Prologue, Chapters 1–3, and Epilogue

Unifying Mysticism and Mathematics

To Realize Love, Peace, Wholeness, and the Truth

Paul Hague

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Paragonian Publications

Svenshögen, Sweden

Published by:

Paragonian Publications
Hällungevägen 60
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Sweden

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The most up-to-date, searchable version of this book will be available at:
http://mysticalpragmatics.net/documents/unifying_mysticism_mathematics.pdf

First edition, 2019

ISBN: 978-91-975176-9-0, 91-975176-9-0

Typeset in Adobe Caslon 12/15

The image on the front cover is a symbol of Indra's Net of Jewels or Pearls in Huayan Buddhism, visualized as a dewy spider's web in which every dewdrop contains the reflection of the light emanating from all the other dewdrops, like nodes in a mathematical graph.

Contents

Having shown how the business modelling methods used by information systems architects have evolved into Integral Relational Logic, this edition of my final book uses this commonsensical science of thought and consciousness to map the entire number system, from Zero to Transfinity.

As these correspond to *Shūnyāta* 'Emptiness' and *Satchitānanda* 'Bliss of Absolute Truth and Consciousness', this third chapter of *Unifying Mysticism and Mathematics* marks the culmination of my life's work to heal the cultural psychic split between science and spirituality.

Even though this book now feels complete, as it is, I plan to spend the first few months of 2019 writing the two remaining chapters I have sketched out.

The first on 'Sequences and Series' will describe many beautiful formations that emerge naturally when we view mathematics as a generative science of patterns and relationships, often seen in the natural world around us.

The fifth chapter, on 'Universal Algebra', goes beyond the number system, describing some of the mathematical abstractions that have now evolved into Integral Relational Logic: the Cosmic Context, Gnostic Foundation, and coordinating framework for all knowledge, which I call Panosophy or the Unified Relationships Theory, better known as the elusive Theory of Everything.

This January-2019 edition also includes a 14-page Epilogue on how I currently view the great existential crisis that humanity faces at the moment and how we could perhaps resolve it, realizing that the Divine Essence we all share is Love.

For humanity

Prologue

This book marks the glorious culmination of my lifelong quest to realize Love, Peace, Wholeness, and the Truth, which I set out to bring to light as a seven-year-old in 1949. For having been born near London in the middle of the Second World War, I could see that we humans would never be able to live in love, peace, and harmony with each other unless we could end the long-running war between science and religion, arising from a deep wound in the cultural and collective psyche.

Healing this wound in my own personal psyche has meant making radical changes to the concepts of God and the Universe, which I introjected into consciousness in childhood from the culture into which I was born. For just as I was beginning to think for myself, I could see that it did not make sense to say that God—as the Supreme Being—should reside somewhere in outer space, as indicated by the first words of the Lord's Prayer in the Anglican *Book of Common Prayer*, established in 1662, as the result of the English Reformation: "Our Father, which art in heaven, …"

The concepts of God and Universe provide the overall context for religion and science, respectively, yet they are incompatible with each other. And as we need a context to form concepts—as mental images of the world we live in—without an overall context for all our learning, we have no way of determining what is true or false. As a consequence, I learned almost nothing during my formal education. It was only after the age of thirty-eight that I began to make sense of my experiences and the world that I live in, as an autodidact.

As I have discovered through seventy years of questioning the fundamental assumptions of Western civilization and forty years of profound self-inquiry, the discordant concepts of God and the Universe can only be reconciled by plunging into the Nondual, mystical foundations of mathematics, recognizing Consciousness as Ultimate Reality and that none of us is ever separate from any other being, including the Supreme Being, for an instant.

I have come to this realization as the result of a life-changing death and rebirth process I went through in the spring of 1980. When working in an innovative marketing job for IBM in London, having some influence in Europe, I realized that the global economy is inherently unstable and that my children were not being educated to live in the world that would exist when they came to be bringing up children of their own, as my daughter, at least, is doing.

What triggered me to abandon my family and business career in search of Wholeness and the Truth was an apocalyptic awakening that I experienced at 11:30 on Sunday 27th April 1980, as I was strolling across Wimbledon Common to the pub for lunch. Puzzling about what is causing technologists like myself to drive the pace of change in society at exponential rates of acceleration, I had the idea that there are nonphysical mental energies at work within our psyches, as well as the physical energies I learned about in school. For *apocalypse* derives from Greek *apokalupsis*, from *apokaluptein* 'to uncover' or 'to reveal', from the prefix *apo* 'from, away' and *kaluptra* 'veil'. So *apocalypse* literally means 'draw the veil away from', indicating

the disclosure of something hidden from the mass of humanity: what the Universe is and how it is intelligently designed.

It is not easy to describe or explain such an epiphany to those who have not had a similar experience. I can best say that a big bang erupted in the utmost depth of my psyche, leading me to create a fundamentally different view of the Universe from that defined in dictionaries and taught in physics at university. In effect, I entered the world known to mystics through the ages, the ninety-nine percent of the Cosmos inaccessible to our five physical senses of sight, hearing, taste, smell, and touch.

It is only within the context of the mystical worldview that I have discovered how my children and I should have been educated in order to adapt intelligently to the unprecedented rate of evolutionary development we are experiencing today. Such an understanding is essential if we are to harmoniously manage our business affairs, collectively making radical changes to the education and economic systems.

