and communicates to us that which up to now has eluded our efforts, great will be our gratitude." (*difficilior est, quàm quis expectaverit … Si quis inventai nobisque communicet, quod industrial nostrum elusit hactenus, magnas de nobis gratias ferret.*)[484] As the Bernoullis lived in Basel, also Euler's birthplace, it came to be known as the 'Basel problem' in the eighteenth century.

Before looking at how Euler solved the Basel problem over half a century later and what this solution led to, we need to note that this power series is just a special case of this series, where *m* can be any positive integer,

$$\sum_{k=1}^{\infty} \frac{1}{k^m} = 1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \frac{1}{5^m} + \cdots + \frac{1}{k^m} + \cdots$$

So, when $m = 1$, we have the divergent harmonic series, as a special case. This series is further generalized when *m* is real, where it is sometimes called a *p*-series, with *m* becoming *p*, which converges if

$p > 1$. In turn, the p-series is a special case of the Riemann zeta function, $\zeta(s)$, where s can be a complex number, which converges when the real part of $s > 1$. However, Riemann also considered the mysterious case when $\Re(s) \leq 1$, which is valid through analytic continuation, which we look at later in this subsection.

Euler began his studies of the Basel problem in 1730, shortly after moving to St Petersburg to be with Daniel Bernoulli, at the suggestion of Johann, Daniel's father, who had been his tutor in Basel. Following his first attempt to interpolate integer sequences, which was to lead to the gamma function, as we see later, he made an estimate of the solution to the Basel problem to six decimal places of 1.644934.[485] Apparently he was not aware that James Stirling had computed $\zeta(2)$ to nine places, eight of which were correct, in 1730.[486]

Then in a paper that he presented to the St Petersburg Academy in 1735, but not published until 1740, he found three solutions to the Basel problem, even if they were of dubious mathematical validity at the time. He began with the quadrature of the circle, which led him to find "for six times the sum of this series to be equal to the square of the perimeter of a circle whose diameter is 1", estimating the sum as 1.644934066842264364.

People were especially sceptical about Euler's third solution—the simplest and most elegant—because he believed that he could apply the rules for finite polynomials to the infinite series for sine, well familiar to mathematicians, at the time. For instance, if we know that the roots or zeros of a cubic are -1, 0, and 1, we have this relationship, representing a series as a product:

$$x^3 - x = x(x^2 - 1) = x(x + 1)(x - 1) = (x - 1)x(x + 1)$$

Similarly, Euler knew that the zeroes of the sine function are $n\pi$, where $n \in \mathbb{Z}$, which enabled him to form this relationship between an infinite series and an infinite product:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)\left(1 - \frac{x^2}{(3\pi)^2}\right)\left(1 - \frac{x^2}{(4\pi)^2}\right)\ldots$$

In the event, it was not until a century later that Weierstrass proved with his Weierstrass Factorization Theorem,[487] also called Product Theorem,[488] that Euler's factorization is valid. So, we have this infinite product, to add to the infinite series, listed in classic books of formulae along with the product that Viète used to find a value for $\pi/2$,[489] as we see on page 295:

$$\frac{\sin x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{(k\pi)^2}\right)$$

This product formula has profound implications, quite apart from its use in solving the Basel problem. For instance, inspired by a one-page article that Paul Levrie wrote in 2012,[490] Mathologer showed in May 2020 in one of his brilliant animated YouTube videos that the three expressions for π on page 294 follow directly from the product formula for $\sin(x)$.[491] First, defining

$$\sin x = x\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\ldots$$

and setting $x = \pi/2$, gives

$$\sin\frac{\pi}{2} = \frac{\pi}{2}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\ldots$$

and hence,

$$\frac{2}{\pi} = \frac{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot}\ldots$$

which is the reciprocal of the Wallis product.

To turn a product into a sum is a little trickier, involving taking logs and removing them through differentiation with the chain rule, like this:

$$\ln(\sin x) = \ln x + \ln\left(1 - \frac{x}{\pi}\right) + \ln\left(1 + \frac{x}{\pi}\right) + \ln\left(1 - \frac{x}{2\pi}\right) + \ln\left(1 + \frac{x}{2\pi}\right) + \ln\left(1 - \frac{x}{3\pi}\right) + \ln\left(1 + \frac{x}{3\pi}\right) + \cdots$$

and

$$\frac{\cos x}{\sin x} = \frac{1}{x} - \frac{1}{\pi - x} + \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x} - \frac{1}{3\pi - x} + \frac{1}{3\pi + x} - \cdots$$

Now setting x = π/4, gives:

$$\frac{\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = 1 = \frac{1}{\frac{\pi}{4}} - \frac{1}{\frac{3\pi}{4}} + \frac{1}{\frac{5\pi}{4}} - \frac{1}{\frac{7\pi}{4}} + \frac{1}{\frac{9\pi}{4}} - \frac{1}{\frac{11\pi}{4}} + \frac{1}{\frac{13\pi}{4}} - \cdots$$

Multiplying each side by $x = \pi/4$ gives the Madhava-Gregory-Leibniz formula for π, although Euler attributed it to Leibniz only. In doing so, he felt confident that even though he was uncertain about the validity of his innovative method, it, nevertheless, gave the correct result for $\zeta(2)$.[492]

Levrie also showed how Brouncker's continued fraction for π could be derived from the Madhava-Gregory-Leibniz formula, which could not have been the way that Brouncker found his expression, for he did so before both Gregory and Leibniz had published their results. So, he must have done so from the Wallis product, as Wallis apparently claimed. Either way, deriving Brouncker's continued fraction involves some rather intricate manipulations, which Mathologer has brilliantly animated.

So, how did the product formula for sine enable Euler to find the value of $\zeta(2)$? Well, he saw that if he multiplied out the product, taking 1 from each term in the product and the second element from all the other terms, the resulting coefficient of x^2 would be equal to the coefficient of x^2 in the series. Hence:

$$-\frac{1}{3!} = -\frac{1}{\pi^2} - \frac{1}{(2\pi)^2} - \frac{1}{(3\pi)^2} - \frac{1}{(4\pi)^2} \cdots$$

From which we find this amazing result:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

With this discovery, Euler became world famous, among mathematicians in Europe, at least, but still not known to the general public, unlike Isaac Newton, similarly having a major influence in the evolution of mathematics and science. Indeed, by 1733 he had already became professor of mathematics at the St Petersburg Academy when Daniel Bernoulli left Russia to return to Basel.[493]

But Euler did not stop there. As a generalist, constantly seeking generative patterns within infinite series, Euler found the values for six terms in the sequence of the reciprocals of even powers, at least. He did so in 'On the sums of series of reciprocals', E41 in the Eneström Index, by defining two sets of variables, finding repetitive relationships between them. He defined *P*, *Q*, *R*, *S*, *T*, and *V* etc., which I'll denote with P_k, and α, β, γ, δ, ε, and ζ etc., which can be denoted with a_k. With these revised notations, we have, for $k \geq 1$:

$$P_k = \frac{1}{(\pi)^{2k}} + \frac{1}{(2\pi)^{2k}} + \frac{1}{(3\pi)^{2k}} + \cdots$$

and

$$a_k = \frac{1}{(2k+1)!}$$

With these definitions, Euler then defined P_k in terms of all preceding values of P_k from 1 to $k-1$, thus:

$P_1 = a_1 = \frac{1}{3!} = \frac{1}{6}$

$P_2 = P_1a_1 - 2a_2 = \frac{1}{6}\cdot\frac{1}{6} - \frac{2}{120} = \frac{1}{90}$

$P_3 = P_2a_1 - P_1a_2 + 3a_3 = \frac{1}{90}\cdot\frac{1}{6} - \frac{1}{6}\cdot\frac{1}{120} + \frac{3}{5040} = \frac{1}{945}$

$P_4 = P_3a_1 - P_2a_2 + P_1a_3 - 4a_4 = \frac{1}{945}\cdot\frac{1}{6} - \frac{1}{90}\cdot\frac{1}{5040} + \frac{1}{6}\cdot\frac{1}{5040} - \frac{4}{362880} = \frac{1}{9450}$

$P_5 = P_4a_1 - P_3a_2 + P_2a_3 - P_1a_4 + 5a_5 = \frac{1}{9450}\cdot\frac{1}{6} - \frac{1}{945}\cdot\frac{1}{120} + \frac{1}{90}\cdot\frac{1}{5040} - \frac{1}{6}\cdot\frac{1}{362880} + \frac{5}{39916800} = \frac{1}{93555}$

$P_6 = P_5a_1 - P_4a_2 + P_3a_3 - P_2a_4 + P_1a_5 - 6a_6 = \frac{1}{93555}\cdot\frac{1}{6} - \frac{1}{9450}\cdot\frac{1}{120} + \frac{1}{945}\cdot\frac{1}{5040} - \frac{1}{90}\cdot\frac{1}{362880} + \frac{1}{6}\cdot\frac{1}{39916800} - \frac{6}{6227020800} = \frac{691}{638512875}$

At this point, Euler not surprisingly said, "a fair deal of work [is needed] for the higher powers," not having available to him WolframAlpha or other similar tools. Nevertheless, there is a clear pattern here, whose underlying principles Euler came closer to finding in 1739 in a rather rambling document titled (E130) that extended this sequence to *k* = 13. This was presented more succinctly in 1748 in Chapter X of Volume 1 of his brilliant textbook *Introduction to the Analysis of the Infinite*, known simply as *Introductio*. As he said in this book, "We could continue with more of these, but we have gone far enough to see the sequence which at first seems quite irregular, $1, \frac{1}{3}, \frac{1}{3}, \frac{3}{5}, \frac{5}{3}, \frac{691}{105}, \frac{35}{1}$, ..., but it is of extraordinary usefulness in several places."

Nevertheless, in 1749, in the paper in which Euler began to study what we today call the eta function ($\eta(x)$)—as the alternating form of the zeta function—he tells us that he had calculated the sum of the reciprocals of the even powers up to 34.[494] However, as he said in this paper, "in the cases where *n* is an odd number, all my effort to find their sum is a failure up to now. Nevertheless, it is certain that they do not depend in a similar way on the powers of the number π."

Yet, even though Euler was finding patterns among the even powers, I have not found in any of his writings that I have looked at an explicit reference to this general expression, which he supposedly discovered, in terms of Bernoulli numbers:[495]

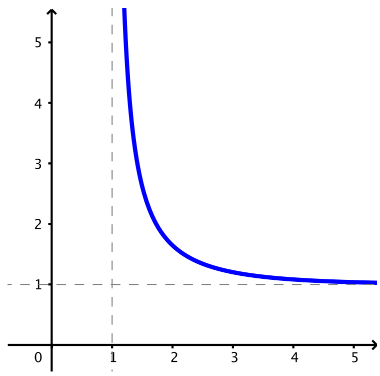
$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}$$

Perhaps we should not be surprised that the Bernoulli numbers pop up, for Jakob Bernoulli found these when studying the sums of the powers themselves, rather than their reciprocals, through a rather complex recurrence equation that we see on page 240. Here, then, is a table of the first thirteen even powers and the first ten decimal digits of their decimal expansions, the OEIS giving up recording them after the first ten.

Power	E41	Ch X, Introductio	Bernoulli nos.	Decimal	OEIS
2	$\frac{\pi^2}{6}$	$\frac{2^0}{3!}\cdot\frac{1}{1}\pi^2$	$\pi^2\cdot\frac{1}{6}$	1.6449340668…	A013661
4	$\frac{\pi^4}{90}$	$\frac{2^2}{5!}\cdot\frac{1}{3}\pi^4$	$\frac{\pi^4}{3}\cdot\frac{1}{30}$	1.0823232337…	A013662
6	$\frac{\pi^6}{945}$	$\frac{2^4}{7!}\cdot\frac{1}{3}\pi^6$	$\frac{2\pi^6}{45}\cdot\frac{1}{42}$	1.0173430619…	A013664
8	$\frac{\pi^8}{9450}$	$\frac{2^6}{9!}\cdot\frac{3}{5}\pi^8$	$\frac{\pi^8}{315}\cdot\frac{1}{30}$	1.0040773561…	A013666
10	$\frac{\pi^{10}}{93555}$	$\frac{2^8}{11!}\cdot\frac{5}{3}\pi^{10}$	$\frac{2\pi^{10}}{14176}\cdot\frac{5}{66}$	1.0009945751…	A013668
12	$\frac{691\pi^{12}}{6825\cdot 93555} = \frac{691\pi^{12}}{638512875}$	$\frac{2^{10}}{13!}\cdot\frac{691}{105}\pi^{12}$	$\frac{2\pi^{12}}{467775}\cdot\frac{691}{2730}$	1.0002460865…	A013670
14	$\frac{2}{18243225}\pi^{14}$	$\frac{2^{12}}{15!}\cdot\frac{35}{1}\pi^{14}$	$\frac{4\pi^{14}}{42567525}\cdot\frac{7}{6}$	1.0000612481…	A013672
16	$\frac{3617\pi^{16}}{325641566250}$	$\frac{2^{14}}{17!}\cdot\frac{3617}{15}\pi^{16}$	$\frac{\pi^{16}}{638512875}\cdot\frac{3617}{510}$	1.0000152822…	A013674
18	$\frac{43867\,\pi^{18}}{38979295480125}$	$\frac{2^{16}}{19!}\cdot\frac{43867}{21}\pi^{18}$	$\frac{2\pi^{18}}{97692469875}\cdot\frac{43867}{798}$	1.0000038172…	A013676

Power	E41	Ch X, Introductio	Bernoulli nos.	Decimal	OEIS
20	$\frac{174611\,\pi^{20}}{1531329465290625}$	$\frac{2^{18}}{21!}\cdot\frac{122277}{55}\pi^{20}$	$\frac{2\pi^{20}}{9280784638125}\cdot\frac{174611}{330}$	1.0000009539…	A013678
22	$\frac{155366\,\pi^{22}}{13447856940643125}$	$\frac{2^{20}}{23!}\cdot\frac{854513}{3}\pi^{22}$	$\frac{4\pi^{22}}{2143861251406875}\cdot\frac{854513}{138}$	1.0000002384…	—
24	$\frac{236364091\,\pi^{24}}{201919571963756521875}$	$\frac{2^{22}}{25!}\cdot\frac{1181820455}{273}\pi^{24}$	$\frac{2\pi^{24}}{147926426347074375}\cdot\frac{236364091}{2730}$	1.0000000596…	—
26	$\frac{1315862\,\pi^{26}}{11094481976030578125}$	$\frac{2^{24}}{27!}\cdot\frac{76977927}{1}\pi^{26}$	$\frac{4\pi^{26}}{48076088562799171875}\cdot\frac{8553103}{6}$	1.0000000149…	—

As no one since Euler has found a closed-form expression for the sums of the reciprocals of the odd powers of integers, we should not be surprised that mathematicians have been unable to find a general formula for the *p*-series of the sums of the reciprocals of the powers of the reals, which is $\zeta(x)$, for $x > 1$. Nevertheless, we can plot this function, showing how rapidly the curve approaches its two asymptotes at $x = 1$ and $y = 1$. On page 304, we show how this chart is extended into the complex plane for $\zeta(z)$, with $\Re(z) > 1$, and, through analytic continuation, into both the Cartesian and complex planes for $\Re(z) \leq 1$, on pages 306 and 308.



In 1737, Euler found yet another amazing relationship involving the zeta function with natural numbers as exponents, published in *Variae observationes circa series infinitas* (Various observations about infinite series) in 1744. However, he presented this relationship much more lucidly in Section 283 of Chapter XV of the *Introductio*. Beginning with

$$\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \frac{1}{9^n} + \cdots$$

form

$$\frac{1}{2^n}\zeta(n) = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \cdots$$

and subtract the second series from the first, giving, as Mathologer delightfully explains with his animated formulae on YouTube:[496]

$$\left(1 - \frac{1}{2^n}\right)\zeta(n) = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \cdots$$

The result is a series in which no terms are divisible by 2. Now form

$$\frac{1}{3^n}\left(1 - \frac{1}{2^n}\right)\zeta(n) = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \cdots$$

and subtract again giving:

$$\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{2^n}\right)\zeta(n) = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \cdots$$

Now we have a series that is not divisible by 2 or 3, the first two primes. This process can be continued indefinitely, rather like the sieve of Eratosthenes used to separate the prime numbers from composite ones. If we continue Euler's process to its ultimate conclusion, all multiples of primes are eliminated from the right-hand side and we are left with only 1, giving:

$$\zeta(n)\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right)\ldots = 1$$

and so

$$\zeta(n) = \frac{1}{\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right)\ldots}$$

In other words, we have this relationship between the zeta function as both an infinite series and an infinite product:

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \prod_{k=1}^{\infty} \frac{1}{1 + \frac{1}{p_k^n}}$$

where p_k is the kth prime.

In *Prime Obsession* from 2003, John Derbyshire calls this amazing relationship the 'Golden Key',[497] which causes mathematicians to go all a flutter. For primes are the atoms of number theory, all integers being uniquely expressible as the product of prime numbers—the fundamental theorem of arithmetic, as we see in Chapter 3. It was this formula that led Riemann in 1859 to explore the possibility that it could tell us something about the distribution of the primes, which have no obvious repeating pattern, as we see in a moment.

What is especially amazing about this result is that there is a relationship between the prime numbers and π, the ratio of the circumference of a circle to its diameter, just one other example where π pops up in the most unexpected places. For instance, for $n = 2$, we have, as 3Blue1Brown brilliantly illustrates geometrically on one of his YouTube videos:[498]

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{2^2}{2^2+1} \times \frac{3^2}{3^2+1} \times \frac{5^2}{5^2+1} \times \frac{7^2}{7^2+1} \times \frac{11^2}{11^2+1} \times \ldots = \frac{\pi^2}{6}$$

Before looking at how Euler's product formula involving just the primes led to the Riemann zeta function, two other functions that Euler introduced are closely related to these developments. These are the Gamma (Γ) and eta (η) functions, which we'll look at briefly before exploring their extensions into the complex domain.

In 1959, Philip J. Davis wrote an informative essay on how the Gamma function emerged in the history of mathematical development, viewing mathematics as a growth process, rather than a static one,[499] as this book is endeavouring to demonstrate, but starting from the Divine Origin of the Universe, rather than with some axioms or assumptions, which deny the paradoxical truth of the fundamental law of the Universe.

At the time that Euler wrote his paper on the Gamma function, mathematicians were exploring how to use interpolation to extend formulae that are valid for natural numbers into fractional values and hence into real numbers. For instance, as n^2 is the formula for squares, we can calculate $3½^2$ as 12¼. Similarly, we know that the nth triangular number is $n(n + 1)/2$, so we know that the sum of the first 3½ 'natural' numbers is 7⅞, lying between 6 and 10.

But what is 3½! lying between 6 and 24? These are the third and fourth of what Euler called the hypergeometric series, for the successive terms increase faster than geometric series, where there is a constant ratio between terms. This is what Euler endeavoured to find in 1729, much inspired by correspondence with Christian Goldbach (1690–1764),[500] soon after moving to St Petersburg. Here is the general formula that Euler discovered:[501]

$$x! = \int_0^1 (-\ln t)^x \, dt$$

which is valid for noninteger values. So, not only does the function give us 3! = 6 and 4! = 24, we have:

$$3½! = \frac{105\sqrt{\pi}}{16} \approx 11.6317$$

Once again, we see π appearing, this time as a square root rather than as a square, this value being derivable from ½! = $\sqrt{\pi}/2$. For

$$3½! = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

However, Euler's formula is not what Adrien-Marie Legendre (1752–1833) called the Gamma function in 1809, with a capital Γ:[502]

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

Rather, if we set u = -ln(t) in Euler's formula, we obtain what Gauss called the Pi function:

$$\Pi(x) = \int_0^\infty u^x e^{-u}\,du$$

This expression has the advantage that

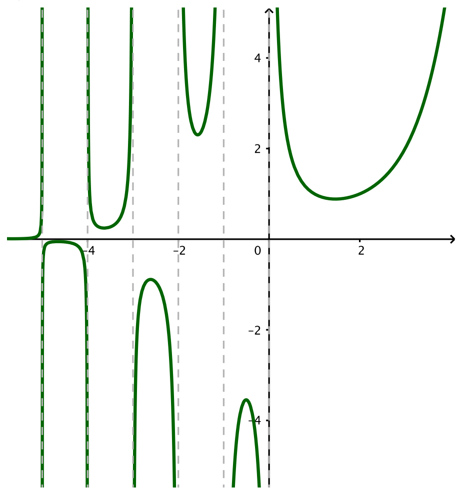
$$\Pi(n) = n!$$

as this is more natural, as H. M. Edwards points out in *Riemann's Zeta Function*,[503] whereas

$$\Gamma(n) = (n-1)!$$

So, we have:

$$\Gamma(x+1) = \Pi(x) = x\Gamma(x) = (x-1)\Pi(x-1)$$



which has no zeroes, as this diagram illustrates, but poles at zero and negative integers. This means that the reciprocal of the Gamma function is an entire one over its whole domain, somewhat easier to deal with in analysis.

However, the symmetry of the Pi function in the factorial doesn't necessarily hold in the many other uses of the Gamma function, which mathematicians prefer to use today.