However, as comparatively few are aware of the mystical worldview underlying the physical universe even when psychologists, for instance, seek to be free of the materialistic, mechanistic paradigm, I'm not sure to what extent this book is understandable. One challenge here is that the English language, like other European languages, has evolved over the years to represent a fragmented view of the Universe. David Bohm, my principal scientific mentor, suggested to me at one of our meetings in the 1980s that we could overcome this problem by studying the archaeology of language, especially its roots in the putative Proto-Indo-European (PIE) language. For these show that our forebears lived closer to Reality than most do today. So, accompanying this book is a Glossary on the website for the Alliance for Mystical Pragmatics, indicating some of the changes I have needed to make to the language I learned as a child.

With such linguistic changes, if this book were ever published, along with my other writings after editing, it could complete the final revolution in science, just as Johannes Kepler and Isaac Newton completed the first in 1609, 1619, and 1687, with *New Astronomy*, *The Harmony of the World*, and *Mathematical Principles of Natural Philosophy*, respectively.

This is the vision that I have been living with since my eureka moment in 1980. But from the feedback I have received over the years from those I have sought to work with, I have now come to the realization that such a project is too ambitious and overwhelming to those who have been educated within Western civilization in the conventional manner, generally attached to the social environment they feel comfortable with. So, for myself, all I can do during the last few years of my life is write out what might have been, trusting that these words could mirror the experiences of those who read them to some extent or other.

In brief, to discover how we humans could live together in love, peace, and harmony, beyond conflict and suffering, I have needed to study humanity's relationship to technology with great profundity, unifying holographic mathematical logic and depth psychology.

In the late 1940s, scientists, mathematicians, and engineers in England designed and built the first stored-program computers, in which programs that instruct the operations of such universal machines are held within memory along with the data elements they process, rather than outside, as in the first electromechanical computers to be built earlier in the decade.

This invention marked the greatest turning point in the history of human learning and technological development. The computer is a machine quite unlike any other that the *Homo genus* has invented during the past two thousand millennia. Unlike the flint axe, wheel, printing press, telescope, steam engine, and telephone, for instance, which extend our rather limited physical abilities, *the computer is a tool of thought, able to extend the human mind, even in some cases replacing it*.

But what exactly is a computer? As I realized nearly forty years ago, I could not answer this question because I did not know myself; I did not understand what it truly means to be human, in contrast to the other animals and machines, like computers. Nothing in my formal education and from sixteen years working in the data-processing industry could answer the fundamental questions of human existence. As as Stephen W. Hawking, generally regarded as the foremost scientist of the second half of the twentieth century wrote in *A Brief History of Time*, perhaps with tongue in cheek, "we have, as yet, had little success in predicting human behavior from mathematical equations!"[1]

Specifically, to understand what it means to be human, I first needed to answer two of the most critical unanswered questions in science. First, *what is causing scientists and technologists, aided and abetted by computer technology, to drive the pace of scientific discovery and technological development at unprecedented exponential rates of acceleration?*

Secondly, what is the relationship of human intelligence to so-called artificial intelligence? Of course, as I set out on my voyage of discovery, I did not know the answer to either of these questions. To answer them, I have needed to awaken my intelligence, which the education system tends to stultify in favour of the intellect, a quite different human faculty, despite the similarities in their roots. In this healing manner, I have also been able to answer some of the Big Questions of human existence, such as, "Who are we?", "Where have we come from?", and "Where are we all heading at unprecedented breakneck speeds?"

But there is no separate being called Paul, with social security numbers in Sweden and the UK, who has been the agent of this wonderful awakening, healing, and liberating adventure. Rather, I have been guided throughout my life, like everyone else, by creative energies over which I have no control as a supposedly separate being.

We humans have given many names to the ultimate creative energy within us over the years, which I most simply call Life, bubbling up from the Divine Origin of the Universe, like a fountain. To understand how our forebears viewed the world they live in, in contrast to how scientists generally do today, I like to study the roots of languages to see how our ancestors experienced and thought about life.

In particular, we see similar relationships between breath, life, soul, and spirit in many different languages. For instance, *animate* derives from the Latin *animalis* 'having a soul', from *anima* 'breath, soul', which, of course, is the root of *animal*. These words are related to the Swedish *anda* 'breath, spirit' and *ande* 'spirit, soul', cognate with *aniti* 'breathe' in Sanskrit. In turn, *spirit* derives from the Latin *spīritus* 'breath, spirit', from *spīrāre* 'to breathe'. So the roots of our language clearly indicate that the ancients were well aware of the role that Spirit, arising from the Soul of the Cosmos, plays in breathing animals, such as humans.

We also see these etymological relationships in other languages. For instance, in the Old Testament, the Hebrew words *nephesh* or *nepeš* 'breath; life, life force, soul' and *rûah* 'breath, wind; spirit, mind, heart' are translated as 'soul' and 'spirit', respectively. Similarly, in the New Testament, the Greek words *psūkhē* 'breath, spirit; life, soul; heart, mind' and *pnuema* 'wind, breath' are also translated as 'soul' and 'spirit', respectively. As *The Strongest Strong's Exhaustive Concordance of the Bible* tells us, all these words denote 'the immaterial part of the inner person that can respond to God'.[2]

And in the East, *Atman*, "the real immortal self of human beings, known in the West as the soul",[3] derives from Sanskrit *ātman* 'breath, spirit; soul, essence, self'. Similarly, *qi* (*ch'i*), a central concept in Taoism and Chinese medicine, denotes "the vital energy, the life force, the cosmic spirit that pervades and enlivens all things", literally 'air, breath, gas'.