There is a link between the gamma function and γ, the Euler-Mascheroni constant, which Weierstrass discovered using what is called the digamma function, known as Psi (Ψ), where:[504]

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \frac{1}{x} - \gamma + \sum_{r=1}^{\infty}\left(\frac{1}{r} - \frac{1}{r+x}\right)$$

So, setting x = 1, gives this elegant result:

$$\Gamma'(1) = -\gamma$$

This identity, which Euler presented in 1755 in *Foundations of Differential Calculus*, Sandifer tells us,[505] gives us the geometrically pleasing result that -γ is the gradient of the gamma function at x = 1.[506]

To see the link between the Gamma and zeta functions, we first need to form what is called Euler's reflection formula, which he discovered, as he tells us in E352, although I don't know in which previous document. A reflection formula is one that relates $f(x)$ to $f(a - x)$ for some constant a. Setting a = 1 in the reflection formula for the Gamma function gives the Complement Formula, also often called 'Euler's functional equation':[507]

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

From which we can derive this beautiful formula, which Riemann showed has far-reaching consequences, although in his path-breaking paper he used Π rather than γ, in an unsymmetrical way:[508]

$$\zeta(x)\Gamma(x) = \int_0^\infty \frac{u^{x-1}}{e^u - 1}\,du$$

In 1749, the year after *Introductio* was published, Euler wrote a far-reaching paper in French—the preferred language of Frederick the Great, his boss at the Prussian Academy of Science in Berlin—which was not published until 1768 and whose significance was not recognized until 1894.[509] The paper was titled *Remarques sur un beau rapport entre les series des puissances tant directes que reciproques* (Remarks on a beautiful relation between direct as well as reciprocal power series), E352 in the Eneström Index.[510] It was translated in 2006 by Lucas Willis, an undergraduate student in mathematics and French, and Tom Osler, his mathematics professor, who, they say, 'struggled to understand this brilliant work'. Three years later, Osler then further explained how this paper prefigured Riemann's functional equation for the zeta function.[511]

Euler began his paper, not with the power series, as such, but with two alternating power series, calling them 'direct' and 'reciprocal', respectively:

$$1^m - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + \cdots$$
$$\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \frac{1}{7^n} - \frac{1}{8^n} + \cdots$$

Taken together, these infinite series with integer exponents are a special case of the Dirichlet eta function, named after Peter Gustav Lejeune Dirichlet (1805–1859), although this eta function covering the complex domain is not mentioned in biographies I have browsed:

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \cdots$$

where s is any complex number. Reverting to positive real exponents, we can find the relationship of the eta function to the zeta one, as Mathologer explains,[512] by:

$$\zeta(x) - \eta(x) = \frac{2}{2^x} + \frac{2}{4^x} + \frac{2}{6^x} + \frac{2}{8^x} + \cdots = \frac{2}{2^x}\left(1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \frac{1}{5^x}\right) = \frac{2}{2^x} \cdot \zeta(x)$$

From which we find:

$$\eta(x) = (1 - 2^{1-x})\zeta(x)$$

This means that we can find $\eta(2n)$ corresponding to the first even powers of $\zeta(2n)$ that Euler found in 1736, after $\eta(1) = \ln(2) = 0.693147$, the alternating harmonic series:

Power	$\eta(2n)$	Decimal	OEIS
2	$\frac{1}{2} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{12}$	0.8224670334…	A072691
4	$\frac{7}{8} \cdot \frac{\pi^4}{90} = \frac{7\pi^4}{720}$	0.9470328294…	A267315
6	$\frac{31}{32} \cdot \frac{\pi^6}{945} = \frac{31\pi^6}{30240}$	0.9855510912…	A275703
8	$\frac{127}{128} \cdot \frac{\pi^8}{9450} = \frac{127\pi^8}{1209600}$	0.9962330018…	—
10	$\frac{511}{512} \cdot \frac{\pi^{10}}{93555} = \frac{73\pi^{10}}{6842880}$	0.9990395075…	—
12	$\frac{2047}{2048} \cdot \frac{691\pi^{12}}{638512875} = \frac{1414477\pi^{12}}{1307674368000}$	0.9997576851…	—

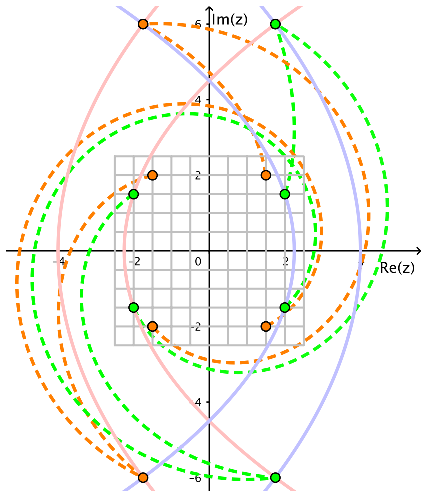
As you can see, the plot of the eta function for $x > 0$ does not make a very interesting chart, converging rapidly to 1 from below from $\eta(0) = ½$, the Cesaro sum of Grandi's series. However, the eta function could play a more important role in the complex domain, the zeta function being defined in terms of the eta one:

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}$$

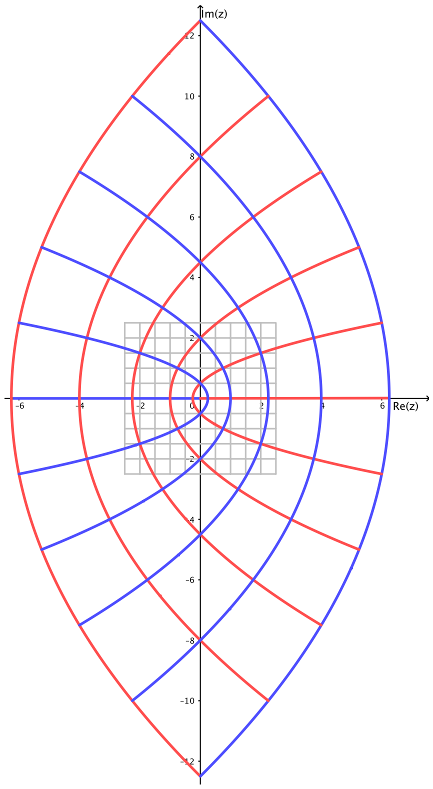
The important point about this relationship when it comes to extending the zeta function is that the eta function is convergent for $\Re(s) > 0$, whereas the zeta function is only convergent for $\Re(s) > 1$, with a discontinuity at $\Re(s) = 1$, enabling it to directly cover the entire complex plane. The continuity of the eta function is illustrated on the Cartesian plane on page 306.

Before we look at what the Riemann zeta function looks like on the complex plane, I find it useful to look at the basic principles of complex functions compared with those restricted to the real domain. With real functions, the independent variable x varies on the number line from $-\infty$ to $+\infty$ to give $y = f(x)$. So, if $y = x^2$, the function depicts the familiar parabola, whose vertex is at point (0, 0).

However, with complex functions, the real number line becomes the entire complex plane as input to functions. The x- and y-axes in the Cartesian x-y plane thus denote the real and imaginary components of complex numbers, both ranging from $-\infty$ to $+\infty$. The output from a complex function is then represented in another complex plane. So, complex functions map an ordered pair, as a 2-tuple, to another pair.



For instance, in the case of the complex square function, $a + bi$ becomes $(a^2 - b^2) + 2abi$. This is easier to understand in polar notation, where $re^{i\theta}$ becomes $r^2e^{2i\theta}$, where $r = \sqrt{a^2 + b^2}$ and $\theta = \arctan(b/a)$. In other words, in this simple complex function, the argument is doubled and the modulus is squared.



The right-hand diagram shows the way that the complex square function transforms individual points $\pm 2 \pm 1.5i$ and $\pm 1.5 \pm 2i$, coloured green and orange, respectively, presenting the results on the same complex plane. 3Blue1Brown represents this transformation dynamically on his YouTube channel,[513] graphically depicting how far points in the third and fourth quadrants must 'travel' to reach their destinations.

However, to get a more complete sense of such transformations, the left-hand diagram shows how a grid—where the real and imaginary dimensions range from -2.5 to +2.5 at 0.5 intervals—maps to nested sets of parabolas. The blue and red ones represent the vertical and horizontal grid lines, where the real and imaginary components are constant, respectively.

Notice that the parabolas intersect at right angles, like the lines in the generating grid. This is a characteristic of the transformation of complex analytic functions, which are angle-preserving, in what is called conformal mapping, which Zeev Nehari defined in his classic book on the subject in 1952. "We say that a mapping is *conformal* [for a regular analytic function] if it preserves the angle between two differentiable arcs."[514]

Denoting the entire complex plane with $z = x + iy$, in the mapping from the vertical lines, where the real component is a constant R, the equation for the blue parabolas is:

$$f(z) = z^2 = (R^2 - y^2) + 2yRi$$

Similarly, when the imaginary component is a constant I, the horizontal lines are transformed into the red parabolas with this equation.

$$f(z) = z^2 = (x^2 - I^2) + 2xIi$$

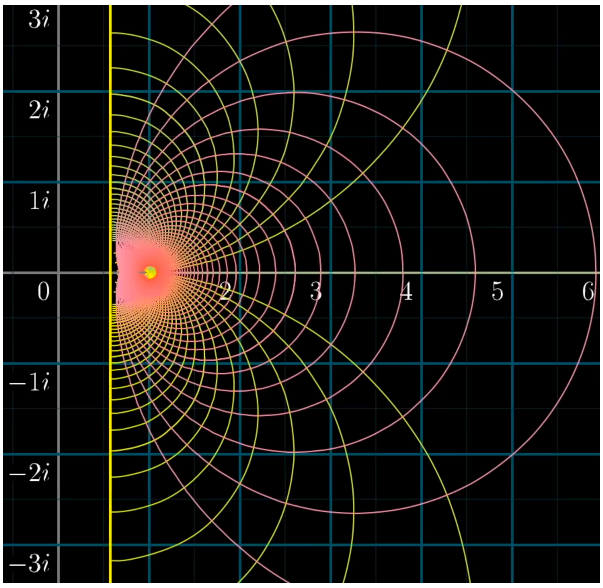
Thus, the vertices are R^2 and $-I^2$, where $y = 0$ and $x = 0$, respectively, and the curves cross the imaginary axis at the points $\pm 2C^2$, where C is R or I, where $y = R$ and $x = I$, respectively. The complex square function transforms the real and imaginary axes into the positive and negative real axes, respectively, where $C = 0$, as expected. So, the familiar parabola in the Cartesian plane of real values is depicted as a straight line from 0 to ∞ in the complex plane.

Treating the complex equations as parametric ones and rotating by $\pm\pi/2$ gives this equation for the spectrum of parabolas in the more familiar Cartesian plane.

$$y = \frac{x^2}{4C^2} - C^2$$

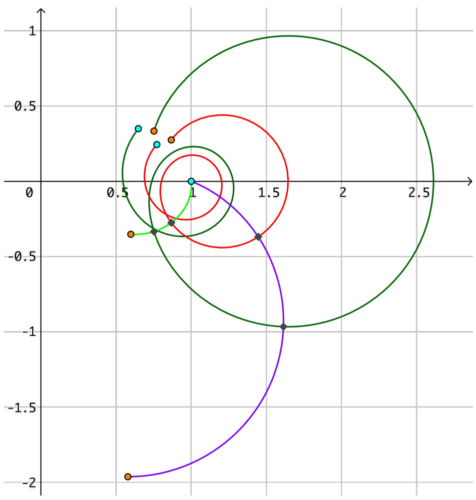
This basic introduction to conformal mapping illustrates how much more complicated complex analysis is than real analysis, requiring more advanced mathematical skills to reveal the beautiful hidden patterns. This is especially the case with the Riemann function in the complex domain, to which we now need to turn our attention.

We now come to the Riemann zeta function itself, which I don't really understand because of my rather limited mathematical skills and experience. Bernhard Riemann wrote his short paper in 1859 with the title '*Über die Anzahl der Primzahlen unter einer gegebenen Größe*', which David R. Wilkins has translated as 'On the Number of Primes Less Than a Given Magnitude'. Having had Dirichlet as one of his teachers,[515] Riemann was inspired to take Euler's product formula into the complex domain. For one of the major problems that mathematicians have long wrestled with is to find patterns in the distribution of the primes, the basic building blocks of the number system.



But before briefly looking at this critical issue, here is a picture of the extension of the chart of the p-series on page 299 into the complex domain, which is a screen capture from one of 3Blue1Brown's brilliant YouTube videos. In a similar manner in which I have shown how rectangular grid lines can be transformed using the square function in the complex plane, this beautiful diagram illustrates how a rectangular grid is similarly transformed with the zeta function, where $\Re(z) > 1$.

The curves intersecting orthogonally illustrate the projections from the grid lines, with the yellow and pink ones being the horizontal and vertical, respectively. To visualize what is happening here, I've drawn just a tiny section of the conformal mapping of two horizontal and vertical grid lines, at positions $\Im(z) = 0.5$ and 2 and $\Re(z) =$ 1.5 and 2. The horizontal grid lines, with constant imaginary values, begin and end at $\Re(z) = 1.00001$ and 15, and the vertical grid lines, with constant real values, begin and end at $\Im(z) = -2$ and 15. They intersect orthogonally at the points in black marked with a diamond symbol. The start and end points are marked in orange and cyan, respectively.



What is particularly interesting is that the starting points for the horizontal grid lines begin close to the yellow vertical line in 3Blue1Brown's diagram at (γ + yi), where γ is the Euler-Mascheroni constant, ~0.577. The closer to the real axis, the closer they are to γ, a fact that no doubt emerges from the mathematics, for, as 3Blue1Brown says in a note, "it's kind of fun to puzzle about why this is the case."[516] Regarding the vertical grid lines, as the zeta function maps the sums of reciprocals of powers, the smaller they are, the larger the curve emanating from the starting point. Not surprising, therefore, that most of the complex plane converges on the red blob in the diagram of the zeta function.

When $\Re(z) \leq 1$, the zeta function diverges. Nevertheless, mathematicians have devised a way in which such divergent series can be given a finite value through a technique call 'analytic continuation'. Julian Havil provides a simple introduction to this mysterious subject with this relationship:[517]

$$f_1(z) = 1 + z + z^2 + z^2 + \cdots = f_2(z) = \frac{1}{1 - z}$$

which is only valid within a circular radius of convergence where $|z| < 1$.[518] However, because $f_2(z)$ can be defined by a power series expansion, which is valid within a larger-than-expected radius of convergence, this power series can be used to define the function outside its original domain of definition. What is more, this extension into the entire complex domain is uniquely defined. As Eric Weisstein says on his Wolfram *MathWorld* website, "This uniqueness of analytic continuation is a rather amazing and extremely powerful statement. It says in effect that knowing the value of a complex function in some finite complex domain uniquely determines the value of the function at every other point."[519]

While analytic continuation is generally seen as a fairly modern invention, Euler effectively introduced this technique in E352, in which he defined the Dirichlet eta function. He began with the 'direct' form of this function, which is clearly divergent when $m \geq 0$.

$$1^m - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + \cdots$$

From this, he then formed a sequence of polynomials, starting with the basic geometric function with $r = -x$, with the initial value of $m = 0$:

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1 + x}$$

He then showed that successive polynomials could be formed with the differential calculus, whose details Tom Osler explains. Euler multiplied this equation by x and differentiated, giving this closed-form expression for the generating function for the next in the sequence, when $m = 1$:

$$1 - 2x + 3x^2 - 4x^3 + \cdots = \frac{1}{(1 + x)^2}$$

Euler then repeated this exercise of first multiplying by x and then differentiating to give, in turn:

$$1 - 2^2x + 3^2x^2 - 4^2x^3 + \cdots = \frac{1 - x}{(1 + x)^3}$$

$$1 - 2^3x + 3^3x^2 - 4^3x^3 + \cdots = \frac{1 - 4x + x^2}{(1 + x)^4}$$

$$1 - 2^4x + 3^4x^2 - 4^4x^3 + \cdots = \frac{1 - 11x + 11x^2 - x^3}{(1 + x)^5}$$

$$1 - 2^5x + 3^5x^2 - 4^5x^3 + \cdots = \frac{1 - 26x + 66x^2 - 26x^3 + x^4}{(1 + x)^6}$$

$$1 - 2^6x + 3^6x^2 - 4^6x^3 + \cdots = \frac{1 - 57x + 302x^2 - 302x^3 + 57x^4 - x^5}{(1 + x)^7}$$

As you can see, these are the generating functions for the sums of the powers on page 246, with x replaced with $-x$. Euler then set $x = 1$ in these polynomials, extending them to include all rows in the triangle of Eulerian numbers on page 234, but omitting the alternating even powers whose numerators sum to zero:

$$1 - 2^0 + 3^0 - 4^0 + 5^0 - 6^0 + \cdots = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$$

$$1 - 2^1 + 3^1 - 4^1 + 5^1 - 6^1 + \cdots = 1 - 2 + 3 - 4 + 5 - 6 + = \frac{1}{2^2} = \frac{1}{4}$$

$$1 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + \cdots = \frac{1 - 4 + 1}{2^4} = -\frac{2}{16} = -\frac{1}{8}$$

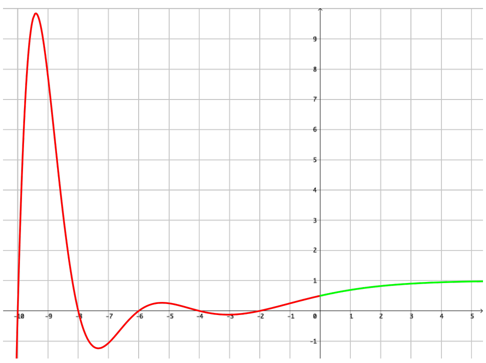
$$1 - 2^5 + 3^5 - 4^5 + 5^5 - 6^5 + \cdots = \frac{1 - 26 + 66 - 26 + 1}{2^6} = \frac{16}{64} = \frac{1}{4}$$

$$1 - 2^7 + 3^7 - 4^7 + 5^7 - 6^7 + \cdots = \frac{1 - 120 + 1191 - 2416 + 1191 - 120 + 1}{2^8} = -\frac{272}{256} = -\frac{17}{16}$$

$$1 - 2^9 + 3^9 - 4^9 + 5^9 - 6^9 + \cdots = \frac{1 - 502 + 14608 - 88234 + 156190 - 88234 + 14608 - 502 + 1}{2^{10}} = \frac{6856}{1024} = \frac{31}{4}$$

When looking at these strange results, giving finite values to divergent series, Euler said, "it is necessary to give to the word *sum* a more extended meaning. We understand the sum to be the numerical value, or analytical relationship which is arrived at according to principles of analysis, that generate the same series for which we seek the sum." Today, such methods of totalling a divergent infinite series are called Abel summation, after Niels Henrik Abel (1802–1829), more powerful than all levels of Cesaro summations,[520] which we introduced when looking at Grandi's series on page 282. Let us now look at how analytical continuation can be applied to the zeta and eta functions in the complex domain.

To analytically continue the zeta and eta functions, Riemann devised a functional equation, which links the domains where the functions are convergent and divergent. In general, a functional equation is an equation that refers to itself, usually with another value of the independent variable(s), perhaps along with other functions. So, recurrence equations in the first section of this chapter are comparatively simple functional equations. However, in general, they are not easy to solve, not being reducible to algebraic or differential equations.[521]



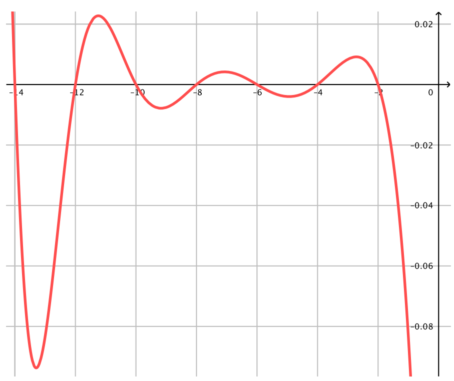
In this regard, the eta function is somewhat easier to deal with, for, as this chart of just the real domain indicates, the convergent and divergent regions are continuous, coloured green and red, respectively, while there is a pole at $\Re(1)$ in the zeta function.

Nevertheless, it is the following functional equation that is most quoted in the literature, which Riemann derived from the beautiful relationship between the zeta and gamma functions on page 301.

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

By some magic, the link between the negative integer values in the zeta function and the Bernoulli numbers then becomes extremely simple:

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$



When the complex number has no imaginary part, the analytical continuation of the *p*-series of real numbers can be plotted in the Cartesian plane, as depicted here, at a quite different scale. And from the relationship between the eta and zeta functions, we are led to a relationship between the Eulerian and Bernoulli numbers, which was not apparent before, showing the amazing interconnectivity between the various types of infinite series and finite sequences:

$$\eta(1-n) = \frac{2^n - 1}{n} B_n$$

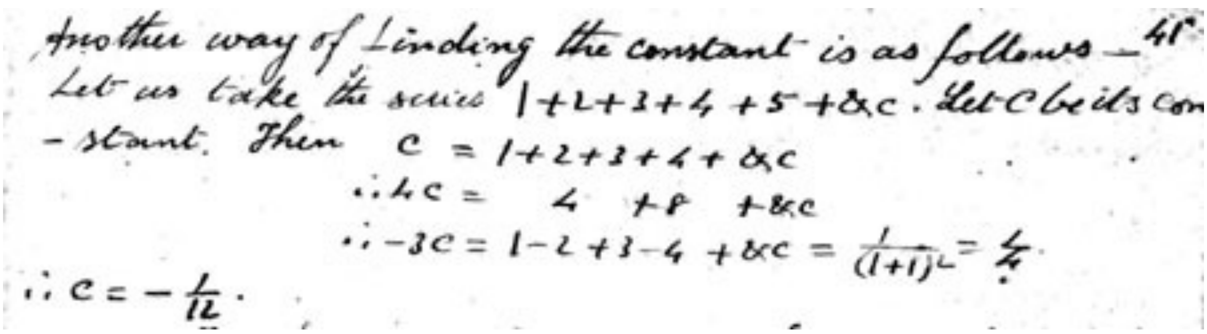
In summary, here is a table of the negative integer values of the zeta and eta functions, the latter being what Euler discovered. The first twenty Bernoulli numbers are given on pages 242 and 298, with the odd ones other than B_1 being zero.

Zeta	Formula	Fraction	Decimal	Eta	Formula	Fraction	Decimal
$\zeta(0)$	$-B_1$	$-\frac{1}{2}$	-0.5	$\eta(0)$	B_1	$\frac{1}{2}$	0.5
$\zeta(-1)$	$-B_2/2$	$-\frac{1}{12}$	$-0.08\overline{3}$	$\eta(-1)$	$3B_2/2$	$\frac{1}{4}$	0.25
$\zeta(-3)$	$-B_4/4$	$\frac{1}{120}$	$0.008\overline{3}$	$\eta(-3)$	$15B_4/4$	$-\frac{1}{8}$	-0.125
$\zeta(-5)$	$-B_6/6$	$-\frac{1}{252}$	$-0.00\overline{396825}$	$\eta(-5)$	$63B_6/6$	$\frac{1}{4}$	0.25
$\zeta(-7)$	$-B_8/8$	$\frac{1}{240}$	$0.0041\overline{6}$	$\eta(-7)$	$255B_8/8$	$-\frac{17}{16}$	-1.0625
$\zeta(-9)$	$-B_{10}/10$	$-\frac{1}{132}$	$-0.00\overline{75}$	$\eta(-9)$	$1023B_{10}/10$	$\frac{31}{4}$	7.75
$\zeta(-11)$	$-B_{12}/12$	$\frac{691}{32760}$	$0.021\overline{092796}$	$\eta(-11)$	$4095B_{12}/12$	$-\frac{691}{8}$	-86.375
$\zeta(-13)$	$-B_{14}/14$	$-\frac{1}{12}$	$-0.08\overline{3}$	$\eta(-13)$	$16383B_{14}/14$	$\frac{5461}{4}$	1365.25
$\zeta(-15)$	$-B_{16}/16$	$\frac{3617}{8160}$	$0.44325\overline{9803921568627450}$	$\eta(-15)$	$65535B_{16}/16$	$-\frac{929569}{32}$	-29049.03125
$\zeta(-17)$	$-B_{18}/18$	$-\frac{43867}{14364}$	$-3.05\overline{395433027011974380}$	$\eta(-17)$	$262143B_{18}/18$	$\frac{3202291}{4}$	800572.75
$\zeta(-19)$	$-B_{20}/20$	$\frac{174611}{6600}$	$26.456\overline{21}$	$\eta(-19)$	$1048575B_{20}/20$	$-\frac{221930581}{8}$	-27741322.625

The analytical continuation of $\zeta(-1)$ is of particular interest, for this is often presented as

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots = -\frac{1}{12}$$

with the implication that the equals sign has its conventional meaning. It is a pity that mathematicians have not devised an alternative symbol to denote Abel summation. For instance, Ramanujan used the conventional sign in the first letter he sent to G. H. Hardy in 1913, giving an over-simplified derivation of this apparent equality, whose context is presumably on the previous page.