Also in Sanskrit, *prana* means 'breath, vital life', from verbal root *prā* 'to fill', from PIE base **pelə-* 'to fill', also root of *fill*, *plenty*, and *plus*. For instance, Vivekananda wrote in *Raja Yoga*: "Everything that has form, everything that is the result of combination, is evolved out of this *Akasha*. … Just as *Akasha* is the infinite, omnipresent material of this universe, so is this *Prana* the infinite, omnipresent manifesting power of this universe."[4]

The word *Akasha* derives from Sanskrit *Ākāsha*, corresponding to Greek *aither* 'pure, fresh air', in Latin *æther*, "the pure essence where the gods lived and which they breathed", which is *quintessence*, the fifth element, the others being fire, air, earth, and water, of course. The systems philosopher Ervin Laszlo invokes Akasha to establish Consciousness as Ultimate Reality in what he calls the 'Akashic paradigm', using the word *Akasha* to refer to the Universal Quantum Field.[5] So, while Albert Michelson and Edward Morley showed in 1887 in a famous experiment that an 'æther wind' could not be physically detected as the Earth passed through the supposed æther,[6] such a substrate does exist in the nonphysical, or better to say beyond the physical and nonphysical realms and all other opposites.

The nonphysical energy of Life has been known throughout the ages as a vital principle underlying human experience, encapsulated in Henri Bergson's concept of *élan vital*, normally translated as 'vital impetus' or disparagingly as 'vital force', which Bergson called the '*original impetus* of life'.[7] Yet this vital force is "the energy or spirit which animates living creatures", as my dictionary says.

Similarly, Reginald O. Kapp, Professor of Engineering at London University, said in 1940 in *Science versus Materialism*, it is utterly amazing that vitalism is not so much dead, as it was claimed at the time, as a taboo. This iconoclastic book, which his son John has published on the Web, courageously made a commonsensical claim for the obvious, saying, "Any evidence which proves the organic world to be subject to laws from which the inorganic world is free is evidence for vitalism," for "as an engineer, we know that it is not in the nature of Matter unaided to fall into the form of machines."[8]

Despite such wise insights, biologists—so-called scientists of life, from Greek *bios* 'life; course, way, or mode of living; livelihood'—regard life as a property of the DNA molecule, which needs water to survive and procreate. So, mechanistic scientists searching for life and intelligent beings on other planets in the physical universe do not know that these creative human qualities are to be found within us, in the psyche. Without bringing Life into science, how can we possibly explain what caused Mozart to write his last three magnificent symphonies in just six weeks in the summer of 1788?

Even psychologists, seeking to understand consciousness, sometimes do so within the framework of the belief that the Universe is the physical universe, accessible through our five senses.[9] Yet, in the words of Kabbalah—the mystical core of Judaism—there is a curtain that divides our reality into two realms, 1% being our physical world, while the other 99% "is the source of all lasting fulfilment. All knowledge, wisdom, and joy dwell in this realm. This is the domain that Kabbalists call *Light*."[10]

Even mathematical objects do not exist in the physical universe, such as the number line, with infinite length and zero width. So, while mathematics is a language that helps scientists map the physical universe, physicists can tell us little about mathematics, itself. To map the Cosmic Psyche, including pure mathematics as a coherent whole, we need a quite different view of the Universe and an integral, holistic scientific method to study it.

For *universe*, like *university*, derives from Latin *universitās* 'the total', from *universus* 'combined in one, whole, entire' from *unus* 'one' and *versus*, the past participle of *vertere* 'to turn'. So, if a university lived up to its name, it would teach its students how to turn their view of themselves and the world they live in into

a single coherent whole. But universities are very far from doing this. They are divided into fragmented fields, surrounded by high hedges to protect these divisions, which do not exist in Reality.

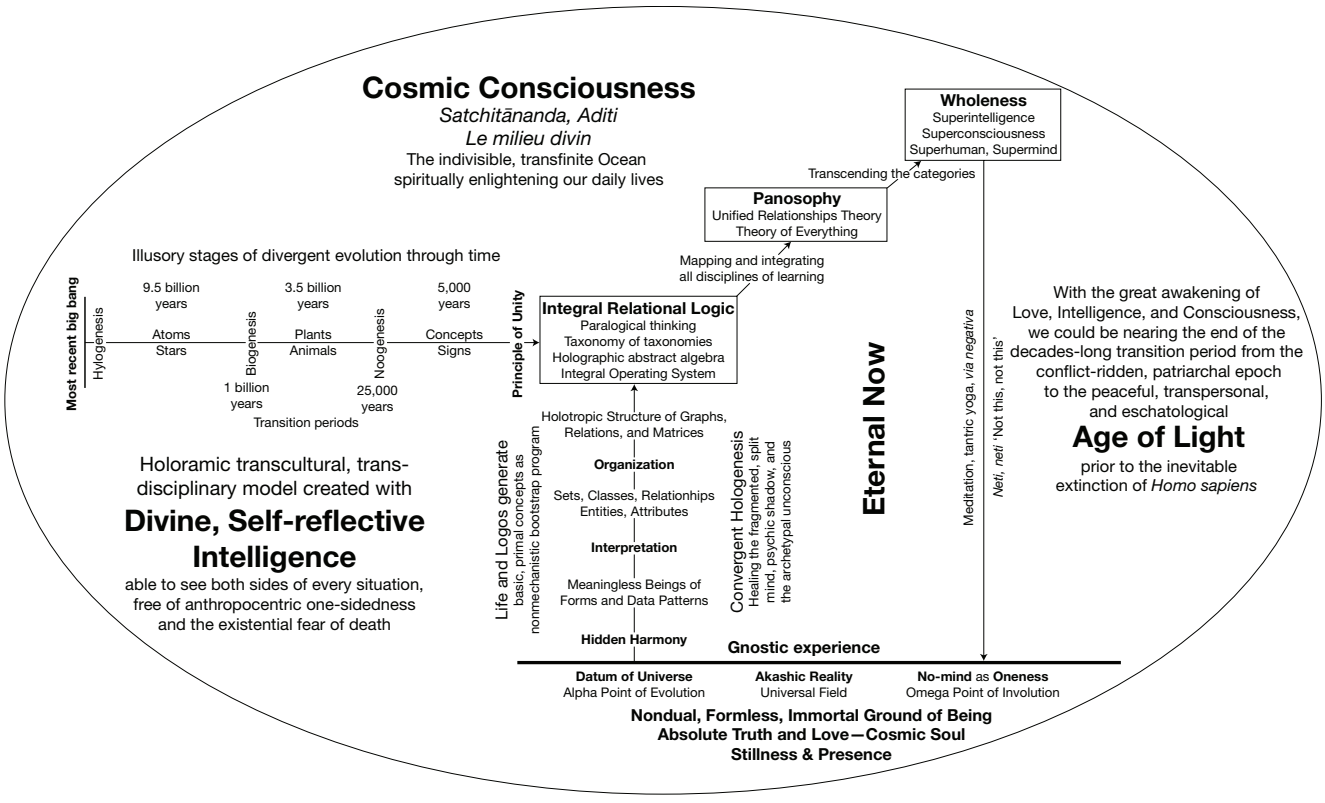
Understanding the ineffable Field, which both encompasses and underlies the Totality of Existence, I have needed to base my scientific studies of humanity directly on experience. For a scientific method that has evolved from the way that planets (as wandering stars) circle the Sun in ellipses and objects fall to Earth can tell us little about the patterns of human behaviour, which are far from mechanistic and therefore predictable.