Another way of finding the constant is as follows — 41
Let us take the series 1+2+3+4+5+&c. Let C be its con-
-stant. Then C = 1+2+3+4+&c
∴ 4C = 4 + 8 + &c
∴ −3C = 1−2+3−4+&c = $\frac{1}{(1+1)^2}$ = $\frac{1}{4}$
∴ C = $-\frac{1}{12}$.

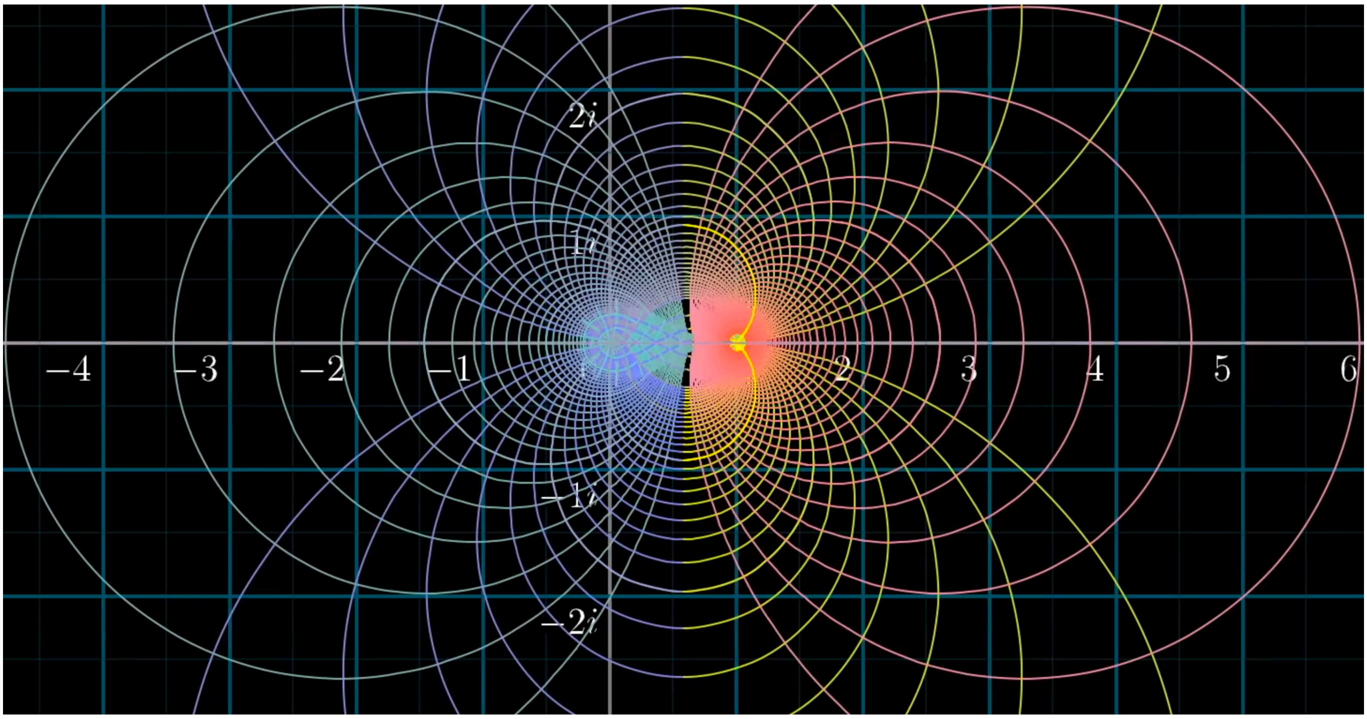
The Numberphile video mentioned in page 282 used a similar derivation, of great affront to pure mathematicians, even though physicists use this summation in string theory, with the mistaken belief that this could lead to the elusive theory of everything. This YouTube video was so startling that the *New York Times* published an article on 3rd February 2014 titled 'In the End, It All Adds Up to 1/12',[522] leading to much confusion among the general public, not understanding that the notion of summability can have other meanings than the conventional one.

We can see further evidence of such limitations from a comment that someone made of 3Blue1Brown's video on the subject: "Mathematics should not be mystical, such that it becomes ungraspable; it should be explained and appreciated for its rigor and intuitive creativity combined." 3Blue1Brown replied:

> I definitely agree with the statement 'Mathematics should not be mystical'. It seems commonplace in outreach to use surprising facts to capture an audience that might not usually care about math, and insofar as this brings in more people who wouldn't otherwise be looking, that might be a net positive. But I do worry that tossing out only mysteries without arguments might have an overall negative effect on the public perception of mathematics.[523]

Yet, we can only fully appreciate the beautiful patterns in mathematics by basing our understanding on our mystical experiences, as I have realized during the last forty years. Furthermore, without admitting the Formless Absolute into our learning, mathematics does not have a sound foundation, and hence neither does society, as a whole. So, separating mysticism and mathematics can cause much heartache at these critical times we live in.

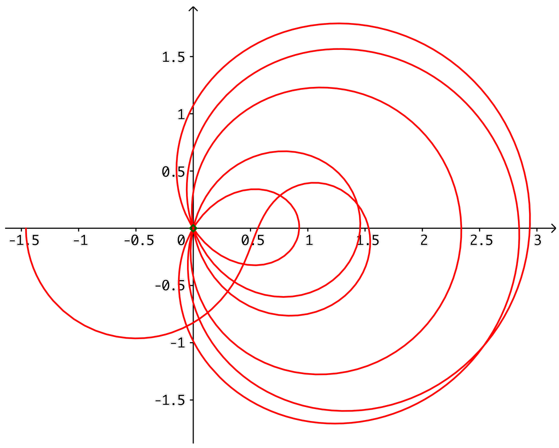
Now extending the zeta function into the entire complex plane, it is not easy to visualize the way that the function behaves as this requires many years of practice,[524] requiring four dimensions to plot the real and imaginary inputs and outputs from the function.[525] Nevertheless, 3Blue1Brown did manage to present this diagram on his YouTube channel, mapping grid lines at various distances apart:



While the symmetry between the convergent and divergent regions is most interesting, mathematicians are usually more focused on the way that the zeta function behaves in the critical strip between 0 + *it* and 1 + *it*. For, in addition to the trivial zeros at negative integer values, because odd Bernoulli numbers are zero, Riemann discovered that there are many non-trivial zeros either lying on the centre line of the critical strip at ½ + *it*, or symmetrically about this line, mirrored at the complements of these zeroing points.

Here is a plot of this mapping from the real axis to the sixth zero on this line at ½ + 37.586178i, beginning at -1.46035, which is zeta(½). The first six at ½ + it_n are given in this table, where the OEIS sequences are the decimal expansions of the imaginary parts. And here are the nearest integers to the imaginary parts of the non-trivial zeros under 100 (OEIS A002410):

n	t_n	OEIS
1	14.134725	A058303
2	21.022040	A065434
3	25.010858	A065452
4	30.424876	A065453
5	32.935062	A192492
6	37.586178	A305741



14, 21, 25, 30, 33, 38, 41, 43, 48, 50, 53, 56, 59, 61, 65, 67, 70, 72, 76, 77, 79, 83, 85, 87, 89, 92, 95, 96, 99

Now there is a great puzzle here. Starting with Euler's product formula, relating the primes to the zeta function as an infinite series, Riemann thought that there is a relationship between the distribution of the non-trivial zeros and the prime-counting function, giving the number of primes less than or equal to n,[526] which Edmund Landau (1877–1938) denoted with π(n) in 1909, nothing to do with π as a constant. As John Derbyshire wrote in *Prime Obsession*, "I am sorry about this; it is not my fault; the notation is perfectly standard; you'll just have to put up with it."[527]

Despite the fact that there appears to be little regularity among the distribution of the primes, since Riemann mathematicians have created the Prime Number Theorem (PNT), formalizing the intuitive idea that primes become less common as they become larger by precisely quantifying the rate at which this occurs. As a first approximation, Gauss and Legendre conjectured at the end of the eighteenth century that:[528]

$$\pi(n) \approx \frac{n}{\ln n}$$

So, the Prime Number Theorem initially states:

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\ln n} = 1$$

One consequence of the PNT is that if a random integer is selected in the range of zero to some large integer n, the probability that the selected integer is prime is about 1/ln(n).[529]

However, how is the distribution of the non-trivial zeros in the Riemann zeta function related to π(n) and the PNT? Browsing through lists of thousands of non-trivial zeros on the Web, these have a quite different distribution pattern from the primes, themselves. After the first few, from the first at about 14, the differences between the non-trivial zeros are sometimes even less than one.

So, why is the Riemann Hypothesis—called the 'greatest unsolved problem in mathematics'—so important? Riemann conjectured that all non-trivial zero points lie on the critical line, ½ + it, but was unable to prove it, saying, "Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation."[530] If proving the Riemann Hypothesis was unnecessary to investigate the prime-counting function, what is all the fuss about?

Well, Riemann's conjecture has played a central role in the development of mathematics since the beginning of the last century. In particular, David Hilbert included the Riemann Hypothesis as the eighth unsolved problem in his 1900 presentation in Paris, including Goldbach's conjecture that every even number is the sum of two primes.[531] And in May 2000, at the Collège de France in Paris, the Clay Mathematics Institute of Cambridge, Massachusetts (CMI) established seven Millennium Prize

Problems, including the Riemann Hypothesis, with $1 million allocated to the solution of each problem.[532]

Back in 1900, it was known that all zero points would fall in the critical strip between $0 + it$ and $1 + it$.[533] Then in 1914, G. H. Hardy proved that there are an infinite number of zeros on the critical line at $\frac{1}{2} + it$.[534] But he did not prove that there are none outside the line and no one has done so since. However, in 1973, Hugh L. Montgomery—noting that the zero points line up in relatively uniform intervals, far more regularly than the primes themselves—found that the differences between the zeros seem to have a distribution given by this formula:[535]

$$1 - \left(\frac{\sin \pi u}{\pi u}\right)^2$$

Now the year before, Montgomery had met Freeman Dyson by chance at the Princeton Institute of Advanced Studies, the latter pointing out that Montgomery's pair correlation conjecture has the same form as the distribution function of the energy levels of subatomic particles.[536] This really got the mathematicians excited, for this similarity seems to indicate a link between the distribution of the prime numbers and quantum physics.

Then in 1996, Alain Connes pointed out another surprising relationship: between his non-commutative geometry, for which he was awarded the Field's Prize, and the Riemann function. This connection opened up a quite new approach to proving the Riemann hypothesis, leading some to speculate that non-commutative geometry could form the basis for the discovery of the fundamental law of nature, one that could explain the creation of the universe. As the commentator on a television programme on the Riemann Hypothesis, which Montgomery produced in 2011, enthusiastically proclaimed, as the new geometry is closely related to prime numbers, if the secrets of the primes are clarified using non-commutative geometry, then the theory of everything would be solved. The century-long search for the hidden meaning behind the prime numbers could well turn out to be the theory of everything, the Creator's blueprint for the Universe.[537] Not all mathematicians share this enthusiasm. An anonymous mathematician who doesn't has said, "What Connes has done, basically, is to take an intractable problem and replace it with a different problem that is equally intractable."[538]

Needless to say, if you have kindly read this far, the human longing to solve the ultimate problem of human learning cannot be resolved within axiomatic, linear mathematics. In my experience, Life can only heal our fragmented, deluded minds by starting afresh at the very beginning, taking the abstractions of pure mathematics, which we explore in the next chapter, to their ultimate level of generality.

From this perspective of Wholeness, confirming the truth of the Riemann Hypothesis with conventional mathematical reasoning is utterly irrelevant. Besides, in 1931, Kurt Gödel proved that there are true theorems in mathematics that cannot be proved to be true with axiomatic, deductive reasoning. Maybe the Riemann Hypothesis is one of them. So, as evolution carries us all into the sixth mass extinction of the species on Earth, self-inquiry must now be our top priority.

Spatial dimensions

While Integral Relational Logic shows that the Universe consists of an infinite number of dimensions, as domains of both quantitative and qualitative values, used in measuring, it is of particular interest to look at just the spatial dimensions, even though these are difficult to visualize. As Coxeter pointed out in *Regular Polytopes*, viewing the fourth Euclidean dimension as time, as Minkowski did in Einstein's special theory of relativity, has little to do with how geometers view spatial dimensions.[539] So, these spatial dimensions are not limited by those envisaged by string theorists in physics: ten, eleven, twenty-six, or more?

As I have a rather limited spatial intelligence, I mention just three aspects of the way spatial dimensions expand in this section:

- regular convex polytopes, generated from figurate sequences
- associatopes and permutatopes, generated from the Catalan sequence and factorials, respectively
- and hyperspheres, with a very weird property, which I feel moved to mention to conclude this chapter.

What these show is that the similar patterns that keep appearing in various branches of mathematics and in different classes of sequences can sometimes be seen as an interconnected whole in multidimensional spatial structures, reflecting the interconnectedness of the Cosmic Psyche, as a whole. It's a pity that it is so difficult to visualize even four-dimensional structures, never mind higher dimensions, meaning that it is necessary to turn to the abstractions of Integral Relational Logic to find Peace by healing the fragmented mind in Wholeness.

Regular convex polytopes

In *Regular Polytopes*, Coxeter defines *polytope* as the general term for the sequence *point*, *segment*, *polygon*, *polyhedron*, etc. More formally, a polytope is "a finite convex region of *n*-dimensional space enclosed by a finite number of hyperplanes",[540] whose essential characteristic is that it is flat, as an extension of a plane in two dimensions.

Reinhold Hoppe (1816–1900) coined the word *polytope* in German in 1882 from Greek *poly*- 'many' and *topos* 'place, region, space'.[541] Alicia Boole Stott (1860–1940), the third of George Boole's five talented daughters, introduced the word into English in 1900[542] after meeting Pieter Hendrik Schoute (1846–1923), a leading geometer at the University of Groningen in the Netherlands.

Dimension of element	Term (in an *n*-polytope)
0	Vertex or point
1	Edge, side, or segment
2	Face or polygon
3	Cell or polyhedron
⋮	⋮
k	*k*-unit
⋮	⋮
$n-3$	Peak – $(n-3)$-unit
$n-2$	Ridge – $(n-2)$-unit
$n-1$	Facet – $(n-1)$-unit
n	The polytope itself

To illustrate the way that polytopes grow in spatial dimensions, I use the nomenclature in this table, a slight modification of one in Wikipedia. For '*k*-face' as a generic for elements in a polytope seems misleading to me. I would therefore suggest '*k*-unit', for *unit* can mean both 'part' and 'whole', like Arthur Koestler's notion of *holon*. Norman Johnson, who we met on page 218 in association with associatopes, coined *polychoron* from Greek *poly*- 'many' and *khoros* 'room, space, place, region', to denote a 4-dimensional polytope.[543] In terms of non-geometric abstractions, it is sometimes also convenient to refer to a dimension of -1, to denote a null polytope, like an empty set.[544]

Regarding the history of the subject, Julian Lowell Coolidge (1873–1954) wrote in *A History of Geometrical Methods* in 1940, "The first vague outlines of the idea of higher spaces are blurred in the mists of time."[545] He surmised that François Viète (1540–1603), who introduced the first systematic algebraic notation in his book *In artem analyticam isagoge* (Introduction to the analytic art) in 1591, had considered four or more dimensions of space.[546] For relating the algebraic symbols that he introduced to geometric objects, Viète could see that, if they were to maintain their homogeneity, they would be associated with planar, solid, sursolid, etc. numbers. However, although algebra was becoming generalized through the works of Viète and Descartes, in particular, “the metaphysical implications apparently involved in the assumption of more than three dimensions appeared formidable.”[547]

In the event, it was not until the middle of the 1800s that a great surge occurred in mathematical abstractions across several fronts and these metaphysical inhibitions could be overcome. As mentioned in Chapter 3, further explored in the next chapter, Alfred North Whitehead wrote an overview of some of these nineteenth-century developments in *A Treatise on Universal Algebra* in 1898, referring, in particular, to Hamilton’s Quaternions, Grassmann’s Calculus of Extension, and Boole’s Symbolic Logic.

In the context of this subsection on regular convex polytopes, the most significant figure was Ludwig Schläfli (1814–1895), who, between 1850 and 1852, discovered such polytopes in four and more dimensions, calling them polyschemes.[548] He did so by extending Euclid’s proof that there are just five regular polyhedra.[549] For the maximum number of triangles, squares, and pentagons that can meet at a point with a solid angle is five, three, and three, giving rise to the tetrahedron, octahedron, icosahedron, cube, and dodecahedron, respectively.

To help his investigations, Schläfli devised a coding system for these constructs, first denoting polygons with n sides as $\{n\}$. Extending this notation into three dimensions, $\{p, q\}$ denotes q polygons with p sides meeting at a vertex. The letter q also denotes the vertex figure of the polyhedron, as a polygon, in this instance, formed by joining the midpoints of the edges meeting at a vertex.

Recursively extending this notation into higher dimensions, $\{p, q, r\}$ first denotes four-dimensional regular polytopes, with polyhedra $\{p, q\}$ meeting at a vertex, with vertex figures $\{q, r\}$. For these to be regular polychora, they also need to satisfy this relationship:[550]

$$\sin\frac{\pi}{p}\sin\frac{\pi}{r} \geq \cos\frac{\pi}{q}$$

Schläfli discovered that there are six polychora satisfying these conditions, five being extensions of the Platonic solids, with one extra, which only exists in four dimensions, which John Conway calls the *octaplex* and Johnson the *icositetrachoron*. They have proposed several other names, including the n-cell, to denote the number of facets in each.[551] Charles Howard Hinton (1853–1907), Alicia Boole Stott’s brother-in-law, coined *tessaract* in 1888,[552] from Greek *téssereis aktines* ‘four rays’, changing this to *tesseract* in 1904.[553] This table lists these regular convex polychora, with some of their names.

Polytope	**Schläfli symbol**	**Cells meeting at vertex/edge**	**Vertex figure**	**C**	**F**	**E**	**V**
5-cell, pentachoron	{3, 3, 3}	Tetrahedron, 4/3	Tetrahedron	5	10	10	5
16-cell, 4-orthoplex	{3, 3, 4}	Tetrahedron, 8/4	Octahedron	16	32	24	8
8-cell, tesseract	{4, 3, 3}	Cube, 4/3	Tetrahedron	8	24	32	16
24-cell, octaplex	{3, 4, 3}	Octahedron, 6/3	Cube	24	96	96	24
600-cell, hexacosichoron	{3, 3, 5}	Tetrahedron, 20/5	Icosahedron	600	1200	720	120
120-cell, dodecacontachoron	{5, 3, 3}	Dodecahedron, 4/3	Tetrahedron	120	720	1200	600

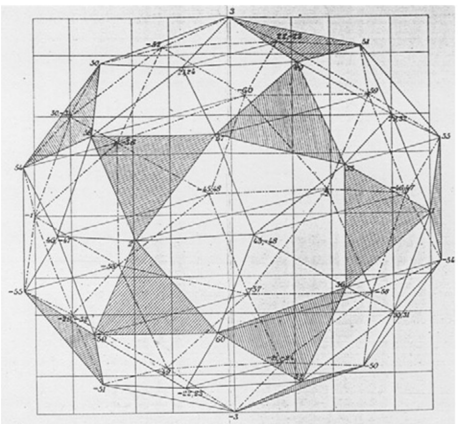
It might seem that all these polyhedra could be extended into higher dimensions, with an increasing number of constructs possible. However, icosahedral/dodecahedral symmetry does not extend beyond four dimensions. By examining the angle criteria for building polyhedra in five or more dimensions, Schläfli discovered that only three are possible, which Coxeter labelled α_n, β_n, and γ_n: {3, 3, …, 3, 3}, {3, 3, …, 3, 4}, and {4, 3, …, 3, 3}, where the ellipses denote n-5 3s. The first of these are self-duals, while the other two are duals of each other. Even though there are four and ten non-convex regular polytopes in three and four dimensions, there are none in five and above.

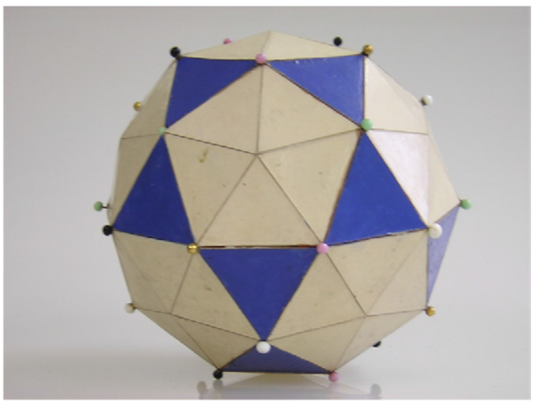
After Schläfli had written a treatise on his innovative discoveries, titled *Theorie der vielfachen Kontinuität* (Theory of Multiple Continuity), he attempted to get it published with the Austrian Academy of Sciences and the Berlin Academy of Science, saying, this is "an attempt to found and to develop a new branch of analysis that would, as it were, be a geometry of *n* dimensions". However, both turned it down, asserting that it was 'too long'.[554] Schläfli did manage to get a portion published in French in 1855,[555] and Arthur Cayley (1821–1895), who corresponded regularly with Schläfli, published a fragment in English in 1858,[556] with the title 'On the multiple integral $\int^n$dx dy … dz', hardly announcing a major new mathematical discovery.[557]

In the event, Schläfli's seminal treatise was not published until 1901 in Switzerland,[558] six years after his death. When reviewing this book in 1904, Schoute wrote in the Dutch mathematical journal *Nieuw Archief voor de Wiskunde*,[559] "This treatise surpasses in scientific value a good portion of everything that has been published up to the present day in the field of multidimensional geometry. The author experienced the sad misfortune of those who are ahead of their time."[560] Coxeter expressed a similar sentiment, likening Schläfli's publications to the art of van Gogh.[561]

In the meantime, W. Irving Stringham (1847–1909) in the USA rediscovered the higher dimensional polytopes in 1880 in his 16-page Ph.D. thesis, titled 'Regular Figures in *n*-dimensional Space',[562] being advised by James Joseph Sylvester. As a consequence, many people believed that Stringham, who incidentally introduced ln(*x*) to denote the natural logarithm,[563] was the discoverer of the regular polytopes. He was followed by several other mathematicians, including Hoppe and Thorold Gosset (1869–1962), who rediscovered the Schläfli symbol.[564]

But perhaps the most remarkable figure in this story was Alicia Boole Stott. Inspired by Hinton, who had married her oldest sister Mary Ellen, she intuitively discovered the six polychora through visualization between 1880 and 1895, making polyhedral cardboard models of their cross sections. She worked alone, knowing nothing of the work of Schläfli and Stringham, until 1895, when her husband Walter, an actuary, suggested that she contact Schoute in the Netherlands. He was astounded by photographs of the models she sent him, for her model of the 600-cell was essentially identical to a drawing he had made of the same cross section:[565]

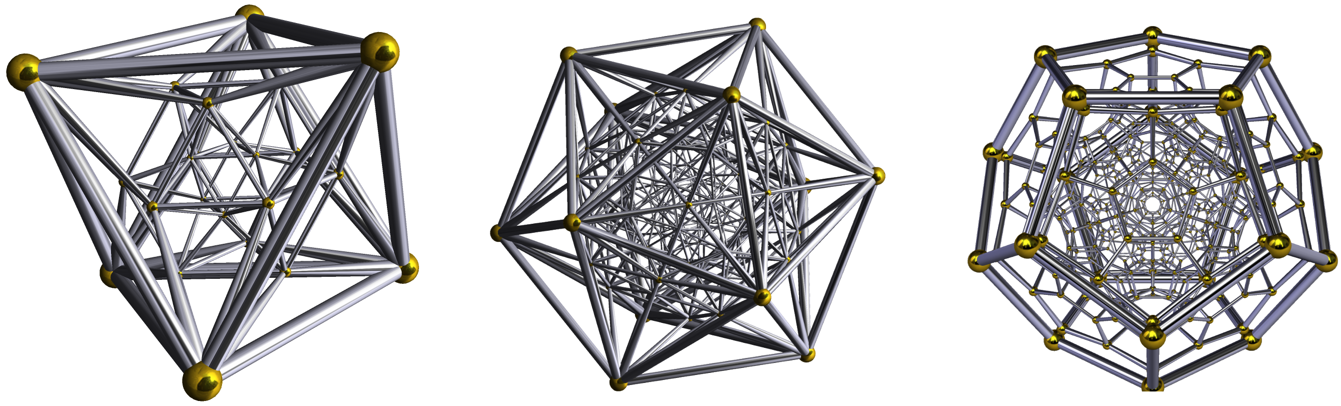




Alicia made her discoveries by synthetic means, in contrast to the analytical methods of professional geometers. As Coxeter said, "Only one or two people have ever attained the ability to visualize hyper-solids as simply and naturally as we ordinary mortals visualize solids."[566] Five years after their first meeting, Schoute arranged for Alicia's discoveries to be published.[567] After this Alicia and Schoute collaborated with their complementary skills for the next twenty years, until his death. In 1930, her distinguished nephew G. I. Taylor introduced her to Coxeter, when he was a graduate student, leading him to collaborate with 'Aunt Alice' until her death. He wrote of her, "The strength and simplicity of her character combined with the diversity of her interests to make her an inspiring friend."[568]

One other figure deserves a mention. To help people visualize polytopes in four dimensions, Victor Schlegel (1843–1905) devised a method in 1883 for projecting a polytope from $\mathbb{R}^d$ to $\mathbb{R}^{d-1}$ from a point just outside one of its facets.[569] Referring to polyhedra, Donald Summerville said, "It is always possible by suitable choice of the centre of projection to make the projection of one face completely contain the projections of all the other faces."[570] Here are Schlegel diagrams in the plane for the three regular polyhedra that can be extended into multiple dimensions: the tetrahedron, cube and octahedron.[571] The face that is omitted from these diagrams is the entire space around them.

Regarding polychora, the Schlegel figures are best created as wire models in three dimensions, which we could hold in our hand, depicting the vertices and edges, such as these for the 24-, 600-, and 120-cell polychora, further projected into two dimensions.



These figures have been uploaded to Wikipedia, created in Robert Webb's brilliant Stella software,[572] originally made possible by a generalized algorithm that Zvi Har'El devised in 1993 for calculating the metrics of all the uniform polyhedra.[573] As I outline in *The Theory of Everything*, this algorithm represents the culmination of the stuttering development of mathematicians' understanding of even three-dimensional polytopes since Euclid around 300 BCE and Kepler in 1619. As I'm primarily interested in this book in the way that mathematical structures grow indefinitely from zero to the infinity of infinities and hence to Transfinity, as the Absolute, I include the Schlegel figures for the three polychora that can be extended beyond four dimensions when we look at them in a little more detail.

In the meantime, it is pertinent to note that Schlegel's projective method can be seen as a metaphor for his relationship with the mainstream mathematical community of his day. For he appears to have been something of an outsider, questioning the scientific methods on which mathematics is based.[574] It is perhaps not surprising that he was an advocate for Hermann Grassmann's Calculus of Extension in which Grassmann sought to establish a sound foundation for mathematics based on a general theory of forms, as I outline in Chapter 3 of this book. Although Grassmann's endeavours were many years ahead of their time, never being fully understood, they have today reached fulfilment in 'Integral Relational Logic, as this book demonstrates. For if we regard the space around a Schlegel figure as Consciousness, by standing outside ourselves with Self-reflective Intelligence, we can visualize the Totality of Existence resident in the Cosmic Psyche as a coherent whole.

One further topic before we look at the way that the three regular convex polytopes grow without limit from tiny seeds. In 1850 and 1851, Euler wrote two papers in which he developed his famous polyhedral formula for convex polyhedra: $V - E + F = 2$,[575] which Ed Sandifer outlined in his 'How Euler Did It' columns in June and July 2004.[576] These are regarded as topology papers, as an extension of Euler's famous paper in 1736 on the bridges of Königsburg,[577] which led to mathematical graph theory.

As graphs in mathematics are ubiquitous, not just restricted to topology, we can use this structure to demonstrate the interconnectedness of all beings, showing that none of us is ever separate from any other being for an instant, including the Supreme Being. However, in this section we are just concerned with the basic topological properties of graphs.

However, Euler's formula does not necessarily hold for the regular non-convex polyhedra or for three-dimensional solids with holes in them, like tori. To generalize the polyhedral formula to accommodate such structures, mathematicians talk about density[578] or genus,[579] which are 2 or 0 for convex polyhedra, regular or not. Then, the Euler polygonal formula becomes the Euler-Poincaré characteristic, denoted with lower-case chi χ, as a topological invariant. For Henri Poincaré (1854–1912) sought what Leibniz called *Analysis Situs*, a multi-dimensional geometry of position, today called algebraic topology, giving his seminal paper on the subject in 1895 this title.[580]

What is especially interesting here is that Poincaré, often described as the 'last universalist' in mathematics, had an unusually high level of self-awareness about his creative processes, albeit within a materialistic worldview, which he described in a famous lecture before the Société de Psychologie in Paris in 1908 entitled 'Mathematical Invention'.[581] This was further expounded in three books *Science and Hypothesis*, *The Value of Science*, and *Science and Method*, collectively published in 1913 in English translation as *Foundations of Science*.[582] It was these that inspired Jacques Hadamard to write his book *The Psychology of Invention in the Mathematical Field* in 1945, for which Einstein wrote an enlightening letter about his own creativity, quoted in the Prologue to this book,[583] which, in turn, has inspired me to develop a comprehensive model of the psychodynamics of society in the context of evolution, as a whole.

However, algebraic topology is a topic about which I know very little and which takes us too far from the primary purpose of this section. What I am more interested in here is the way in which Schläfli, Stringham, and others took the polyhedral formula into the higher dimensions of convex polytopes. If we consider N_k to be the kth element of an n-dimensional polytope, where N_0, N_1, and N_2 denote the vertices, edges, and faces, respectively, while N_n is the polytope itself, as 1, then we have this general formula:[584]

$$\sum_{k=0}^{n} (-1)^k N_k = N_0 - N_1 + N_2 - \cdots \mp N_{n-1} \pm N_n = 1$$

Poincaré pointed out that this formula assumes that all the elements of the polytopes are simply connected.[585] If not, a more complicated formula is required, which he developed in *Analysis Situs* and its first supplement. Keeping things as simple as possible, if we omit N_n, we obtain a sequence of Euler characteristics for the n-dimensional convex polytopes, which Coxeter denotes with Π_n, alternatively numbered 2 and 0:[586]

$\Pi_1: N_0 = 2$

$\Pi_2: N_0 - N_1 = 0$

$\Pi_3: N_0 - N_1 + N_2 = 2$

$\Pi_4: N_0 - N_1 + N_2 - N_3 = 0$

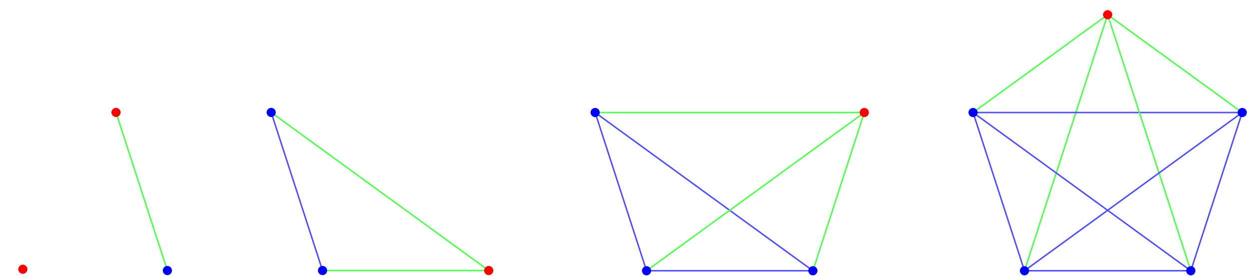
$\Pi_n: N_0 - N_1 + N_2 - N_3 + \cdots (-1)^{n-1} N_{n-1} = 1 - (-1)^n$

What this means is that if we have a generating function that generates a triangle of positive integers with these relationships, they can be visualized as multidimensional polytopes, of which I outline five examples in this section, as special cases of the generative nature of mathematics and hence the Cosmic Psyche, which contains all these structures.

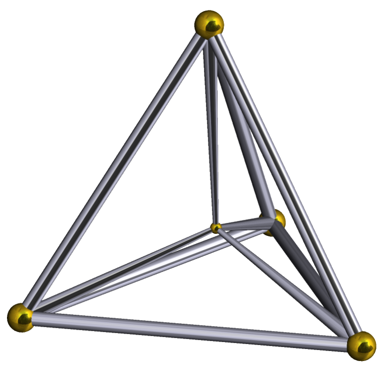
To generate the sequence of simplexes, the initial seed is a point, 'that which has no part', the first definition in Euclid's *Elements*.[587] This is the basic polytope Π_0 with zero dimensions, in Coxeter's notation. Then, at each iteration, a new point is added that is connected to all previous points, giving $n + 1$ in a Π_n. Thus, as edges, faces, and polyhedra connect two, three, and four points, and so on, the general formula for the number of k-units in an n-dimensional simplex is:[588]

$$S(n,k) = \binom{n+1}{k+1} = S(n-1,k) + S(n-1,k-1) \qquad k \le n$$

Topologically, this generative process is most clearly illustrated as coloured variations of Coxeter's own diagrams:



α0 α1 α2 α3 α4



The last of these is the Petrie polygon for the 5-simplex, named after Coxeter's friend John Flinders Petrie (1907–1972), who introduced a general orthogonal projection with which to view polytopes.[589] The Schlegel figure for the 5-cell or 4-simplex is depicted here.[590]

Constructing the simplexes in this generative manner shows that the sequence of numbers that denote the vertices is simply the natural numbers, as the following table illustrates. The next columns are thus the triangular, tetrahedral, and pentachoral, etc. sequences, as we see on page 189. The final column is the total number of elements in each simplex, given by: $2^{n+1} - 1$ (OEIS A000225), the Mersenne numbers, recursively defined on page 250, as an example of Lucas sequences.

$n \backslash N_k$	N_0	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	Σ
0	1											1
1	2	1										3
2	3	3	1									7
3	4	6	4	1								15
4	5	10	10	5	1							31
5	6	15	20	15	6	1						63
6	7	21	35	35	21	7	1					127
7	8	28	56	70	56	28	8	1				255
8	9	36	84	126	126	84	36	9	1			511
9	10	45	120	210	252	210	120	45	10	1		1023
10	11	55	165	330	462	462	330	165	55	11	1	2047

Number of k-*units in each* n-*simplex*

The count of elements in simplexes is thus Pascal's triangle without its left-hand edge, A135278 in the OEIS, which gives this generating function:

$$\frac{1}{(1-x)(1-x-tx)} = 1 + (2+t)x + (3+3t+t^2)x^2 + (4+6t+4t^2+t^3)x^3 + \cdots$$

The coefficients of the *t*-polynomials, as the coefficients of x^n, give the number of *k*-units in each *n*-simplex, as the rows in the triangle. The generating functions for the individual columns, as the sequences of elements in each *k*-dimension, denoted by the placeholder *t*, are, as given on page 245:

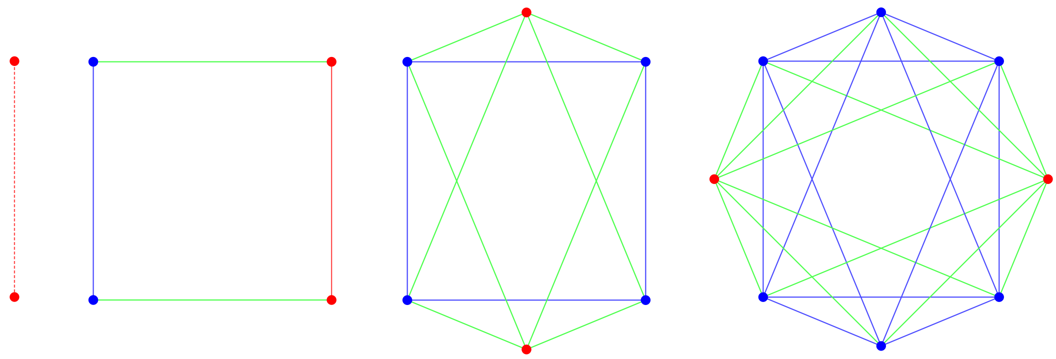
$$\frac{1}{(1-x)^{t+2}} = 1 + (2+t)x + \frac{1}{2}(6+5t+t^2)x^2 + \frac{1}{6}(24+26t+9t^2+t^3)x^3 + \frac{1}{24}(120+154t+71t^2+14t^3+t^4)x^4 + \cdots$$

This is the expansion that Wolfram Alpha gives, different from the generating function in the OEIS, which covers the triangle, as whole, taking note that the sequences in the columns have different offsets. We can reconcile these differences by rewriting the OEIS generating function as a polynomial in *t*:

$$\frac{1}{(1-x)^2} + \frac{x}{(1-x)^3}t + \frac{x^2}{(1-x)^4}t^2 + \frac{x^3}{(1-x)^5}t^3 + \cdots$$

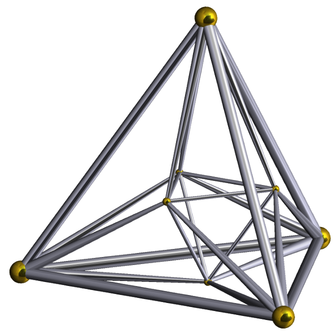
As the number of edges, faces, and cells etc. are the same as the number of facets, ridges, and peaks, etc. respectively, the *n*-simplexes are self-dual.

Cross polytopes are so named because they extend the cross of the orthogonal axes in the Cartesian plane into higher dimensions. Starting with a single axis, pairs of points are placed equidistantly from the origin O as each orthogonal axis is added. So, unlike the other four polytopes outlined in this section, the initial seed from which the orthoplexes are generated is not a point, but a pair of points connected by an implicit segment, made explicit in the next step. β_2 is then formed by adding an explicit segment connecting two points, which are connected to the first two points to form the sides of a square. After this, unconnected pairs of points are added at each step in the sequence and connected to all previous points. The square in β_2 thus becomes the common base for a dipyramid or octahedron, which, in turn provides the base for a four-dimensional dipyramid or 4-orthoplex and so on. Here are coloured versions of Coxeter's diagrams illustrating this generative process.[591]



β_1 β_2 β_3 β_4

The last of these is a Petrie polygon, which is displayed in this Schlegel figure in three dimensions projected into two.[592]



As the sequence of orthoplexes is formed by adding two points at a time, Π_0 is not defined for the orthoplexes, and the base sequence for $X(n, 0)$, from which the $X(n, k)$ are generated in each *n*-orthoplex, is twice the natural numbers, or the number of vertices in each simplex:

$$X(n,0) = 2S(n,0) = 2n$$

After this, "since β_n is a dipyramid based on β_{n-1}, all its elements are either elements of β_{n-1} or pyramids based on such elements. Thus, all are simplexes," as Coxeter defines this growth process, giving this recurrence equation:

$$X(n,k) = X(n-1,k) + 2X(n-1,k-1) \qquad n > k \geq 1$$

The number of k-units in an n-orthoplex or n-dimensional cross polytope is thus, provable by induction:

$$X(n,k) = 2^{k+1}\binom{n}{k+1} \qquad n > k \geq 1$$

Here then is the table of k-units in each n-orthoplex from n = 1 to 10. I have added the implicit values for $X(n, n)$ to ensure that Euler's generalized polytopic formula holds. The total number of elements in each n-orthoplex is then 3^n, the same as for the hypercubes.

$n \backslash N_k$	N_0	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	Σ
0	(1)											1
1	2	(1)										3
2	4	4	(1)									9
3	6	12	8	(1)								27
4	8	24	32	16	(1)							81
5	10	40	80	80	32	(1)						243
6	12	60	160	240	192	64	(1)					729
7	14	84	280	560	672	448	128	(1)				2187
8	16	112	448	1,120	1,792	1,792	1,024	256	(1)			6561
9	18	144	672	2,016	4,032	5,376	4,608	2,304	512	(1)		19683
10	20	180	960	3,360	8,064	13,440	15,360	11,520	5,120	1,024	(1)	59049

Number of k*-units in each* n*-orthoplex*

This triangle is A276985 in the OEIS, which does not give a closed-form expression for this generating function:

$$2x + (4+4t)x^2 + (6+12t+8t^2)x^3 + (8+24t+32t^2+16t^3)x^4 + \cdots$$

Nevertheless, the generating functions for the columns, giving the sequences of elements in each k-dimension, do have closed-form expressions, giving this generating function in terms of t:

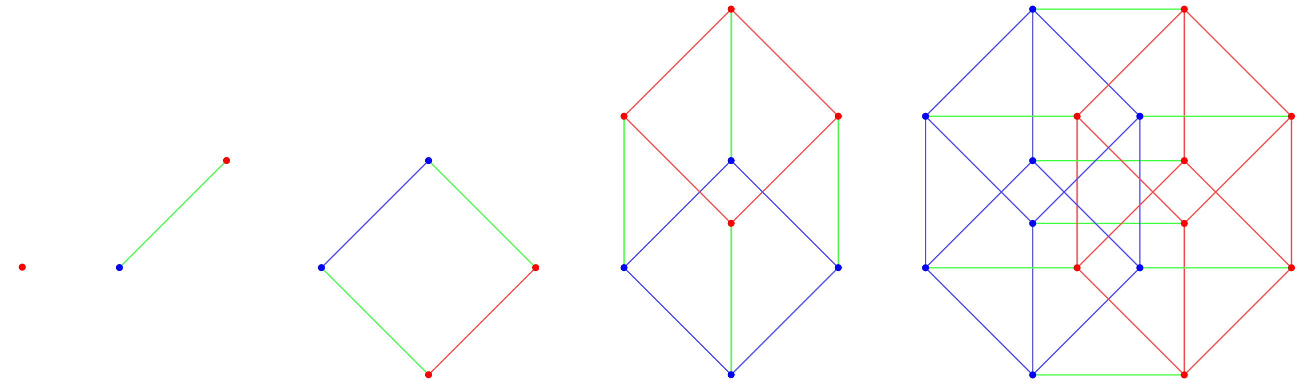
$$\frac{2x}{(1-x)^2} + \frac{4x^2}{(1-x)^3}t + \frac{8x^2}{(1-x)^4}t^2 + \frac{16x^3}{(1-x)^5}t^3 + \frac{32x^4}{(1-x)^6}t^4 + \cdots$$

The OEIS gives A130809, A130810, and A130811 as the sequences for the coefficients of t^2, t^3, and t^4, respectively, with offsets that match the triangle. However, A005843 and A046092, corresponding to the first two coefficients in the t-polynomial, have offsets of 0, rather than 1 and 2, respectively.

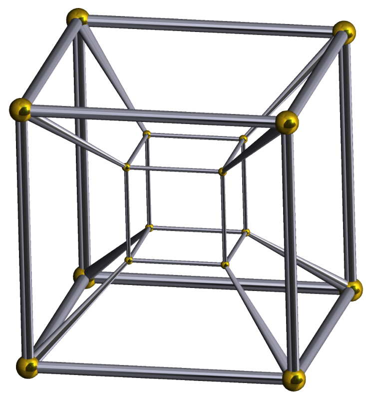
As the number of edges, faces, and cells etc. in the orthoplexes are the same as the number of facets, ridges, and peaks etc. in the hypercubes, and vice versa, these polytopes are duals of each other.

Coxeter calls the hypercubes *measure polytopes* because each one in the sequence acts as a unit with which the content of k-space can be measured. For instance, a square and cube, with sides or edges of length one, measure area and volume, respectively. In other words, hypercubes are honeycombs, tessellating multidimensional space, so named because the hexagons that form bee's honeycombs also fill the plane. In turn, hexagons initiate another sequence of polytopes that fill k-space, as we see in the permutatopes, outlined on page 328.

To provide a unit with which to measure k-space, the polytope at each step is formed by duplicating the previous step and by connecting the corresponding vertices, illustrated in these coloured variations of Coxeter's own diagrams.[593]



γ_0 γ_1 γ_2 γ_3 γ_4



As before, the last of these is a Petrie polygon, which is displayed in this Schlegel figure in three dimensions projected into two.[594]

As the sequence of hypercubes is formed by doubling the number of vertices each time, the base sequence for $M(n, 0)$, from which the $M(n, k)$ are generated in each n-dimensional measure polytope is powers of 2 (OEIS A000079):

$$M(n,0) = 2^n$$

The recurrence equation that then generates the number of k-units in each n-hypercube is:

$$M(n,k) = 2M(n-1,k) + M(n-1,k-1) \qquad n \geq k \geq 0$$

It is then quite easy to prove by induction that the number of k-units in an n-dimensional measure polytope (hypercube) is:[595]

$$M(n,k) = 2^{n-k}\binom{n}{k}$$

Here then is the table of k-units in each n-hypercube from n = 0 to 10. The total number of elements in each measure polytope is then 3^n, the same as for the cross polytopes.