R. D. Laing emphasized the need to base our scientific studies on experience, writing in *The Voice of Experience*, "The scientific objective world is not the world of real life. It is a highly sophisticated artifact, created by multiple operations which effectively and efficiently exclude immediate experience in all its apparent capriciousness from its order of discourse."[11] And in *The Politics of Experience*, he wrote, "In a world where the normal condition is one of alienation, most personal action must be destructive both of one's own experience and that of the other."[12]

Similarly, a central theme running through Erich Fromm's *The Sane Society*, as a successor to his wartime *The Fear of Freedom*, is alienation, from Latin *alius* 'other', defined in this way:

> By alienation is meant a mode of experience in which the person experiences himself as an alien. He has become, one might say, estranged from himself. He does not experience himself as the center of his world, as the creator of his own acts—but his acts and their consequences have become his masters, whom he obeys, or whom he may even worship. The alienated person is out of touch with himself as he is out of touch with any other person.[13]

Similarly, again, Rupert Spira, a leading teacher of Nonduality, writes, contemplating the nature of experience, "our conventional ways of seeing … bear little relation to our actual moment to moment experience," going on to say, "the idea that the body and world exist as objects in time and space, independent and separate from Consciousness … is not based on experience."[14]



In the event, it took nearly seventy years of life experience before I was able to see the Big Picture, unifying the incompatible concepts of God and Universe in Consciousness, depicted in this diagram of the

Cosmic Context, Gnostic Foundation, and coordinating framework of the Grand Design of the Universe, revealed by evolution becoming fully conscious of itself.

This diagram illustrates the mystical worldview underlying the physical universe—as the ancient wisdom underlying all the religions—known to Isaac Newton as *prisca sapientia*, as it was called during the Humanistic Renaissance,[15] and Gottfried Leibniz as *philosophia perennis*,[16] co-discovers of the infinitesimal calculus in the mid 1600s.[17] In words, extracted from the website for the Alliance for Mystical Pragmatics, which Anne Baring quotes on her website[18] and in a presentation she gave at Schumacher College:[19]

> It is from the Formless Absolute—as the Divine Datum of the Cosmos—that the entire relativistic world of form emerges, like waves and currents on and beneath the surface of an ocean, never separate from the ocean itself. This union of form and Formlessness is the Ocean of Consciousness, the centre of which is Love, the Divine Essence we all share, providing the Cosmic Context for all beings in the Universe, including all of us human beings.[20]
>
> Consciousness is Ultimate Reality; physical universes and their components, including the brain, emerge from Consciousness; all beings in the manifest Universe are related to each other, never separate from God, Nature, or any other being for an instant.[21]

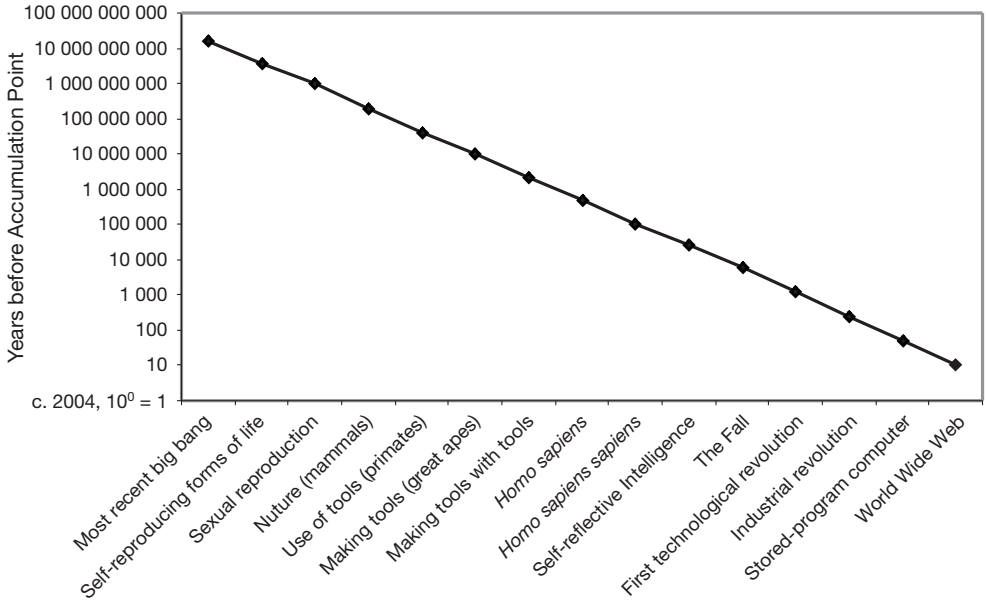
Integral Relational Logic (IRL), in the centre of this diagram, is the scientific method that leads to this cosmology of cosmologies, having evolved from the transcultural, transdisciplinary, and transindustrial modelling methods underlying the Internet. Specifically, IRL is the commonsensical art and science of reason and consciousness that we all implicitly use every day to form concepts and organize our ideas in tables or relations and semantic networks or mathematical graphs. As you can see, while this holotropic, nonaxiomatic system of thought is the product of some fourteen billion yeas of evolution, it has actually come into existence through the creative power of Life, emerging directly from the Divine.