$n \backslash N_k$	N_0	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	Σ
0	1											1
1	2	1										3
2	4	4	1									9
3	8	12	6	1								27
4	16	32	24	8	1							81
5	32	80	80	40	10	1						243
6	64	192	240	160	60	12	1					729
7	128	448	672	560	280	84	14	1				2187
8	256	1,024	1,792	1,792	1,120	448	112	16	1			6561
9	512	2,304	4,608	5,376	4,032	2,016	672	144	18	1		19683
10	1,024	5,120	11,520	15,360	13,440	8,064	3,360	960	180	20	1	59049

Number of k-*units in each* n-*hypercube*

This triangle is A038207 in the OEIS, which gives this closed-form expression for the generating function that generates the entries in the table:

$$\frac{1}{1-2x-tx} = 1 + (2+t)x + (2+t)^2x^2 + (2+t)^3x^3 + (2+t)^4x^4 + \cdots$$

Taking account of the correct offsets, the generating functions for the columns, giving the sequences of elements in each k-dimension, have this generating function in terms of t:

$$\frac{1}{(1-2x)}+\frac{x}{(1-2x)^2}t+\frac{x^2}{(1-2x)^3}t^2+\frac{x^3}{(1-2x)^4}t^3+\frac{x^4}{(1-2x)^5}t^4+\cdots$$

The coefficients in this t-polynomial expand into the sequences A000079, A001787, A001788, A001789, and A003472, not necessarily with the same offsets as those in the table.

Associatopes and permutatopes

Just as simplexes and hypercubes extend triangles and squares into higher dimensions of space, associatopes and permutatopes extend pentagons[596] and hexagons.[597] They are not regular polytopes. However, they do have the common property that each vertex is connected to the same number of vertices, marking, in these cases, binary transpositions of associations and permutations. These are central concepts in group theory, which we turn to in the next chapter, as we move ever closer to the abstractions of Integral Relational Logic.

I mentioned on page 218 that Mark Haiman coined *associahedron* in 1984, deriving the word from *association*, first used in English about 1535 to mean 'action of coming together for a common purpose', from Latin *associāre* 'to join with', from *ad-* 'to' and *sociāre* 'unite, combine', from *socius* 'sharing, allied; companion'. The word *association* thus has a much broader meaning than that in mathematics, key to creating a comprehensive model of the psychodynamics of society. For it denotes the fact that all beings in the Universe are interconnected, as encapsulated in Integral Relational Logic, the system of thought we all implicitly use every day to bring a modicum of order to our lives. Of particular relevance is that association is a basic construct in the Unified Modeling Language (UML), playing a key role in modelling information systems in business, as mentioned in Chapter 1.

So the algebraic, combinatorial, and topological meaning of *association* represents a special, rather technical instance, of a universal principle. As already mentioned, Tamari and Stasheff independently discovered the associahedron when studying Catalan lattices and homotopic structures, respectively, homotopy being the continuous deformation between two continuous maps, such as the transformation of a coffee cup with a handle into a doughnut or torus, which are topologically equivalent and homeomorphic.[598] Although Tamari visualized the potential of higher dimensions, saying in his thesis: "*Généralement, on aura des hyperpolyèdres*" (Generally, we shall have hyperpolyhedra),[599] it seems that it was Stasheff who first articulated them, albeit initially as curvilinear polytopes, which are today sometimes called Stasheff polytopes.[600]

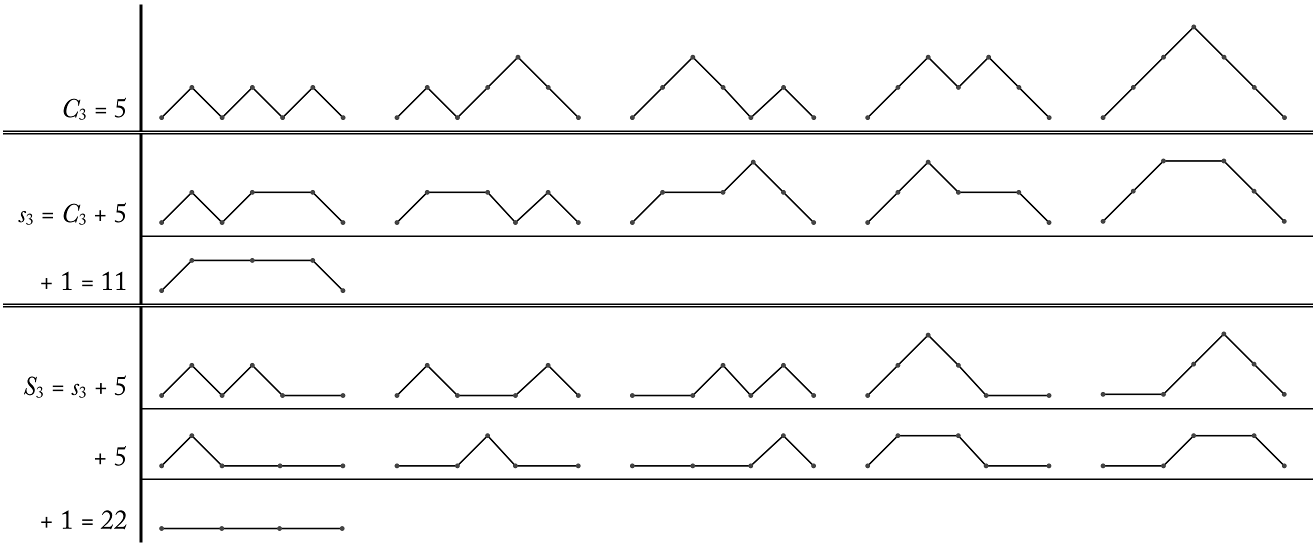
However, as with so many discoveries in mathematics, they were not the first to discover the sequences of integers encapsulated in what should be more properly called *associatopes*, the associahedron being the three-dimensional member of the class. In 1870, when generalizing Catalan's bracketings, as the second of four combinatorial problems,[601] Ernst Schröder (1841–1902) discovered what are today called super-Catalan numbers or small Schröder numbers (s_n) (OEIS A001003), the large ones being twice the size, except the first (S_n), the solution to the first of Schröder's problems (OEIS A006318). In *Advanced Combinatorics* in 1970, Comtet described what these generalizations mean, giving this recurrence equation:[602]

$$(n+1)s_{n+1}=3(2n-1)s_n-(n-2)s_{n-1} \qquad n\geq 2; \quad s_1=s_2=1$$

which generates this sequence with offset 1 rather than 0, as in the OEIS.

n	1	2	3	4	5	6	7	8	9	10	11	12
s_n	1	1	3	11	45	197	903	4279	20793	103049	518859	2646723

However, to visualize the large and small Schröder numbers and their relationships to each other, it is easier to view Schröder paths as a generalization of Dyck paths, of which an example is presented on page 214. In the way that they are depicted there, Dyck paths of semilength n are defined as those with diagonal steps (1, 1) and (1, -1) from (0, 0) to ($2n$, 0) that do not fall below the x-axis. The Schröder paths add a horizontal step (2, 0), the small Schröder paths not including any steps on the x-axis. Here are diagrams of Schröder paths enumerated as Catalan, super-Catalan, and Schröder numbers for n = 3.



Interestingly, there is a recursive equation that simply illustrates the relationship between the small and large Schröder numbers. It arises from an extension of Pascal's triangle arranged in columns of sequences of simplexes, such as the triangular and tetrahedral numbers on page 198. In this triangle (OEIS A033877), each cell after the first column of ones is the sum of all three cells above and to its left, not just the sum of the cells above it, as in Pascal's triangle. Then, the main diagonal is the large Schröder numbers, while the sum of each row is the small Schröder numbers, offset by one position.[603]

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	Σ
0	1											1
1	1	2										3
2	1	4	6									11
3	1	6	16	22								45
4	1	8	30	68	90							197
5	1	10	48	146	304	394						903
6	1	12	70	264	714	1412	1806					4279
7	1	14	96	430	1408	3534	6752	8558				20793
8	1	16	126	652	2490	7432	17718	33028	41586			103049
9	1	18	160	938	4080	14002	39152	89898	164512	206098		518859
10	1	20	198	1296	6314	24396	77550	206600	461010	831620	1037718	2646723

The recurrence equation that generates this triangle is

$$T(n,k) = T(n,k-1) + T(n-1,k-1) + T(n-1,k) \qquad T(1,k) = 1,\ \ T(n,k) = 0, k > n$$

giving

$$T(n,n) = S_n$$

and

$$\sum_{k=0}^{n} T(n,k) = s_{n+1}$$

Nor is this all. There is some circumstantial evidence that Hipparchus of Nicaea (c. 190–c. 120 BCE) was aware of the Schröder sequence, for Plutarch wrote this in *De Stoicorum repugnantiis* 'On Stoic self-contradictions', the fourth of six essays on Stoic philosophy:[604]

> But now he [Chrysippus] says himself that the number of conjunctions produced by means of ten propositions exceeds a million, though he had neither investigated the matter carefully by himself nor sought out the truth with the help of experts. … Chrysippus is refuted by all the arithmeticians, among them Hipparchus himself who proves that his error in calculation is enormous if in fact affirmation gives 103,049 conjoined propositions and negation 310,952.

This Plutarchean passage is repeated almost verbatim in *Quaestiones Conviviales* 'Table-talk'.[605] But, where do these numbers come from? Harold Cherniss, the translator of *De Stoicorum repugnantiis* said that if Hipparchus was familiar with the technical terms in Stoic logic, these comparatively large numbers must have meant something to him, for he "was celebrated for his industry and accuracy". Nevertheless, when this translation was made in 1976, even any approximation to them remained an unsolved mystery.[606] Similarly, Thomas Heath, the leading authority on Greek mathematics in the early twentieth century, wrote in 1921, "it seems impossible to make anything of these figures."[607]

This was the situation until 1994, when David Hough, who only began a career in mathematics in 1992, "noticed that the mysterious number 103,049 of Plutarch, i.e. the number of compound propositions that can be formed from ten simple propositions, is just the tenth Schröder number [s_{10}]," Richard P. Stanley told us in 1997, giving the quote from Plutarch's 'Table-talk'. But, if this is the case, as the number 103,049 "is much too large to have been computed by a direct enumeration of all the cases", he must have used a simple recurrence relationship, far simpler that Comtet's sophisticated recurrence equation.[608]

However, this does not explain where 310,952 comes from. In the following year, Laurent Habsieger, Maxim Kazarian, and Sergei K. Lando proposed a possible solution to this problem. They suggested that 310,952 was a 'misprint' for 310,954, for[609]

$$\frac{s_{10} + s_{11}}{2} = \frac{103{,}049 + 518{,}859}{2} = 310{,}954$$

How this number enumerates combinations of negative propositions is far from clear to me. Nevertheless, the possibility that Hipparchus had found a way of calculating at least 103,049 led Fabio Acerbi to reappraise ancient Greek combinatorics. In a scholarly paper titled 'On the Shoulders of Hipparchus' in 2003, he said that the evidence that he had discovered supports at least the plausibility of the assumption that Hipparchus had grasped the recursive character of the calculations, as the basis of combinatorial techniques.[610]

However, Susanne Bobzien, a leading authority on Stoic logic, says that while she does not doubt Acerbi's assessment of Hipparchus's arithmetical skills, she doubts that Hipparchus really understood Stoic logic. In another scholarly paper in 2012 titled 'The Combinatorics of Stoic Conjunction: Hipparchus refuted, Chrysippus vindicated', she says that Chrysippus of Soli (c. 279–c. 206 BCE) , as the third head of the Stoa and one of the two greatest logicians in antiquity, "not only got his Stoic logic right (which would not be that surprising), but also got his mathematics right; in other words, that, within the context of Stoic logic, 'the number of conjunctions [constructible] from ten assertibles exceeds one million'."[611]

An 'assertible' here corresponds to what we would call *propositions* or *axioms* today, Plutarch's statement being a reference to an attempt to enumerate the many different relationships or associations between them. However, far from clarifying the situation, there still seems to be much confusion in mathematical circles about the association of the Greeks to these numbers. For instance, while Eric W.

Weisstein refers to 103,049 and 310,952 as 'Plutarch numbers' on his *MathWorld* site,[612] Wikipedia refers to the super-Catalan sequence as Schröder–Hipparchus numbers.[613] Yet, how much did the ancient Greeks really know about combinatorics? While they had some understanding of triangular numbers, I have seen no evidence that they were aware of Pascal's triangle, related to the triangle that generates the Schröder numbers. As the connection of Hipparchus to Schröder is tenuous, at best, I prefer to refer to the small Schröder numbers as the super-Catalan sequence, simplifying large Schröder numbers as Schröder numbers.

Besides, debating the relationship between logic and mathematics, as the ancient Greeks saw it, is not a subject that I wish to engage in for not only is it beyond my ability to do so, their logical assumptions have led Western reasoning into an evolutionary dead end, from which Integral Relational Logic has extricated the author of this book, at least. This has happened because I am more focused on *meaning* than on *counting*, revealing the Contextual Foundation of the Universe, necessary to build a comprehensive model of all evolutionary processes and hence explain what causes mathematicians and others to be creative.

Schröder was also much concerned with the foundations, a subject that few mathematicians study, even today. He was a logician, seeking the patterns that underlie human reasoning and hence the Universe. As such, he was naturally attracted to those mathematicians working outside the mainstream, producing such theories as Grassmann's Calculus of Extension and Cantor's set theory, and those of Charles Sanders Peirce and his students in developing the logic of relatives and first-order predicate logic.

Geraldine Brady highlights Schröder's role in the development of mathematical logic in her 1996 Ph.D. thesis 'The Contributions of Peirce, Schröder, Löwenheim, and Skolem to the Development of First-Order Logic', later expanded as a book *From Peirce to Skolem: A Neglected Chapter in the History of Logic*. As she points out, Peirce and Schröder's contribution in these developments has been neglected in favour of Frege's better-known *Begriffsschrift* 'concept writing', with its rather obscure notation, never further developed.

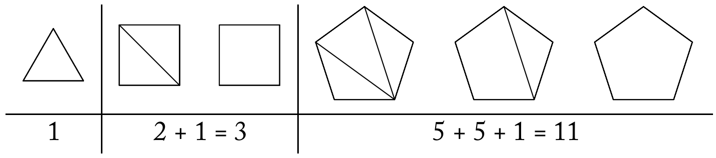
While I haven't read Schröder's *Über die formalen Elemente der absoluten Algebra* (On the Formal Elements of the Absolute Algebra) from 1874, *Der Operationskreis des Logikkalkuls* (The Operation of the Logical Calculus) from 1877, in which he emphasized the principle of duality in Boolean algebra, or his 3-volume masterwork *Vorlesungen über die Algebra der Logik* (Lectures on the Algebra of Logic), from 1890–1905,[614] not the least because they are in German, it is clear that he was way ahead of his time, not able to fulfil his great ambition because direct-access storage devices attached to computers, which led Ted Codd to develop his relational model of data, were not invented until the 1950s.

In turn, this led to Integral Relational Logic, which everyone, including logicians, have implicitly applied throughout the history of learning, showing the close connection of this universal system of thought to abstract algebra, combinatorics, and topology, for instance, for they all reveal that the underlying structure of the Universe is a multidimensional network of hierarchical relationships.

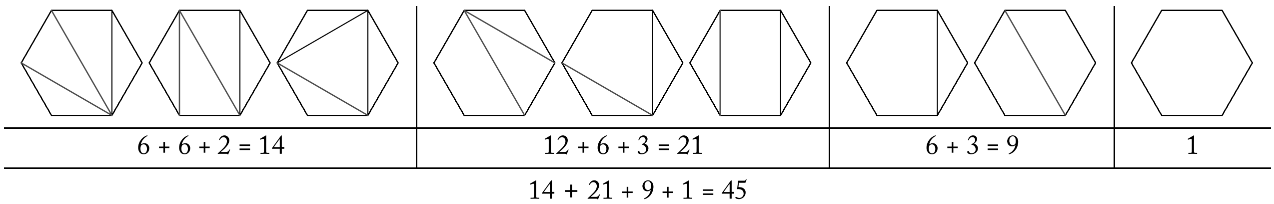
It is also interesting to note that in 1871, the year after Schröder wrote his paper on combinatorics, he wrote '*Über iterierte Funktionen*' (About Iterated Functions),[615] which is often cited as a basis of modern dynamical systems theory.[616] Indeed, as my book *Through Evolution's Accumulation Point: Towards Its Glorious Culmination* from 2016 describes, it is possible to use a nonlinear difference equation in systems dynamics to map the whole of evolution since the most recent big bang, explaining why society is degenerating into chaos at the moment, essentially because most focus on particulars, in contrast to universals, which enable us to fulfil our potential as universal humans, in contrast to Alan Turing's universal machines.

With this rather long preamble, we now come to the associatopic sequences themselves, which are somewhat more complex than those for the regular convex polyhedra. For while the initial seed is just a point, like that for simplexes and hypercubes, the generating sequence of vertices in the zero dimension is that of the Catalan numbers, with their intricate patterns manifesting in many different ways.

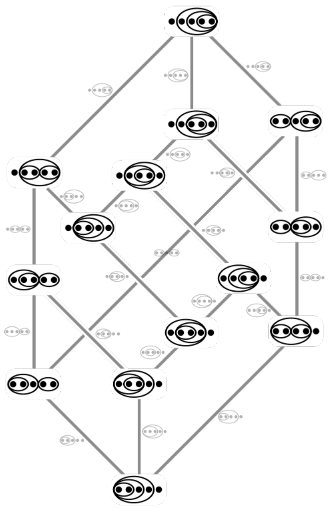
Combinatorially, another way of looking at this generative process is as a generalization of Euler's original question on the subject: into how many regions can a polygon be divided with non-crossing diagonals, where the regions are not just triangles? Here are diagrams of the number of ways in which the triangle, square, and pentagon can be segmented with non-crossing diagonals, including none, the third corresponding to the Dyck paths and first set of Schröder paths above.



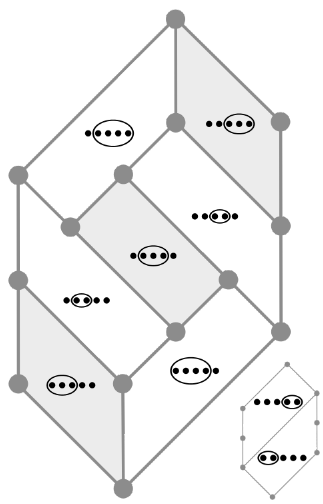
Dividing the hexagon into regions is a little more complicated, for the number of different patterns, counting rotational and reflective symmetries as the same, is different for each number of diagonals, illustrated in these diagrams:



As you can see from the rapid increase of the super-Catalan numbers, the complexity of these patterns increases swiftly as the polygons gain sides, making it virtually impossible to draw all of them. Nevertheless, enumerating regions created from non-crossing diagonals in polygons is the primary interpretation of these sequences in the OEIS.



Topologically, figures for the two dimensions after the initial zero-dimensional point are given on page 217 and that for the Tamari lattice of order four (T_4) on page 218. To visualize this as a three-dimensional associahedron (K_5) with edges and faces, denoting generalized parentheses, Wikipedia presents these two diagrams.[617] Here, the parentheses have been replaced by ovals and the distinct letter symbols, which have no meaning in this context, by bullets. On the left, the 21 connections between the 14 initial nodes are marked with two rather than three sets of ovals, indicating what is common to the two vertices that they connect. Then, on the right, the ovals on the nine faces are reduced to one, denoting what they



share with their three vertices. The entire associahedron is then denoted by the set of five elements with no parentheses, indicating the uniqueness of all fourteen vertices, with nothing in common.

Extending this associatope into multi-dimensional polytopes, like the three regular polytopes, here is a table of the number of *k*-units in each associatope, with their final column being the super-Catalan sequence, denoting the total number of elements in each associatope. There is some difficulty with indexing here. The first Catalan number C_1 corresponds to the zeroth dimension. So, unless we include a dimension of -1, C_0 is omitted as a generator. And neither of the ways of indexing the super-Catalan numbers (with offset 0 or 1) aligns with the number of dimensions of the associatopes. Jean-Louis Loday, a specialist in associatopes, tells us that they are often indexed as K_{n+2} because "the set of vertices is in one-to-one correspondence with the planar binary trees having $n + 2$ leaves,"[618] another interpretation of Catalan numbers.

n	$K_{n+2}\backslash N_k$	N_0	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	Σ
0	K_2	1											1
1	K_3	2	1										3
2	K_4	5	5	1									11
3	K_5	14	21	9	1								45
4	K_6	42	84	56	14	1							197
5	K_7	132	330	300	120	20	1						903
6	K_8	429	1287	1485	825	225	27	1					4279
7	K_9	1430	5005	7007	5005	1925	385	35	1				20793
8	K_{10}	4862	19448	32032	28028	14014	4004	616	44	1			103049
9	K_{11}	16796	75582	143208	148512	91728	34398	7644	936	54	1		518859
10	K_{12}	58786	293930	629850	755820	556920	259896	76440	13650	1365	65	1	2646723

Number of k*-units in each* n*-associatope*

This triangle $T(n, k)$ is A126216 in the OEIS, but with rows at offset 1, defined as "the number of Schröder paths of semilength n containing exactly k peaks but no peaks at level one ($n \geq 1$; $0 \leq k \leq n - 1$)". However, as the generating sequence of vertices is the Catalan numbers, there are many other ways in which to visualize this lower triangular matrix. One is that each column, denoting edges, faces, and cells, etc., counts the number of ways in which convex $(n + 3)$-gons can be divided with non-crossing diagonals into $n + 1 - k$ regions.

The mirror image of the associatopic triangle, where the rows are reversed, is A033282 in the OEIS, thus defined as "Triangle read by rows: $T(n, k)$ is the number of diagonal dissections of a convex n-gon into $k + 1$ regions." This does not directly represent the dual of associatopes, where vertices and facets, edges and ridges, etc. are interchanged, like in the relationship between orthoplexes and hypercubes. As a linear integer sequence, such a triangle would omit the first term in A033282. For the number of vertices in the first column of the triaugmented triangular prism, the dual of the associahedron, is $3 \times 3 = 9$.