However, developing this synthesis of all knowledge, healing my fragmented mind and split psyche in Wholeness, is not for my benefit alone. The line on the left of this diagram shows the conventional view of evolution in the horizontal dimension of time, extended from the biological into Pierre Teilhard de Chardin's first three stages in his four-stage evolutionary model, following his law of complexity-consciousness, the greater the complexity the greater the consciousness.[22] My book *The Four Spheres: Healing the Split between Mysticism and Science* updates Teilhard's model with the very latest scientific discoveries, showing how Integral Relational Logic can map those disciplines that study the spiritual, mental, biological, and physical aspects of our lives in the Numinosphere, noosphere, biosphere, and hylosphere.

Through Evolution's Accumulation Point: Towards Its Glorious Culmination then shows how we can apply the logistic map in nonlinear systems dynamics to mathematically model evolution under constraint, showing why the fourteen billion years of evolution passed through its Accumulation Point in 2004, give or take a couple of years. Comparing evolutionary bifurcations to a dripping tap, the evolutionary tap is now turned full on, with no further turning points to be discerned. This explains why society is blindly accelerating exponentially into chaos right now, with a few oases of self-similar order amongst the turmoil, depicted in the upper diagram on the next page.

Conversely, during the last three or four millennia, mystics and spiritual seekers have written countless words to describe their experiences, as they have come ever closer to the Divine, depicted in the downward path in the diagram of the Grand Design of the Universe. Viewed as a whole, the literature is pretty confusing, as people use different words to describe what is essentially the same experience and similar words to describe what might be dissimilar psychic phenomena. While a rose is a rose is a rose in our outside world, the languages we use to describe our inner worlds are in much disarray.

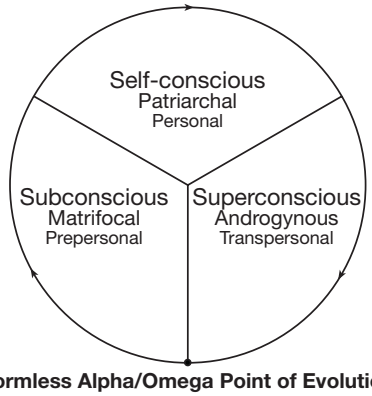
Major Turning Points before Evolution's Accumulation Point, c. 2004



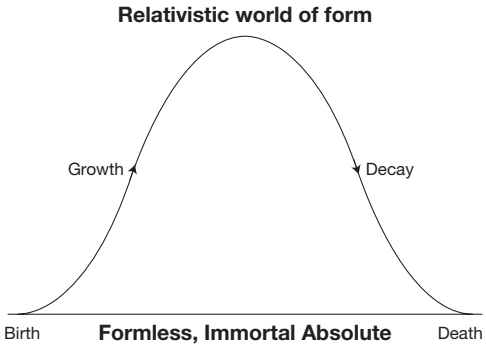
So, to heal the deep wound in the cultural and collective psyche between mysticism and mathematics, and hence science and spirituality, this book focuses attention on the upward path in the centre of the Cosmological diagram. It thus unifies the traditional scientific and spiritual paths with a coherent language defined in an evolving Glossary of terms on my website for the Alliance for Mystical Pragmatics, whose motto is 'Harmonizing evolutionary convergence'.

Specifically, this book provides a more concise description of Integral Relational Logic (in Chapter 2) than Part I of my trilogy on *Wholeness*, titled *Integral Relational Logic*, written in the early years of this decade after thirty years developing and applying IRL to organize all knowledge into a coherent whole. It then shows in the later chapters how it can be used to map mathematics, viewed as a generative science of patterns and relationships emerging directly from the Divine Origin of the Universe, rather than as an axiomatic, deductive proof system, which eschews self-contradictions. This book thus shows how to welcome paradoxes into our reasoning, healing the deep wound in our collective and hence personal psyches.

This is absolutely essential if humanity is to complete the transition into the third and final phase of Ken Wilber's model of human phylogeny, first presented in *Up from Eden*,[23] depicting the transition stage between the biosphere and noosphere and the final two stages in Teilhard's evolutionary model.