Now, while the recurrence equation for the Catalan numbers is

$$C_{n+1} = \frac{2(2n+1)}{n+2} C_n \qquad\qquad C_0 = 1$$

I have not yet found such a recursive relationship for the triangle, as a whole, like those for the regular polytopes. However, the OEIS provides this formula for $T(n, k)$, which I'll denote with A for associatope, adjusted for offset 0 rather than 1:

$$A(n,k) = \frac{1}{n+1}\binom{n+1}{k}\binom{2(n+1)-k}{n+2} \qquad 0 \leq k \leq n$$

Adjusting the offset for the generating function that the OEIS provides gives:

$$\frac{1-2x-tx-\sqrt{1-4x-2tx+t^2x^2}}{2(1+t)x^2} = 1+(2+t)x+(5+5t+t^2)x^2+(14+21t+9t^2+t^3)x^3+\cdots$$

Like the regular convex polyhedra, setting t = 0 gives the first column, as the Catalan numbers in this case, and setting t = 1 gives the super-Catalan numbers, as the sums of each row, without C_0 and s_0. Here is the effect on the generating function of omitting the first term in each sequence:

Numbers	With 0th term	Adjusting 0th term
Catalan	$\frac{1-\sqrt{1-4x}}{2x} = 1+x+2x^2+5x^3+14x^4+\cdots$	$\frac{1-2x-\sqrt{1-4x}}{2x^2} = 1+2x+5x^2+14x^3+42x^4+\cdots$
Super-Catalan	$\frac{1+x-\sqrt{1-6x+x^2}}{4x} = 1+x+3x^2+11x^3+45x^4+\cdots$	$\frac{1-3x-\sqrt{1-6x+x^2}}{4x^2} = 1+3x+11x^2+45x^3+197x^4 \ldots$

Setting t greater than 1 naturally generates other sequences, which do not appear to have any special significance in this context. Expressing the generating function as a polynomial in t, like those for the convex regular polytopes, requires us to find closed-form formulae for the columns N_k, as the coefficients of t^n. However, this is not so easy, for these are rather complex and do not seem to follow a regular pattern. On the other hand, finding the nth term in each column for each k is relatively straightforward, for all we need to do is successively set k = 1, 2, 3 … in the function for $A(n, k)$.

For instance, entry A002054 in the OEIS gives this generating function for the number of edges in each associatope, where C is the generating function for the Catalan numbers:

$$\frac{xC^4}{2-C} \quad \text{where} \quad C = \frac{1-\sqrt{1-4x}}{2x}$$

In full, Wolfram Alpha gives:

$$\frac{(1-\sqrt{1-4x})^4}{16\left(2-\frac{1-\sqrt{1-4x}}{2x}\right)x^3} = x+5x^2+21x^3+84x^4+330x^5+\cdots$$

OEIS A002055, defined as 'Number of diagonal dissections of a convex n-gon into n - 4 regions' gives this generating function, with its offset adjusted for the faces in associatopes:

$$\frac{16x^2(x+\sqrt{1-4x})}{(\sqrt{1-4x}+1)^2(1-4x)^{3/2}} = x^2+9x^3+56x^4+300x^5+1485x^6+\cdots$$

In turn, OEIS A002056 gives this generating function for the cells or polyhedra in associatopes, with its offset again suitably adjusted:

$$\frac{\frac{10x^4-50x^3+40x^2-11x+1}{(1-4x)^{5/2}}+x-1}{2x^2} = x^3+14x^4+120x^5+825x^6+5005x^7+\cdots$$

As there does not seem to be any obvious pattern in these generating functions, it is perhaps not surprising that the OEIS does not provide such a function for the next in the sequence.

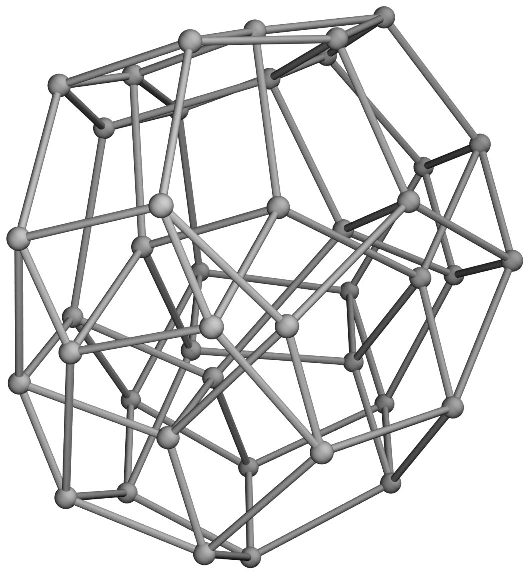
On the other hand, the diagonals, denoting the facets, ridges, peaks, and so on, enumerating the diagonal dissections of a convex (n + 3)-gon into 2, 3, and 4, etc. regions, respectively, are somewhat simpler. Essentially, this is because setting $k = n - 1$, $n - 2$, etc. in $A(n, k)$ frees the sequences from the complications of the central binomial coefficients, which are closely related to the Catalan numbers, as we have seen. The OEIS entries in this table after the first have different offsets because they are related to the dissections of polygons, not to diagonals in the associatopic triangle.

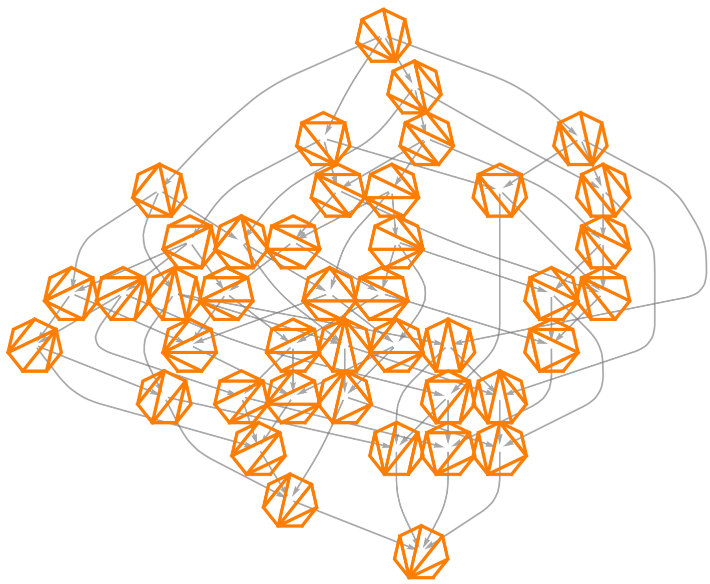
Elements	OEIS	nth term	Generating function
Facets	A000096	$n(n+3)/2$	$\frac{x(2-x)}{(1-x)^3} = 2x+5x^2+9x^3+20x^4+27x^5+\cdots$
Ridges	A033275	$(n-1)n(n+3)(n+4)/12$	$\frac{x^2(5-4x+x^2)}{(1-x)^5} = 5x^2+21x^3+56x^4+120x^5+225x^6+\cdots$

Elements	OEIS	*n*th term	Generating function
Peaks	A033276	$(n-2)(n-1)n(n+3)(n+4)(n+5)/144$	$\frac{x^3(14-14x+6x^2-x^3)}{(1-x)^7} = 14x^3+84x^4+300x^5+825x^6+1925x^7+\cdots$

Turning now to a more topological perspective, algebraic topologists view associatopes somewhat differently from combinatorialists. As I am not a specialist in either of these fields, or in any other discipline of human learning, for that matter, all I can do here is mention the little that I have discovered so far. In 'The Multiple Facets of the Associahedron', Jean-Louis Loday says that it is 'fairly simple' to construct a $K_n + 1$ from a K_n along the lines of the simplexes or hypercubes, as described above, but 'slightly more involved'.[619] Well, such a construction method may be fairly simple, but I have not yet understood the complications involved.

To illustrate the challenge of understanding the higher dimensional associatopes, here are diagrams of a three-dimensional wire-frame model of K_6 in four dimensions and of the Tamari lattice T_5, which Robert Dickau created with *Mathematica* and presented as a Wolfram Demonstrations Project.[620] I don't think the wire frame is a Schlegel model, viewing the associachoron through one face, a 3D printed version of which is available from shapeways.com.[621] The sequence of heptagonal triangularizations has a bijective correspondence to the ordering of Dyck paths, viewed as binary numbers (OEIS A063171), which Antti Karttunen presents in a software-generated pdf file.[622] Hence, paths through the lattice can also be traced with Tamari's encoding of the vertices and in the decimals corresponding to the binary ordering (OEIS A014486).





What I know from the associatopic triangle above is that these figures have 42 vertices, 84 edges, 56 polygons, and 14 polyhedra, as cells or facets of the associachoron. But, how many polygons are squares and pentagons? Well, the wire frame seems to suggest that these are the only types of polygon in K_6. But, how many of each? Well, on the assumption that the polygons are all either squares or pentagons, a simple calculation shows their distribution in the first several associatopes is listed in this table:

Polygon\\K_n	OEIS	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}
Pentagons	A002694	1	6	28	120	495	2002	8008	31824	125970
Squares	A074922	0	3	28	180	990	5005	24024	111384	503880
Total	A002055	1	9	56	300	1485	7007	32032	143208	629850

What this table indicates is that the polyhedral cells in K_6 are not all K_5. We can also see this from the wire-frame, which shows squares sharing edges with squares, edges that do not exist in the associahedron.

I did explore this a little further, and found that fourteen vertices at the beginning and end of the lattice *do* form 9-face, 21-edge associahedra, which are connected directly to each other by two squares. However, how the remaining fourteen vertices in the associachoron are connected through the remaining ten edges in both of these associahedra and to each other, thus forming twelve more polyhedra with unknown numbers of vertices, edges, and faces, is far from clear. No doubt it would be possible to investigate further, but I have neither the patience nor the mathematical or programming skills to do so.

Rather, we can simply make a connection with the combinatorics through the sequences in the OEIS. The number of pentagons in each associatope is given by the third factor in $A(n, k)$, for $k = 2$ and $n = n - 1$:

$$\binom{2(n-1)}{n+1}$$

So the number of squares in n-dimensional associatopes, for $n \geq 2$, is:

$$\frac{1}{n+1}\binom{n+1}{2}\binom{2n}{n+2} - \binom{2(n-1)}{n+1} = \frac{n-3}{2}\binom{2(n-1)}{n-3}$$

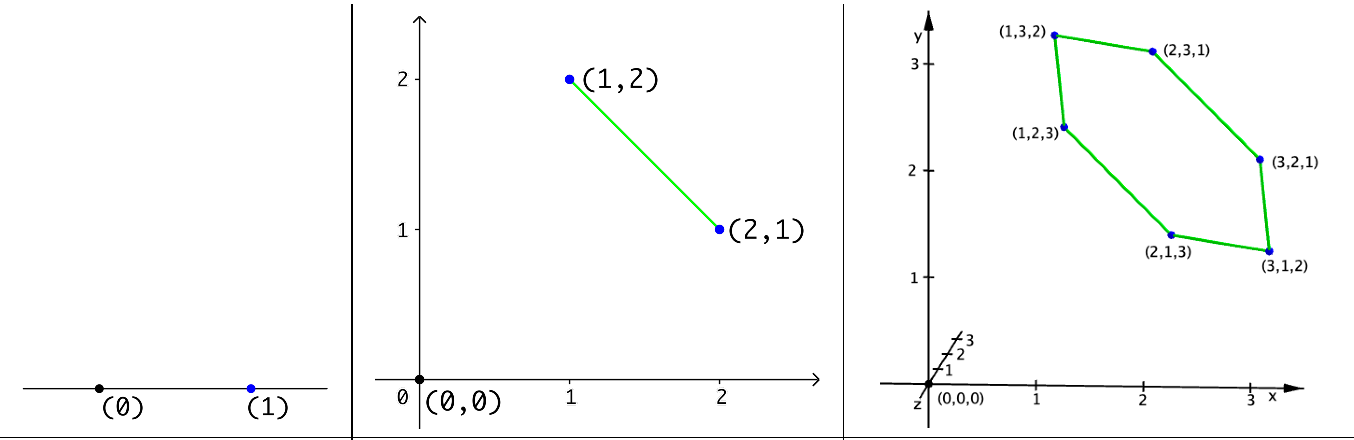
There are two other interesting features of associatopes that I could come back to at another time. First, as the Narayana numbers enumerate the peaks in Dyck paths, as mentioned on page 214, and as the Schröder paths either flatten the peaks or chop them off, there is a relationship between associatopes and the Narayana numbers, which I don't yet understand. Secondly, associatopes can be formed, together with strange beasts called cyclohedra or cyclotopes, by removing facets from permutatopes.[623] Perhaps I could gain a little insight into what this means by exploring how hexagons appear to generate permutatopes in higher dimensions.

The word *permutatope* derives from *permutation*, from Latin *permūtātio*(*n*-) 'a complete change', from *permūtāre* 'change thoroughly', from *per*- 'thoroughly' and *mūtāre* 'to change'. This is different from the conventional term *permutohedron*, from French *permutoèdre*, which Georges-Théodule Guilbaud (1912–2008) and Pierre Rosenstiehl coined in 1963, saying '*le mot* permutoèdre *est barbare, mais il est facile à retenir; soumettons-le aux critiques des lecteurs*," "the word *permutohedron* is barbaric, but it is easy to remember; let's submit it to readers' criticism."[624]

Well, *permutohedron* seems unsuitable on two counts. First, the suffix -*hedron* refers to 3-dimensional polytopes, so it is misleading to extend words with this suffix into higher dimensions, as has also been done with *associahedron*. A suffix -*tope* is more appropriate. Secondly, the root of the prefix is *permūtātio*. So why change -*a*- to -*o*-, the conventional infix for Greek words? After all, we already have *tetrahedron* and *icosahedron*, for instance, not *tetrohedron* and *icosohedron*.

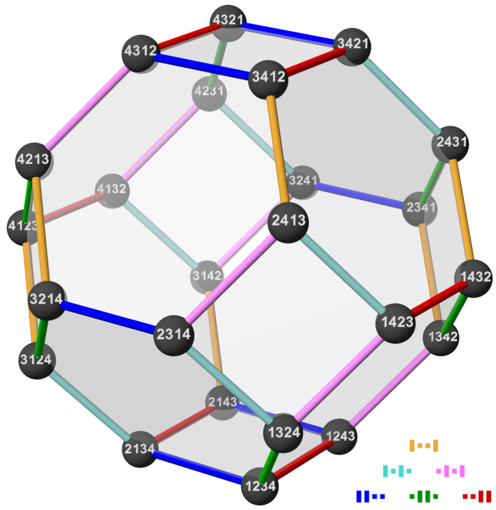
However, Günter M. Ziegler says that permutatopes were first studied as far back as 1911,[625] when Pieter Hendrik Schoute analytically extended uniform polyhedra into higher dimensions[626] with the assistance of Alicia Boole Stott's geometric treatment from the previous year.[627] Using a more conventional mathematical approach, Schoute explained how she had shown how the uniform polyhedra in any number of dimensions could be expanded by a systematic method from the regular polytopes, including the truncated octahedron, which is the form of a permutahedron in three dimensions.

However, permutatopes are not ordered in terms of their dimensions, for the vertices of permutahedron, for instance, mark all permutations of four characters. So, permutatopes are more readily seen as hyperplanes of a one-higher dimensional space, where sets of vertices, numbered as permutations of (1), (1, 2), (1, 2, 3), etc., denote coordinates in Euclidean geometry, in relationship to the origin. Here then are diagrams of the first three orders of permutatopes, corresponding to α_0, α_1, and α_2, and γ_0, γ_1, and γ_2, for the simplexes and hypercubes, respectively.

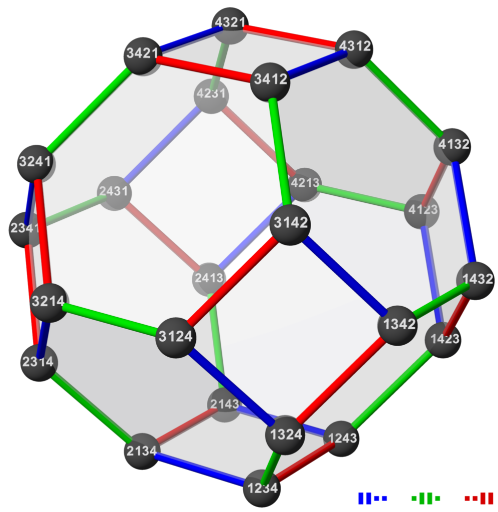


P_1 | P_2 | P_3

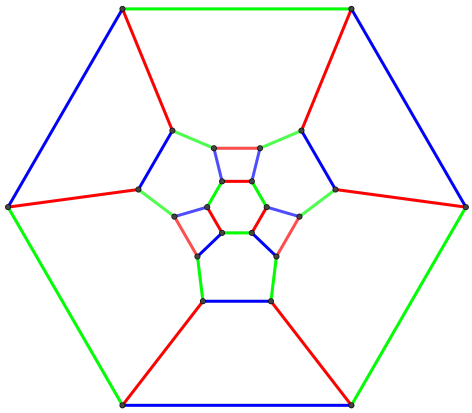
However, extending hexagons, rather than triangles and squares, into higher dimensions is a little more complicated. Thankfully, it is somewhat simpler than the growth of pentagons in associatopes because there is a higher degree of symmetry, generating uniform polytopes. They are based on the symmetric group S_n, which encompasses all groups of order *n*, as we see in the next chapter on 'Universal Algebra'. As the number of ways of permuting *n* elements is *n*!, the number of vertices in P_n is *n* times the number in P_{n-1}.

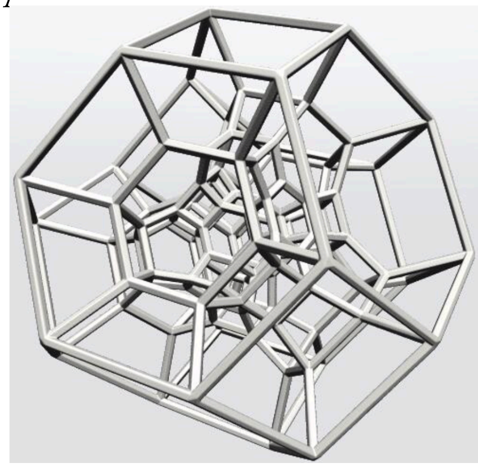


Here is a Wikipedia model of P_4, showing the hexagon in P_3 in the top-left, behind the front faces, with a fourth element added in the first position. The edges represent the relationships between permutations where just two positions are swapped. The legend in the bottom right-hand corner indicates that there are six ways of doing this in the permatahedron, suitably coloured in the diagram. The six squares and eight hexagons thus represent 4- and 6-cycles of permutations, with two and three pairs of transitions. However, these are not the only possible cycles in pentagonal faces. Cycles exist where only adjacent elements are swapped, as this second diagram from Wikipedia illustrates, corresponding to a Cayley graph of the symmetric group S_4.



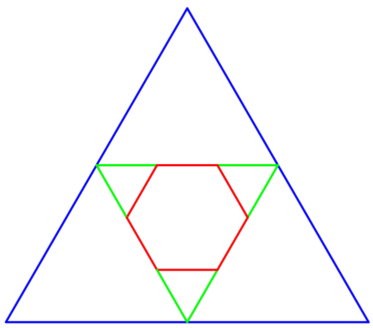
Moving into higher dimensions, labelled graphs of vertices and edges become difficult to decipher, as diagrams on hexnet.org [628] and Wikipedia illustrate. [629] So, it is easier to visualize the permutachoron as a 3-dimensional projection using 3D printing, available from shapeways.com to purchase. As the next best thing, this model can also be explored from various angles interactively on the Shapeways website, as can the associachoron. [630] Here is a screenshot of their model, together with a Schlegel diagram for the permutohedron, coloured as a Cayley graph, as P_4 and P_5:





Another name for the permutachoron is an omnitruncated 5-cell and for P_n for $n > 5$ the conventional names are omnitruncated (n - 1)-simplexes. To understand how permutatopes grow indefinitely, we need to understand how P_3 and P_4 are formed from triangles and tetrahedra, as 2- and 3-simplexes. What 'omnitruncated' refers to here is a two-step process, not just simple truncation, which is a secondary step. The first involves a more drastic truncation, called rectification, which is the process of marking the midpoints of all a polytope's edges, and cutting off its vertices at those points.

For instance, the rectified cube and octahedron is the cuboctahedron, which is the polyhedron at the centre of the compound of these polyhedra, as duals of each other. In the case of a tetrahedron, which is self-dual, the result of rectification is the octahedron, at the centre of what Kepler called the stella octangular. It is this rectified tetrahedron that is then truncated to form the truncated octahedron, as the omnitruncated 3-simplex or permutahedron. In extended Schläfli notation, P_4 is denoted as tr{3,3}. It is even possible to form a hexagon, as P_3 or tr{3}, from a triangle, as this diagram illustrates. In general, P_n is a tr{3,3,…3,3}, with (n – 2) 3's.



One fascinating feature of permutatopes is that they tessellate the space in which they live as honeycombs, filling it entirely. This characteristic is illustrated for hexagons and truncated octahedra on page 195. However, it holds for any number of dimensions because of the way that permutatopes are formed with vertices at all permutations of a tuple (x_1, x_2, x_3, … x_n) in modular arithmetic, as Wikipedia explains quite well.[631]

Now, as the generating column of vertices is $(n + 1)!$, starting from a single point, and as the Stirling numbers are concerned with permutations, the number of each type of element in permutatopes is found by multiplying the mirror of the Stirling numbers of the second kind, listed on page 230, (OEIS A008278)—adjusted to offset 0, to include zero-dimensional polytopes—by the factorials of the reverse dimensions (k) in each n-permutatope, giving:

$$P(n,k) = (n-k+1)!\,S(n+1,n-k+1) \qquad 0 \le k \le n$$

From this function, we can generate a table of elements in each permutatope, as follows, where the sum of each row is known as 'ordered-Bell numbers', which Comtet also called 'Fubini numbers', from the discrete analogue of a theorem for multiple integrals that Guido Fubini (1879–1943) had developed.[632]

$n\backslash N_k$	N_0	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	Σ
0	1											1
1	2	1										3
2	6	6	1									13
3	24	36	14	1								75
4	120	240	150	30	1							541
5	720	1800	1560	540	62	1						4683
6	5040	15120	16800	8400	1806	126	1					47293
7	40320	141120	191520	126000	40824	5796	254	1				545835
8	362880	1451520	2328480	1905120	834120	186480	18150	510	1			7087261
9	3628800	16329600	30240000	29635200	16435440	5103000	818520	55980	1022	1		102247563
10	39916800	199584000	419126400	479001600	322494480	129230640	29607600	3498000	171006	2046	1	1622632573

Number of k*-units in each* n*-permutatope*

This permutatopic lower triangular matrix is A090582 in the OEIS, but indirectly related to probabilities, not labelled as the elements in permutatopes. Rather, what is of interest here is how many hexagons and squares there are in each permutatope. As there are (n - 1) faces at each edge (E) and as the total number of faces (F) is given by the $P(n, k)$ function, we can calculate the number of hexagons (H), as we did for pentagons in associatopes, as:

$$H = \frac{1}{2}((n-1)E - 4F) \quad n > 1$$

Hence, the number of squares (S) in each permutatope is:

$$S = F - H$$

These formulae generate two sequences, like those for pentagons and squares in associatopes, and their sum, as the number of polygons in each permutatope.