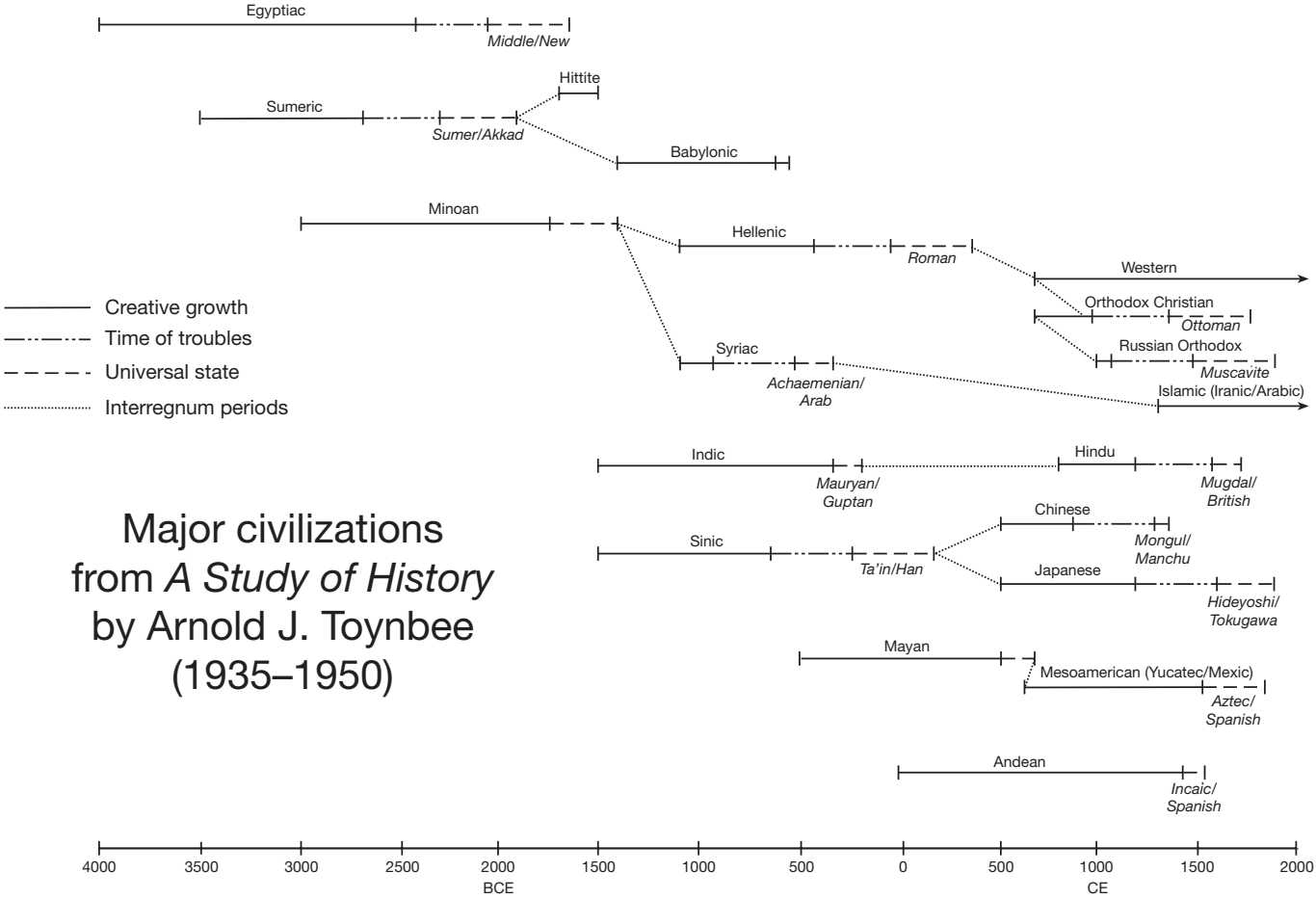
Formless Alpha/Omega Point of Evolution



This schema illustrates Joseph Campbell's Cosmogonic Cycle at the phylogenetic level. Like all other structures in the Universe, *Homo sapiens* emerged from the Formless Ground of Being and is destined to return there at the end of its lifespan. For all beings in the Universe are born to die, with no exceptions. This naturally includes our planet, species, and civilizations, the global economy, and our individual body-mind-soul organisms. In this instance, the base line represents the mystical and formless transfinite, out of which the entire world of form arises.

So, if our scientific methods are to present a coherent view of the Universe, rather than the fragmented views of specialists, we need a methodology that recapitulates the Cosmogonic Cycle, as Integral Relational Logic does.

Another way to see this is in terms of the timeline of the birth and death of civilizations during the five or six thousand years of the patriarchal epoch, depicted here:



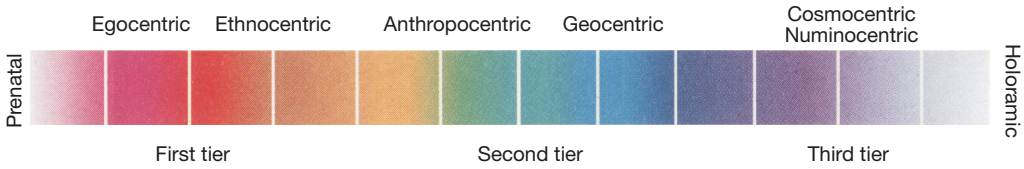
Major civilizations from *A Study of History* by Arnold J. Toynbee (1935–1950)

For myself, while I intuitively felt during my formal education that the civilization I had been born into did not make sense, it was not until the late 1970s that I realized that my children were not being educated to live in the world that would exist when they came to be bringing up children of their own. What I have now realized, after some forty years of profound self-inquiry, is that we are miseducating our children because Western civilization, in particular, is based on the false belief that we humans are separate from God, Nature, and each other.