Polygon\\P_n	OEIS	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
Hexagons	A005990	1	8	60	480	4200	40320	423360	4838400	59875200
Squares	A317487	0	6	90	1080	12600	151200	1905120	25401600	359251200
Total	A037960	1	14	150	1560	16800	191520	2328480	30240000	419126400

There is no need to stop here, as we need to do for associatopes, for the edge lengths of permutatopes are all equal to $\sqrt{2}$, generating uniform polytopes and their constituents. For instance, the first person to explore P_5, as the permutachoron, seems to have been Alicia Boole Stott's brother-in-law C. Howard Hinton in his book *The Fourth Dimension* in 1904.[633] So, Coxeter called the omnitruncated 5-cell Hinton's polytope, forming a uniform honeycomb, as Hinton's honeycomb. Here, just as hexagons do not provide all the facets of the permutahedron, truncated octahedra do not occupy all the facets or cells of the permutachoron. There are just ten of them, with the other twenty uniform polyhedra being hexagonal prisms.[634]

Moving to the next dimension, the omnitruncated 5-simplex, as a P_6, has 360 P_4's, 90 hexagonal prisms, and 90 cubes, as cells or ridges, giving 540 in total. Its 62 facets consist of 12 P_5's, 30 truncated octahedral prisms, and 20 6-6 duoprisms,[635] all uniform polychora. Clearly, the k-units of higher dimensional permutatopes form many different constructs, which do not need to concern us further.

Perhaps the only remaining task is to explore the combinatorics a little. First, Tom Copeland, a regular contributor to the OEIS, gives this exponential generating function for all the elements in permutatopes:

$$\frac{1}{1+\frac{1-e^{xt}}{t}} = \frac{t}{1+t-e^{xt}} = 1 + x + \frac{1}{2!}(2+t)x^2 + \frac{1}{3!}(6+6t+t^2)x^3 + \frac{1}{4!}(24+36t+14t^2+t^3)x^4 + \cdots$$

However, the egf generates a term for dimension -1 at the beginning, with the coefficients of t^n enumerating the elements in P_n, not with n denoting dimension. The egf for the Fubini or ordered-Bell numbers (OEIS A000670) is even simpler, found by setting $t = 1$, again with a redundant term at the beginning:

$$\frac{1}{2-e^x} = 1 + x + \frac{3x^2}{2!} + \frac{13x^3}{3!} + \frac{75x^4}{4!} + \cdots$$

The individual columns, counting vertices, edges, faces, and cells, etc., seem to have few combinatorial interpretations. First, the generating function for the vertices—as the sum of the Stirling numbers of the first kind (OEIS A000142)—is the most basic of all, but viewed as an exponential generating function, where the factorials in the denominators are ignored, not as an ordinary one.

$$\frac{1}{1-x} = \frac{1}{0!} + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{6x^3}{3!} + \frac{24x^4}{4!} + \cdots$$

Edges in permutatopes are best known combinatorially as unsigned Lah numbers $L(n, 2)$, defined on page 232, also related to the Eulerian numbers on page 236, as OEIS A001286, whose nth term is:

$$(n-1)n!/2$$

It has this exponential generating function:[636]

$$\frac{2x+1}{(1-x)^4} = 1 + 6x + 18x^2 + 40x^3 + 75x^4 \ldots = \frac{1}{0!} + \frac{6x}{1!} + \frac{36x^2}{2!} + \frac{240x^3}{3!} + \frac{1800x^4}{4!} + \cdots$$

After this, the number of higher dimensional elements in permutatopes expands even faster than the columns in the lower triangular matrix of the Lah numbers. However, they don't have the simple, repeating pattern that we see in the columns of the Lah numbers on page 253. For here are the relevant formulae for the faces and cells.

As already mentioned, the number of faces in permutatopes is OEIS A037960, with this nth term:

$$\frac{1}{24}n(3n+1)(n+2)!$$

It has this exponential generating function:

$$\frac{x(1+2x)}{(1-x)^5} = x + 7x^2 + 25x^3 + 65x^4 + 140x^5 \ldots = \frac{x}{1!} + \frac{14x^2}{2!} + \frac{150x^3}{3!} + \frac{1560x^4}{4!} + \frac{16800x^5}{5!} + \cdots$$

In turn, the number of polyhedral cells in permutatopes is OEIS A037961, with this nth term:

$$\frac{1}{48}n^2(n+1)(n+3)!$$

It has this exponential generating function:

$$\frac{x(6x^2+8x+1)}{(1-x)^7} = x + 15x^2 + 90x^3 + 350x^4 + 1050x^5 \ldots = \frac{x}{1!} + \frac{30x^2}{2!} + \frac{540x^3}{3!} + \frac{8400x^4}{4!} + \frac{126000x^5}{5!} + \cdots$$

Although it is not easy to see a pattern in these functions, looking at the diagonals, as $P(n, n - j)$, with $j > 0$, a pattern does appear, as we see in this table.

Elements	OEIS	nth term	EGF
Facets	A000918	$2^{n+1} - 2$	$(e^x - 1)^2 = \frac{x^2}{2!} + \frac{6x^3}{3!} + \frac{14x^4}{4!} + \frac{62x^5}{5!} + \frac{126x^6}{6!} + \cdots$
Ridges	A001117	$3^{n+2} - 3 \cdot 2^{n+2} + 3$	$(e^x - 1)^3 = \frac{x^3}{3!} + \frac{24x^4}{4!} + \frac{240x^5}{5!} + \frac{1560x^6}{6!} + \frac{8400x^7}{7!} + \cdots$
Peaks	A000919	$4^{n+3} - 4 \cdot 3^{n+3} + 6 \cdot 3^{n+3} - 4$	$(e^x - 1)^4 = \frac{x^4}{4!} + \frac{120x^5}{5!} + \frac{1800x^6}{6!} + \frac{16800x^7}{7!} + \frac{126000x^8}{8!} + \cdots$

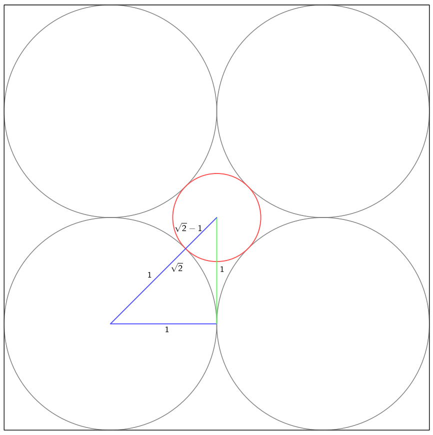
Hyperspheres

Finally, in this chapter, we look briefly at the way that circles and spheres can grow into higher dimensions, although I am not being rigorous with the mathematical terminology here. Strictly speaking, circles and spheres are 1- and 2-dimensional objects enclosing 2- and 3-dimensional objects called disks or discs and balls, respectively. It is sometimes important to make this distinction. For instance, on the surface of a sphere, the sum of the angles is greater than 2π, and we have moved into non-Euclidean geometry. Its dual is hyperbolic geometry, used in Einstein's general theory of relativity, where the angles of triangles total less than 2π.

But here, I'm just concerned with extending Cartesian geometry into higher dimensions. So just as the circle and sphere in 2- and 3-dimensional space are defined as all the points equidistant from a central point, as the origin, we can define a hypersphere in four dimensions with this equation:

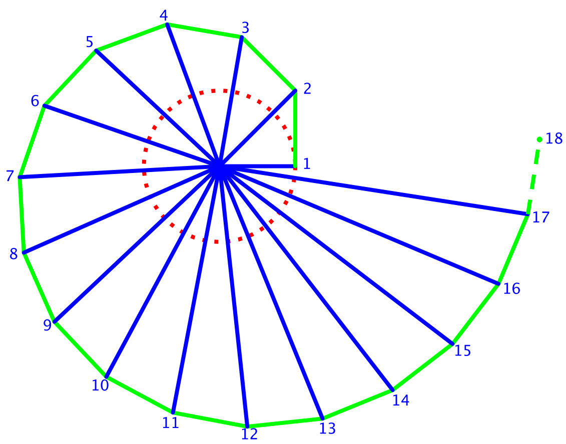
$$r^2 = x^2 + y^2 + z^2 + w^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

This representation of a hypersphere, a word that Duncan Sommerville coined in 1914,[637] can clearly be extended indefinitely into higher dimensions, with sometimes some counterintuitive, weird results, presented in three YouTube videos in 2016 and 2017. First, in the very first video in the PBS Infinite Series, which stopped production eighteen months later,[638] Kelsey Houston-Edwards gave a presentation on the unsolved problems of the optimum way of packing hyperspheres, like oranges in a crate, ending with a very strange problem of hyperspheres,[639] which Grant Sanderson further explained with his brilliant, animated graphics on his 3Blue1Brown channel.[640] Then, Matt Parker, well-known as a stand-up mathematical comedian, further explained the mathematics of this peculiar situation in two Numberphile videos,[641] also described in his book *Things to Make and Do in the Fourth Dimension*.[642]



To look at the counterintuitive way that circles grow in higher dimensions, we can simply start by packing four unit-circles into a 4 × 4 square, as illustrated in this diagram. Then, the question is what is the largest circle that can be fitted into the space in the middle? Well, as you can see, the distance from the centre of a packing disk to the centre of the square is $\sqrt{1^2 + 1^2} = \sqrt{2}$. So, as the radius of the packing disk is 1, the largest circle that can be inscribed in the space in the middle has a radius of $\sqrt{2} - 1 \approx 0.4142$.

If we now regard these four circles as the cross section through four spheres and add four more in a 4 × 4 × 4 cube, by adding another dimension, the centre of the packing spheres moves further away from the centre of the box, by the unit radius of the packing spheres, giving $\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$. So the maximum size sphere that can be fitted in the space in the centre has a radius of $\sqrt{3} - 1$. Then, moving into four dimensions, the maximum radius of the inscribed hypersphere, more properly called a hyperball, is $\sqrt{1^2 + (\sqrt{3})^2} - 1 = \sqrt{4} - 1 = 1$. The size of the central hypersphere is the same size as the packing hyperspheres!

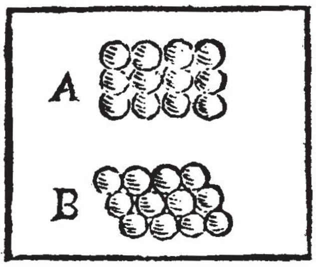


In general, as dimensions are added, the distance from the centre of the hypercube to the centres of the packing hyperspheres becomes $\sqrt{d}$, and the radius of the largest inscribed hypersphere becomes $\sqrt{d} - 1$. So, when $d = 9$, the inner radius is 3 - 1 = 2 and the inscribed hypersphere fills the entire 4^9 hypercube, while still touching the 512 packing hyperspheres contained within the box, quite astonishing. After this, the inscribed hypersphere extends beyond the enclosing hypercube. Above is a diagram of the way that the radii of the internal hyperspheres grow, as the segments outside the unit circle, denoting a packing hyperball. The partial sum of the internal angles is given by this formula:

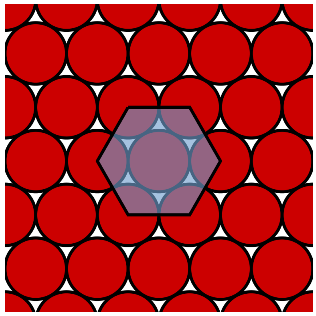
$$\sum_{d=1}^{n} \arctan \frac{1}{\sqrt{d}}$$

With n = 17, the partial sum is greater than 2π and we are back in the first quadrant. Here, we clearly have another divergent infinite series, depicted as a spiral, whose terms get closer and closer to zero, like the harmonic series.

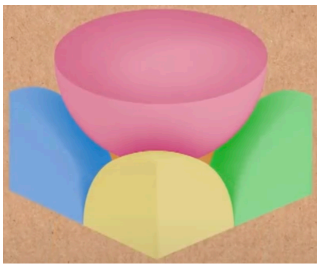
In view of the strange way that spheres pack together in higher dimensions, it is not surprising that mathematicians have had difficulty in determining optimal hypersphere packing. This was a problem that Kepler, my favourite mystical mathematician, pondered in *The Six-Cornered Snowflake* in 1611 in two and three dimensions. After saying that there must be a nonmaterial agent that causes bee honeycombs and pomegranates to form as they do—as the plant's 'soul' or life-principle—he then considered the two ways that pellets could be arranged on the horizontal plane: in square or hexagonal packing, which he illustrated thus. Extending this into three dimensions, he thought that hexagonal packing would be the best way of packing spherical balls of equal size, with each touching twelve others,[643] known as the Kepler conjecture,[644] illustrated on page 189, as tetrahedral numbers stacked in a pyramid.



This diagram from Wikipedia helps us to calculate the packing density of coins on a table, for instance. In the hexagon, there are six one-third sectors of circles plus a complete one, giving three circles in total, with an area of 3π, if the radius of each circle is 1. The hexagon consists of six equilateral triangles of base 2, giving the total area of 6√3. Thus, the packing density δ_2 is:



$$\delta_2 = \frac{3\pi}{6\sqrt{3}} = \frac{\sqrt{3}\pi}{6} = 0.9069 = 90.69\%$$



In a similar manner, James Grimes shows in a Numberphile YouTube video how a sphere can be cut up to fill a square-rectangular box that can tessellate three-dimensional space.[645] The volume of a sphere of radius one is 4π/3 and the volume of the box is 2 × 2 × √2. So, the packing density δ_3 is:

$$\delta_3 = \frac{\frac{4\pi}{3}}{4\sqrt{2}} = \frac{\sqrt{2}\pi}{6} = 0.7405 = 74.05\%$$

Sphere packing in such a lattice can be extended into higher dimensions, as this table illustrates.[646] However, it was not until 1940 that Fejes Tóth proved that the hexagonal lattice is the densest of *all* possible plane packings.[647] The three-dimensional problem, known as 'cannonball packing' took until 1998 for Thomas C. Hales to prove, eventually accepted in 2017 after an exhaustive computer search.[648] However, because regular lattice packing is not the only way of packing hyperspheres, until 2016 there was no known proof for the optimal non-lattice packing in four or more dimensions.

n	δ_n	%
2	$\sqrt{3}\pi/6$	90.69%
3	$\sqrt{2}\pi/6$	74.05%
4	$\pi^2/16$	61.69%
5	$\sqrt{2}\pi^2/30$	46.53%
6	$\sqrt{3}\pi^3/144$	37.29%
7	$\pi^3/105$	29.53%
8	$\pi^4/384$	8.07%

This situation changed when Maryna Viazovska announced a proof that the E_8 lattice provides the optimal packing in eight-dimensional space. Because the space between packed hyperspheres is constantly increasing, a space appears that is large enough to fit in another hypersphere, thereby increasing the 'kissing number', the number of hyperspheres that can touch any such hypersphere. Very shortly thereafter, Viazovska and collaborators announced a similar proof that the Leech lattice is optimal in 24 dimensions.[649]

We look further at lattices in the next chapter on 'Universal Algebra', as instances of ubiquitous graphs, illustrated by Indra's Net on the front cover of this book, depicting that none of us is ever separate from any other being, including the Supreme Being, for an instant.

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[1] Thomas Heath, *A History of Greek Mathematics, Volume I: From Thales to Euclid*, originally published, Oxford University Press, 1921, New York: Dover Publications, 1981, pp. 76–84.
[2] Eric W. Weisstein, 'Figurate Number'. From *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/FigurateNumber.html. Contains a table listing the most common types of figurate numbers.
[3] https://en.wikipedia.org/wiki/Figurate_number.
[4] Eric W. Weisstein, 'Gnomonic Number'. From *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/GnomonicNumber.html.
[5] Euclid, *The Thirteen Books of Euclid's Elements*, Vol. I, tr. Thomas L. Heath, New York: Dover Publications, Book II, Definition 2, pp. 370–372.
[6] Heath, *History of Greek Mathematics, Vol. I*, pp. 78–79.
[7] Leonard Eugene Dickson, *History of the Theory of Numbers, Volume II: Diophantine Analysis*, Carnegie Institute of Washington, 1920, p. 1.
[8] Eric W. Weisstein. 'Recurrence Equation'. From *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/RecurrenceEquation.html. Recurrence equations can be solved with the RSolve operator in WolframAlpha and *Mathematica*. https://reference.wolfram.com/language/ref/RSolve.html.
[9] Elena Deza and Michel Marie Deza, *Figurate Numbers*, Singapore: World Scientific, 2012, p. 5.
[10] Eric W. Weisstein. 'Polygonal Number'. From *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/PolygonalNumber.html.
[11] Deza & Deza, *Figurate Numbers*, p. 91 gives the sequences and formulae of 3-dimensional pyramidal numbers up to triacontagonal pyramidal numbers, i.e. with $m = 30$.
[12] Ibid., p. 50.
[13] Ibid., p. 51 gives the sequences and formulae of centred polyhedral numbers up to centred triacontagonal numbers.
[14] To distinguish centred hexagonal numbers from corner hexagonal numbers, Martin Gardner called the former hex numbers. John H. Conway and Richard Guy, *The Book of Numbers*, New York: Copernicus Books, Springer-Verlag, 1996, p. 41.
[15] Deza and Deza describe these centred m-gonal pyramidal numbers in three dimensions on pp. 138–145.
[16] Conway and Guy, *Book of Numbers*, pp. 42–43.
[17] Ibid., p. 58.
[18] https://en.wikipedia.org/wiki/Squared_triangular_number.
[19] Heath, *History of Greek Mathematics, Vol. I*, pp. 97–99 and 108–110.
[20] http://oeis.org/wiki/Platonic_numbers and http://oeis.org/wiki/Centered_Platonic_numbers
[21] Conway and Guy, *Book of Numbers*, pp. 42–43.
[22] http://oeis.org/wiki/Centered_Platonic_numbers.
[23] Hyun Kwang Kim, 'On Regular Polytope Numbers', *Proceedings of the American Mathematical Society*, Vol. 131, No. 1, 2002, pp. 65-75.
[24] H. S. M. Coxeter, *Regular Polytopes*, 3rd ed., 1st ed. 1963, New York: Dover Publications, 1973, p. 118.

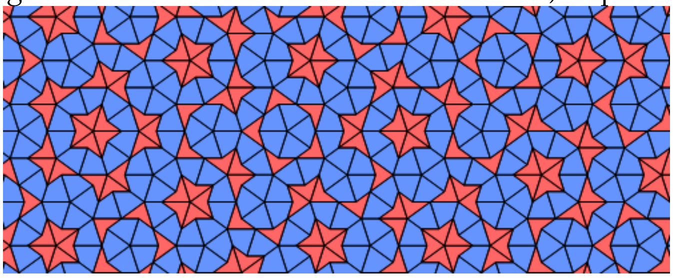
[25] http://oeis.org/wiki/Classifications_of_figurate_numbers.
[26] Conway and Guy, *Book of Numbers*, pp. 46–47. There is a typo in the formula in this book, which is reproduced on Eric W. Weisstein's page on 'Truncated Tetrahedral Number' in Wolfram's MathWorld. Wolfdieter Lang pointed out the error on 9th January 2017 on the OEIS's page for the sequence of truncated tetrahedral numbers (A005906).
[27] Eric W. Weisstein. 'Stella Octangula'. From *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/StellaOctangula.html. However, I do not find this term in either English or Latin in the Kindle version of Johannes Kepler, *The Six-Cornered Snowflake*, Paul Dry Books, 2009, which Kepler alternatively titled *A New Year's Gift*. Neither is this term in *The Harmony of the World*, in which he introduced the small and great stellated dodecahedra.
[28] Conway and Guy, *Book of Numbers*, p. 51.
[29] https://en.wikipedia.org/wiki/File:Grenat_pyrope_1.jpg.
[30] Eric W. Weisstein. 'Rhombic Dodecahedron'. From *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/RhombicDodecahedron.html.
[31] Matt Parker, *Things to Make and Do in the Fourth Dimension*, Penguin, 2014, p. 217.
[32] Deza & Deza, *Figurate Numbers*, pp. 123–124.
[33] Conway and Guy, *Book of Numbers*, p. 53.
[34] Coxeter, *Regular Polytopes*, p. 123.
[35] David M. Burton, *The History of Mathematics: An Introduction*, seventh edition, McGraw-Hill, [1998] 2011, pp. 456–461.
[36] A. W. F. Edwards, *Pascal's Arithmetical Triangle*, London: Charles Griffin, 1987, pp. ix–x.
[37] Ibid., pp. 5–6, 44, and 51.
[38] Burton, *History of Mathematics*, p. 457.
[39] Edwards, *Pascal's Arithmetical Triangle*, pp. 53–54.
[40] David Eugene Smith, ed., 'Pascal, On the Arithmetic Triangle', tr. Anna Savitsky, in *A Source Book in Mathematics*, 1st ed. 1929, Dover Publications, 1959, pp. 67–76.
[41] Edwards, *Pascal's Arithmetical Triangle*, p. 59.
[42] https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics).
[43] https://en.wikipedia.org/wiki/Binomial_coefficient.
[44] Conway and Guy, *Book of Numbers*, p. 94.
[45] https://www.reddit.com/r/askscience/comments/5tnz8e/why_does_pascals_triangle_give_the_powers_of_11/
[46] Conway and Guy, *Book of Numbers*, p. 112. This derivation is in Édouard Lucas, '*Théorie des fonctions numériques simplement périodiques*', *American Journal of Mathematics*, Vol. 1, 1878, pp. 207–208 and Édouard Lucas, 'The Theory of Simply Periodic Numerical Functions', Sections I–XXIII, tr. Sidney Kravitz, ed. Douglas Lind, *Fibonacci Association*, 1969, of pp. 32–33. An elegant proof using an ingenious way of generating Fibonacci numbers is given in Arthur T. Benjamin and Jennifer Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, The Mathematical Association of America, 2003, pp. 4–5.
[47] https://en.wikipedia.org/wiki/Pascal%27s_pyramid.
[48] https://people.hofstra.edu/Eric_Rowland/investigations/pascalssimplices.html gave me this idea.
[49] Édouard Lucas, '*Théorie des fonctions numériques simplement périodiques*', *American Journal of Mathematics*, Vol. 1, 1878, pp. 184–240 (Sections I–XXIII) & 289–321 (Sections XXIV–XXX).
[50] Édouard Lucas, 'The Theory of Simply Periodic Numerical Functions', Sections I–XXIII, tr. Sidney Kravitz, ed. Douglas Lind, *Fibonacci Association*, 1969.
[51] Paulo Ribenboim, *My Numbers, My Friends: Popular Lectures on Number Theory*, New York: Springer, 2000, p. 2.
[52] Ibid., p. 5. The reference is to J. P. M. Binet. 'Mémoire sur l'intégration des équations linéaires aux différences finies, d'un ordre quelconque, á coefficients variables'. *C. R. Acad. Sci. Paris*, 1843, 17:559–567.
[53] Ibid., p. 2.
[54] Charles F. Linn, *The Golden Mean: Mathematics and the Fine Arts*, Garden City, N.Y.: DoubleDay, 1974 is referenced in the literature, but I have not been able to find a copy.