As a consequence, the culture I was born into is built on seven pillars of unwisdom, a term that Arthur Koestler introduced in *The Ghost in the Machine* to highlight the absurdities and limitations of the biological, behavioural, mechanistic, and quantitative sciences.[24] These pillars are misconceptions of God, Universe, Life, humanity, money, justice, and reason. It is thus vitally important that we demolish these seven pillars, rebuilding the entire world of learning on seven pillars of wisdom, briefly given in this table:

No.	Pillars of unwisdom	Pillars of wisdom
1	God is other	Humans are Divine beings
2	The Universe is the physical universe	The Universe is Consciousness
3	Life is a property of the DNA molecule	Life arises from our Divine Source like a fountain
4	Humans are machines and nothing but machines	Humans are creative beings living in the Eternal Now
5	Financial modelling methods	Meaningful information systems modelling methods
6	Individuals have the free will to act independently	There is no doership or ownership
7	Only either-or reasoning is valid	Both-and thinking is the Hidden Harmony

We can see the challenge facing us all as a species from the three-tier, twelve-level model of the spectrum of consciousness that Ken Wilber has been working on since he wrote his first book in the 1970s on *The Spectrum of Consciousness*, which I have modified a little to match my own awakening of intelligence and consciousness.



As he said in his recent ten-module Internet course titled 'Superhuman Operating System', intended to "Install a Revolutionary New Operating System for Your Mind to Illuminate the Full Spectrum of Your Human Potential, and Become the Greatest Possible Version of Yourself", some 95% of the populace are still in the egocentric and ethnocentric first tier, while just 5% have reached the second tier. The third tier indicates "an identification with all life and consciousness, human or otherwise, and a deeply felt responsibility for the evolutionary process as a whole … an emergent capacity, rarely seen anywhere," as Ken defined it in a conversation with Andrew Cohen in the *What is Enlightenment?* magazine in 2007.[25]

So while there are both conservative and progressive liberals in the political spectrum, there is widespread resistance to total liberation, from Latin *liber* 'free'. Vimala Thakar highlights this critical situation in the opening paragraph of *Spirituality and Social Action: A Holistic Approach* with these wise words: "In a time when the survival of the human race is in question, continuing with the status quo is to cooperate with insanity, to contribute to chaos." She therefore asks, "Do we have the vitality to go beyond narrow, one-sided views of human life and to open ourselves to totality, wholeness?" For as she says, "The call of the hour is to move beyond the fragmentary, to awaken to total revolution."[26]

So, whether we shall be able to collectively realize Teilhard's vision before our inevitable demise looks most unlikely at the moment: "The way out for the world, the gates of the future, the entry into the superhuman, will not open ahead to some privileged few, or to a single people, elect among all peoples. They will yield only to the thrust of all together in the direction where all can rejoin and complete one another in a spiritual renewal of the Earth."[27]

Nevertheless, I continue to follow my inner guide regardless, which the Greeks and Romans called *Daimon* and *Genius*, respectively. To unify mysticism and mathematics, at the heart of science and spirituality, healing deep personal and cultural wounds in my psyche, I have made the most radical change in the work ethic since our forebears settled in village communities some ten thousand years ago to cultivate the land and domesticate animals. This has led me to view mathematics in a quite different way from that which I was taught at university in the early 1960s. I have also developed an evolutionary approach to meditation, in contrast to the involutionary the way I was taught to meditate in the late 1990s.

To explain this, we can best begin with Euclid's *Elements*, a standard mathematical textbook for hundreds of years until the end of the nineteenth century, as a model both of a three-dimensional rectilinear, orthogonal universe and of valid reasoning. However, at the beginning of the 1900s, Albert Einstein realized that he needed to use non-Euclidean geometry, systematically studied by Bernhard Riemann in the mid 1800s, to develop his general theory of relativity. As he wrote in the opening paragraph of his introductory book *Relativity* in 1920,

> In your schooldays most of you who read this book made acquaintance with the noble building of Euclid's geometry, and you remember—perhaps with more respect than love—the magnificent structure, on the lofty staircase of which you were

> chased about for uncounted hours by conscientious teachers. By reason of your past experience, you would certainly regard everyone with disdain who should pronounce even the most out-of-the-way proposition of this science to be untrue. But perhaps this feeling of proud certainty would leave you immediately if some one were to ask you: "What, then, do you mean by the assertion that these propositions are true?" Let us proceed to give this question a little consideration.[28]

However, Einstein did not question the linear structure of Euclid's reasoning, starting with some axioms or postulates, as self-evident or assumed truths, proving a sequence of theorems in a step-by-step manner, mostly related to geometric figures. This is a pity for this deductive way of reasoning is essentially mechanistic, like computers executing a sequence of instructions in algorithmic programs, albeit in parallel threads in modern multi-headed central processing units, collectively collaborating in networks, such as the Internet. But this is not how we humans think and organize our ideas.

So, while Einstein spent the last thirty years of his life attempting to find a simple equation at the heart of his unified field theory,[29] he was unable to explain his creative process, described in a letter to Jaques Hadamard in 1945, who was then studying mathematicians' creative experiences:

> The words or the language, as they are written or spoken, do not seem to play any role in my mechanism (sic) of thought. The physical entities (sic) which seem to serve as elements in thought are certain signs and more or less clear images which can be 'voluntarily' reproduced and combined.
>
> There is, of course, a certain connection between those elements and relevant logical concepts. It is also clear that the desire to arrive finally at logically connected concepts is the emotional basis of this rather vague play with the above mentioned elements. But taken from a psychological viewpoint, this combinatory play seems to be the essential feature in productive thought—before there is any connection with logical construction in words or other kinds of signs which can be communicated to others.
>
> The above mentioned elements are, in my case, of visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary stage, when the mentioned associative play is sufficiently established and can be reproduced at will.
>
> According to what has been said, the play with the mentioned elements is aimed to be analogous to certain logical connections one is searching for.
>
> In a stage when words intervene at all, they are, in my case, purely auditive, but they interfere only in a secondary stage as already mentioned.[30]