[55] Euclid, *The Thirteen Books of Euclid's Elements*, Vol. II, tr. Thomas L. Heath, New York: Dover Publications, Book VI, Definition 3, p. 188.
[56] Carl B. Boyer and Uta C. Merzbach, *A History of Mathematics*, third edition, significantly revised, originally published, 1968, New York: John Wiley, 2011, p. 267. The reference is to Piero della Francesca, *Libellus de Quinque Corporibus Regularibus* (*Short Book on the Five Regular Solids*), in which Piero "noted the 'divine proportion' in which the diagonals of a regular pentagon cut each other".
[57] H. S. M. Coxeter, *Introduction to Geometry*, New York: John Wiley, 1961, pp. 160 and 162.
[58] Mario Livio, *The Golden Ratio: The Story of Phi, the Extra ordinary Number of Nature, Art and Beauty*, London: Review, 2002, pp. 6–7. Ohm wrote in a footnote to the second edition of *Die Reine Elementar Mathematik* (*The Pure Elementary Mathematics*), "One also customarily calls this division of an arbitrary line in two such parts the golden section," indicating that the term was commonly accepted.
[59] Livio, *Golden Ratio*, pp. 5–6.
[60] Johannes Kepler, *The Harmony of the World*, translated from Latin with an introduction and notes by E. J. Aiton, A. M. Duncan, and J. V. Field, first published as *Harmonice Mundi*, American Philosophical Society, 1997, pp. 111 and 116-117.
[61] Max Caspar, *Kepler*, tr. C. Doris Hellman, 1st ed., 1959, New York: Dover, 1993, pp. 50–53.
[62] Ibid., pp. 60–71.
[63] Karl Fink, *A Brief History of Mathematics*, tr. Wooster Woodruff Beman and David Eugene Smith from *Geschichte Der Elementar Mathematik*, Chicago: Open Court, 1900, p. 223, requoted in Boyer and Merzbach, *A History of Mathematics*, p. 46. However, this is a paraphrase of A. M. Duncan's translation of Kepler's 1597 book *Mysterium Cosmographicum* (*The Secret of the Universe*), New York: Abaris Books, 1981, p. 143:

> *That is, that there are two treasure houses of geometry.* They are two theorems of infinite usefulness, and so of the greatest value; but yet there is a great difference between the two. For the former-that the squares of the sides of a right triangle are equal to the square of the hypotenuse-that, I say, can rightly be compared to a mass of gold; the second, on proportional division, can be called a jewel. For in itself it is indeed splendid, but without the previous theorem it has no force; but it then takes knowledge further, when the previous one which has carried us so far deserts us; that is, on the derivation and discovery of the side of the decagon and related quantities.

[64] Johannes Kepler, *The Six-Cornered Snowflake*, tr. Colin Hardie of *A New Year's Gift*, or *On the six-cornered Snowflake*, 1611, Oxford University Press, 1966, p. 21.
[65] Ball, *Strange Curves,* p. 155. Also, Eric W. Weisstein. 'Binet's Fibonacci Number Formula'. From *MathWorld*—A Wolfram Web Resource.
http://mathworld.wolfram.com/BinetsFibonacciNumberFormula.html
[66] Lucas, '*Théorie des fonctions numériques simplement périodiques*', p. 186, refers to *Il liber Abbaci di Leonardo Pisano, pubblicato secondo la lezione del Codice Magliabechiano,* da B. BONCOMPAGNI. Roma, 1867. Pag. 283 et 284.
[67] Keith Devlin, *Finding Fibonacci: The Quest to Rediscover the Forgotten Mathematical Genius Who Changed the World*, Princeton University Press, 2017, p. 128.
[68] Susantha Goonatilake, *Toward a Global Science: Mining Civilizational Knowledge*, Indiana University Press, 1998, p. 126.
[69] Keith Devlin, *The Man of Numbers: Fibonacci's Arithmetic Revolution*, London: Bloomsbury, 2011, p. 64.
[70] L. E. Sigler, *Fibonacci's* Liber Abaci*: A Translation into Modern English of Leonardo Pisano's Book of Calculation*, tr. from a 1228 ed., published in 1857 by Baldassarre Boncompagni, New York: Springer-Verlag, 2003, p. 404.
[71] Ibid., pp. 404–405.
[72] Burton, *History of Mathematics*, p. 287.
[73] Sigler, *Fibonacci's* Liber Abaci, p. 404.
[74] Benjamin and Quinn, *Proofs that Really Count*, p. 2.
[75] Conway and Guy, *Book of Numbers*, p. 111.
[76] H. E. Huntley, *The Divine Proportion: A Study in Mathematical Beauty*, Mineola, N.Y.: Dover Publications, 1970, p. 159–160, and Livio, *Golden Ratio*, p. 100.

[77] T. C. Scott and P. Marketos, 'On the Origin of the Fibonacci Sequence', http://www-history.mcs.st-andrews.ac.uk/Publications/fibonacci.pdf.
[78] Sigler, *Fibonacci's* Liber Abaci, p. 15.
[79] Livio, *Golden Ratio*, pp. 79 and 203–206. Inspired by Kepler's studies of the Divine Proportion in *The Harmony of the Universe*, Roger Penrose created a number of nonperiodic tilings, such as this one, which joins pairs of Golden Triangles and Gnomons to form kites and darts, respectively.



Another example is in Chapter 3.
[80] Eric W. Weisstein. 'Pythagoras's Constant'. From *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/PythagorassConstant.html
[81] Leonard Eugene Dickson, *History of the Theory of Numbers, Volume II: Diophantine Analysis*, Carnegie Institution of Washington, 1920, p. 341.
[82] André Weil, *Number Theory: An approach through history from Hammurapi to Legendre*, Boston: Birkhäuser, 1987, p. ix. Fermat was followed by Euler, Lagrange, and Legendre, as the four major contributors to number theory, as described in the book.
[83] J. J. O'Connor and E. F. Robertson, 'Pell's equation', *MacTutor History of Mathematics Archive*, http://www-history.mcs.st-andrews.ac.uk/HistTopics/Pell.html.
[84] Florian Cajori, *A History of Mathematical Notations: Two Volumes Bound As One; Volume I: Notations in Elementary Mathematics*, 1st ed., 1928, New York, Dover Publications, 1993, pp. 270–271. Although ÷ had previously been used as a minus sign, Rahn introduced it as a sign of division in 1659 in a book titled *Teutsche Algebra*, which Thomas Brancker translated into English, with additions by John Pell in 1668. This gave John Collins the impression that Pell had introduced this sign, mistakenly calling it 'Pell's symbol', another case of misattribution.
[85] Even though John Pell had little to do with the solutions to the so-named Pell's equations, he was a fascinating character who worked with Samuel Hartlib, who translated and published *A Reformation of Schooles* by John Amos Comenius in 1642, the foremost promoter of Pansophy, as 'Universal Wisdom', Pell meeting Comenius in London in 1641. (Noel Malcolm, 'The Publications of John Pell, F.R.S. (1611–1685): Some New Light and Some Old Confusions', *Notes and Records of the Royal Society of London*, Vol. 54, No. 3, Sep. 2000, pp. 275–292.)

Like Comenius, Pell was an encyclopaedic thinker, interested in the reformation of teaching methods. Most famously, he wrote a pamphlet titled *An Idea of Mathematicks*, written in the form of a letter to Hartlib, published in both English and Latin in 1638. Pell introduced this set of proposals for the advancement of mathematics in a letter to Thomas Goad that year in this way, "… there are some y^t have seene the delineation of my Idea of y^e perfecting, reforming, advancing & facilitating y^e whole study from y^e first seedes of pure Mathematickes to their highest & noblest, as well as the meanest & most ordinary applications with my designes to y^e end." Apparently, he was seeking patronage for his proposal, envisaging that it could "be performed by one man, without any assistants, provided that he were neither distracted with cares for his maintenance, nor diverted by other employments"!
[86] O'Connor and Robertson, 'Pell's equation'.
[87] H. W. Lenstra Jr. 'Solving the Pell Equation', *Notices of the AMS*, Vol. 49, No. 2.
[88] Derek Smith, 'The Search for an Exhaustive Solution to Pell's Equation', Web Archive.
[89] Eric W. Weisstein. 'Newton's Iteration'. From *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/NewtonsIteration.html.
[90] http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html#moresilver.

[91] http://www.mi.sanu.ac.rs/vismath/spinadel/index.html.
[92] https://en.wikipedia.org/wiki/Metallic_mean.
[93] Keith Ball, *Strange Curves, Counting Rabbits, and Other Mathematical Explorations*, Princeton University Press, 2003, p. 168.
[94] Generated with the MatrixPower function in the symbolic Wolfram language.
[95] https://en.wikipedia.org/wiki/Stigler%27s_law_of_eponymy.
[96] Victor J. Katz, *A History of Mathematics: An Introduction*, Addison Wesley, 2008, p. 251.
[97] Ibid., pp. 293–297.
[98] https://www.math.ucla.edu/~pak/hidden/papers/Quotes/Leibniz-Arte-Combinatoria.pdf.
[99] Eberhard Knobloch, 'The Mathematical Studies of G. W. Leibniz on Combinatorics', *Historia Mathematica*, 1974, pp. 409–430.
[100] Percy A. MacMahon, *Combinatory Analysis: Volume I*, Cambridge University Press, 1915, p. v.
[101] John Riordan, *An Introduction to Combinatorial Analysis*, New York, John Wiley, 1958, pp. vii–viii.
[102] Herbert John Ryser, *Combinatorial Mathematics*, Mathematical Association of America, 1963, pp. 1–3.
[103] Antony Flew, *Philosophy: An Introduction*, Sevenoaks, England: Teach Yourself Books, Hodder & Stoughton: 1979, pp. 3–7.
[104] Igor Pak, 'History of Catalan Numbers', 27th August 2014, reproduced as Appendix B in Richard P. Stanley, *Catalan Numbers*, Cambridge University Press, 2015, p. 186 in pp. 177–189.
[105] Peter J. Larcombe announced the earlier discovery of the Catalan numbers in P. J. Larcombe, 'The 18th century Chinese discovery of the Catalan numbers' *Mathematical Spectrum*, 1999, Vol. 32, No. 1, pp. 5–7. He had discovered this in J. J. Luo, 'Antu Ming, the first inventor of Catalan numbers in the world, *Neimenggu Daxue Xuebao*, Vol. 19, 1988, pp. 239–245, written Chinese. Using Maclaurin series, introduced by a Jesuit priest, Ming found Catalan numbers in expansions of $\sin n\alpha$, which I still have to investigate.
[106] https://oeis.org/search?q=A000108.
[107] N. J. A. Sloane and Simon Plouffe, *The Encyclopedia of Integer Sequences*, San Diego: Academic Press, 1995, Figure M1459 'Catalan numbers'.
[108] Iohannes Andreas de Segner, '*Enumeratio modorum quibus figurae planae rectilineae per diagonales dividuntur in triangula*', *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* 7 (dated 1758/59, published in 1761), pp. 203–210.
[109] Pak, 'History of Catalan Numbers', pp. 181–182 in Stanley, *Catalan Numbers*.
[110] G. Lamé, '*Extrait d'une lettre de M. Lamé à M. Liouville sur cette question: Un polygone convexe étant donné, de combien de manières peut- on le partager en triangles au moyen de diagonales?*', *Journal de mathématiques pures et appliquées 1re série*, tome 3, 1838, pp. 505-507. English translation by David Pengelley in Desh Ranjan, 'Counting Triangulations of a Convex Polygon'.
[111] E. Catalan, '*Note sur une Équation aux différences finies*', *Journal de mathématiques pures et appliquées 1re série, tome* 3, 1838, pp. 508-516.
[112] On page 108 in Thomas Koshy, *Catalan Numbers with Applications*, Oxford University Press, 2008, Koshy tells us that Euler published this formula in the same volume of *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* in which von Segner published his recurrence formula. However, it seems that they were unable to prove the link between them.
[113] John Riordan, *Combinatorial Identities*, New York: John Wiley, 1968, pp. 82, 101, 130, 153, 156, 168, and 238.
[114] Pak, 'History of Catalan Numbers', pp. 186–187 in Stanley, *Catalan Numbers*.
[115] Richard P. Stanley, *Catalan Numbers*, Cambridge University Press, 2015, p. vii.
[116] Ibid., p. 15.
[117] Ibid., pp. 8–11.
[118] http://www.durangobill.com/BinTrees.html.
[119] Richard P. Stanley, *Enumerative Combinatorics: Volume I*, 2nd ed., 1st ed. 1997, Cambridge University Press, 2011, pp. 572–573.
[120] Stanley, *Catalan Numbers*, p. 11.

[121] I discovered this triangle in https://en.wikiversity.org/wiki/Partition_related_number_triangles, which contains a number of other triangles and graphical representations.
[122] http://oeis.org/A014486/a014486.pdf.
[123] https://oeis.org/wiki/Combinatorial_interpretations_of_Catalan_numbers.
[124] Garrett Birkhoff, *Lattice Theory*, 2nd ed., American Mathematical Society, 1948, p. iii.
[125] Wikipedia articles on 'Bartel Leendert van der Waerden' and 'Moderne Algebra'.
[126] George Grätzer, *Lattice Theory: First Concepts and Distributive Lattices*, San Francisco: W. H. Freeman, 1971, p. vii.
[127] Birkhoff, *Lattice Theory*, pp. iii–iv.
[128] Folkert Müller-Hoissen and Hans-Otto Walther, 'Dov Tamari (formerly Bernhard Teitler)', in Folkert Müller-Hoissen, Jean Marcel Pallo, and Jim Stasheff, eds., *Associahedra, Tamari Lattices and Related Structures: Tamari Memorial Festschrift*, Basel, Switzerland: Springer, Birkhäuser, 2012, pp. 1–40.
[129] Dov Tamari, 'The algebra of bracketings and their enumeration', *Nieuw Archief voor Wiskunde*, 1962, Ser. 3, No. 10, pp. 131–146.
[130] Birkhoff, *Lattice Theory*, p. 1.
[131] Samuel Huang and Dov Tamari, 'Problems of Associativity: A Simple Proof for the Lattice Property of Systems Ordered by a Semi-associative Law', *Journal of Combinatorial Theory*, Vol. 13, Ser. A, 1972, pp. 7-13.
[132] Dov Tamari, *'Monoïdes préordonnés et chaînes de Malcev'*, Doctorat ès-Sciences Mathématiques Thèse de Mathématiques, Université de Paris, 1951.
[133] Dov Tamari, *'Monoïdes préordonnés et chaînes de Malcev'*, Bulletin de la Société Mathématique de France, Vol. 82, 1954, pp. 53–96.
[134] Jean-Louis Loday, 'Dichotomy of the Addition of Natural Numbers', in Folkert Müller-Hoissen, et al., *Associahedra*, pp. 73–74 in pp. 65–79.
[135] Birkhoff, *Lattice Theory*, p. 6.
[136] Jim Stasheff, 'How I 'met' Dov Tamari', in Folkert Müller-Hoissen, et al., *Associahedra*, pp. 45 and 48–49 in pp. 45–63.
[137] Mark Haiman, 'Constructing the Associahedron', 1984, handwritten document available from: https://math.berkeley.edu/~mhaiman/ftp/assoc/manuscript.pdf.
[138] https://commons.wikimedia.org/wiki/File:Associahedron_K5.svg.
[139] https://en.wikipedia.org/wiki/Triaugmented_triangular_prism.
[140] I have drawn this figure from *Stella4D*, an amazing program that Robert Webb developed for building models of polyhedra.
[141] https://en.wikipedia.org/wiki/Johnson_solid.
[142] https://levskaya.github.io/polyhedronisme/?recipe=C100A1t4dP3.
[143] https://en.wikipedia.org/wiki/Near-miss_Johnson_solid.
[144] Jean-Louis Loday, 'The Multiple 'Facets of the Associahedron', 25th August 2005, pp. 5–6.
[145] For reference, the fifteen chains in the decimal numbering of the elements of the Dyck lattice are:

170→172→180→184→216→232→240
170→172→180→212→216→232→240
170→172→180→212→228→232→240
170→172→204→212→216→232→240
170→172→204→212→228→232→240
170→178→180→184→216→232→240
170→178→180→212→216→232→240
170→178→180→212→228→232→240
170→178→210→212→216→232→240
170→178→180→212→228→232→240
170→202→204→212→216→232→240
170→202→204→212→228→232→240
170→202→210→212→216→232→240
170→202→210→212→228→232→240

170→202→210→226→228→232→240

146 Dickson, *History of the Theory of Numbers, Volume II*, p. 101.
147 https://www.geni.com/people/Philippe-Naudé-II/6000000002406986587.
148 Ed Sandifer, 'Philip Naudé's problem', in *How Euler Did It*, MAA Online, October 2005, http://eulerarchive.maa.org/hedi/index.html. Also in C. Edward Sandifer, *How Euler Did It*, The Mathematical Association of America, 2007.
149 William Dunham, *Euler: The Master of Us All*, The Mathematical Association of America, 1999, p. 162.
150 Leonhard Euler, *Observationes analyticae variae de combinationibus* (Various analytical observations about combinations), presented to the St. Petersburg Academy on 6th April 1741 and originally published in *Commentarii academiae scientiarum Petropolitanae* 13, 1751, pp. 64-93 (Eneström Index, E158). English translation by Jordan Bell.
151 Sandifer, 'Philip Naudé's problem'.
152 These papers were E175, '*Découverte d'une loi tout extraordinaire des nombres, par rapport à la somme de leurs diviseurs*' (Discovery of an extraordinary law of numbers in relation to the sum of their divisors), 1747/51; E191, '*De partitione numerorum*' (On the partition of numbers), 1750/53; E243, 'Observatio de summis divisorum (An observation on the sums of divisors), 1752/60; E244, '*Demonstratio theorematis circa ordinem in summis divisorum observatum*' (A demonstration of a theorem on the order observed in the sums of divisors), 1760; E394, '*De partitione numerorum in partes tam numero quam specie datas*' (On the partition of numbers into a number of parts of a given type), 1768/70; and E541, '*Evolutio producti infiniti $(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)$ [etc.] in seriem simplicem*' (The expansion of the infinite product $(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)$ [etc.] into a single series), 1780/83.
153 George E. Andrews, *The Theory of Partitions*, Cambridge University Press, 1984, p. 1.
154 J. J. Sylvester and F. Franklin, 'A Constructive Theory of Partitions, Arranged in Three Acts, an Interact and an Exodion', Johns Hopkins University Press, *American Journal of Mathematics*, Vol. 5, No. 1, 1882, pp. 251-330.
155 A. Young, 'On Quantitative Substitutional Analysis', *Proceedings of the London Mathematical Society*, Volume s1–33, Issue 1, 1900, p. 133 in pp. 97–145.
156 Louis Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, tr. from *Analyse Combinatoire*, Presses Universitaires de France, 1970 by J. W. Niehuys, D. Reidel Publishing, 1974, p. 96.
157 Ibid., p. 116.
158 Leonhard Euler, 'On the Partition of Numbers', in *Introduction to Analysis of the Infinite: Book I*, tr. John D. Blanton, 1st ed., *Introductio in analysin infinitorum*, 1748, New York: Springer-Verlag, 1988, §300, pp. 257–258. Eneström Index, E101.
159 Ibid., p. 263.
160 Jordan Bell, English translation of E158.
161 Euler, 'On the Partition of Numbers', p. 263.
162 Ibid., pp. 259–260.
163 Ibid., pp. 275–276.
164 George E. Andrews and Kimmo Eriksson, *Integer Partitions*, Cambridge University Press, 2004, pp. 8–9.
165 https://en.wikipedia.org/wiki/Pentagonal_number_theorem.
166 George Pólya, *Mathematics and Plausible Reasoning, Vol. I: Induction and Analogy in Mathematics*, Princeton University Press, 1954, p. 92.
167 Ibid., p. 93.
168 Jordan Bell, 'Euler and the Pentagonal Number Theorem', 17th August 2006.
169 Eric W. Weisstein, 'Pentagonal Number Theorem', from *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/PentagonalNumberTheorem.html.
170 Herbert W. Wilf, 'What is an Answer?'.
171 L. J. Rogers: 'Second Memoir on the Expansion of certain Infinite Products', *Proceedings of the London Mathematical Society*, Volume s1-25, Issue 1, November 1893, pp. 318–343, also dated 12th April 1893.

[172] https://royalsocietypublishing.org/doi/pdf/10.1098/rsbm.1934.0013.
[173] Percy A. MacMahon, *Combinatory Analysis: Volume II*, Cambridge University Press, 1916, pp. 33–36.
[174] Andrew Sills and Eric W. Weisstein, 'Rogers-Ramanujan Identities', from *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/Rogers-RamanujanIdentities.html.
[175] Andrews and Eriksson, *Integer Partitions*, pp. 3–4.
[176] I don't know if there are any bijective relationships between these sets, like there are for the basic partition identity. Something to be investigated.
[177] https://en.wikipedia.org/wiki/Rogers–Ramanujan_identities.
[178] G. H. Hardy and S. Ramanujan, 'Asymptotic Formulæ in Combinatory Analysis', *Proceedings of the London Mathematical Society*, Volume s2-17, Issue 1, 1918, Pages 75–115.
[179] Andrews, *The Theory of Partitions*, pp. 68–87.
[180] S. Ramanujan, 'Congruence properties of partitions', *Mathematische Zeitschrift*, 1921, Vol. 9. No. 1–2, pp. 147–153. https://zenodo.org/record/1447425/files/article.pdf.
[181] https://www.wikiwand.com/en/Ramanujan%27s_congruences.
[182] Ibid.
[183] Mu Prime Math, 'Explicit Formula for Euler's Totient Function!' 2nd September 2020, https://youtu.be/HgUfBx8Pvz0.
[184] Miran Fattah, 'Euler's Totient Function', *YouTube*, 23rd December 2016, https://youtu.be/AnPFnnpda6c.
[185] C. Edward Sandifer, *The Early Mathematics of Leonhard Euler*, The Mathematical Association of America, 2007, pp. 203–206.
[186] Leonhard Euler, *Theoremata arithmetica nova methodo demonstrate* (Demonstration of a new method in the theory of arithmetic), presented to the Berlin Academy on 8th June 1758 and to the St. Petersburg Academy on 15th October 1759 and originally published in *Novi Commentarii academiae scientiarum Petropolitanae* 8, 1763, pp. 74-104. (Eneström Index, E271). No English translation.
[187] Burton, *History of Mathematics*, p. 478.
[188] Stanley, *Enumerative Combinatorics: Volume I*, p. 26.
[189] Conway and Guy, *Book of Numbers*, pp. 92–93.
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$$a_n = 3a_{n-1} - 2a_{n-2} + 5a_{n-3} \quad a_1 = 1, a_2 = 2, a_3 = 3$$

giving this expanded generating function:

$$1 + 2x + 3x^2 + 10x^2 + 34x^2 + 97x^2 + \cdots$$

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