Similarly, David Bohm pointed out in *Wholeness and the Implicate Order*, which unified quantum and relativity theories, "The word *theory* derives from the Greek *theoria*, which has the same root as *theatre*, in a word meaning 'to view' or 'to make a spectacle'. Thus it might be said that a theory is primarily a form of *insight*, i.e. a way of looking at the world, and not a form of *knowledge* of how the world is."[31]

Indeed, no traditional method of scientific reasoning, whether it be deduction, induction, or abduction, introduced by Aristotle and Euclid,[32] Francis Bacon,[33] and Charles Sanders Peirce,[34] respectively, can explain why we humans behave in the way that we do, not intelligently adapting to the accelerating pace of change in society, being driven by scientists and technologists.

To discover what it truly means to be a human being, I have needed to develop a quite new system of thought that has evolved from the information systems modelling methods that underlie the Internet, which trace their origins to Plato's *Republic* and Aristotle's *Organon* and *Metaphysics*. This has enabled me to admit nonphysical, psychospiritual synergistic energies into science, unifying them with the four forces acknowledged by physicists: electromagnetic, gravitational, and the weak and strong nucleic forces. Indeed, a radically new way of thinking is not only necessary to solve many unsolved scientific problems.

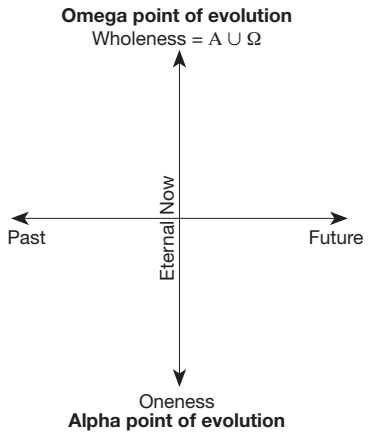
Given the turbulent state of the world, we need to follow Einstein's observation that you cannot solve a problem with the mindset that created it. This is one of many paraphrases of a statement he made in an article titled 'The Real Problem Is in the Hearts of Men', published in the *New York Times Magazine* on 23rd June 1946, which began with these words: "Many persons have inquired concerning a recent message of mine that 'a new type of thinking is essential if mankind is to survive and move to higher levels'." He

then went on to write, "Past thinking and methods did not prevent world wars. Future thinking *must* prevent wars."[35] For, as he said in an address at the fifth Nobel anniversary dinner in New York on 10th December 1945, "The war is won, but the peace is not. The great powers, united in fighting, are now divided over the peace settlements."[36]

In my experience, just one simple step is needed to find Love and Inner Peace, as the Divine Essence we all share. Machines, like computers, function in the horizontal dimension of time of past and future. But to realize what it truly means to be a human being, in contrast to computers, I have needed to develop a system of reasoning in the vertical dimension of time, in the Eternal Now. For as Eckhart Tolle said in his best-selling *The Power of Now*,

> To be identified with your mind is to be trapped in time: the compulsion to live almost exclusively through memory and anticipation. This creates an endless preoccupation with past and future and an unwillingness to honour and acknowledge the present moment and allow it to be. The compulsion arises because the past gives you an identity and the future holds the promise of salvation, of fulfilment in whatever form. Both are illusions.[37]

Now traditionally, spiritual practices—such as Vipassana or Insight meditation in Buddhism or *Jñāna Yoga*, the path of wisdom and abstract knowledge in Advaita—follow the downward direction in the vertical dimension of time. By repeating the mantra *neti neti* 'not this, not that', corresponding to *via negativa* in Christianity, we can discover our True Nature, Authentic Self, and Genuine Identity, answering the question, "Who am I?"



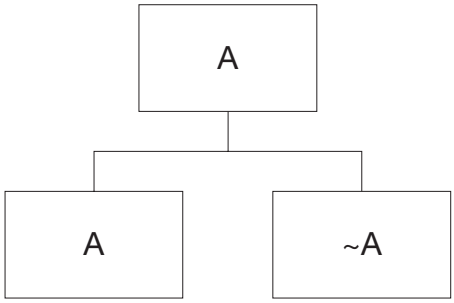
However, this is a one-sided approach to spiritual awakening. It does not take into account the upward movement in the vertical dimension of time, starting at the Divine Origin of the Universe, as the Ultimate Source of the creative power of Life. For evolution to become fully conscious of itself within me, enabling me to discover what causes me to think and behave as I do, free of my mechanistic, cultural conditioning, I have needed to unify both dimensions of time in all directions, as this diagram illustrates.

By starting afresh at the very beginning in an apocalyptic awakening in the spring of 1980, I have been carried from the Alpha Point of evolution to the Omega Point and back again. In other words, all the divergent streams of some fourteen billion years of evolutionary history have converged within me in a megasynthesis of all knowledge, not unlike the way that the scientific mystic Pierre Teilhard de Chardin prophesied in *Le phénomène humain*, posthumously published in 1955.

The two dimensions of time, which are inseparable, like two sides of a coin, are a special case of the fundamental law of the Universe, which I call the *Principle of Unity*: *Wholeness is the union of all opposites*. In the most abstract, I express this universal truth as a theorem in the notation of mathematical logic as the *Cosmic Equation*, where W is any whole, including Wholeness, A is any being, including the Supreme Being and all human beings, $\cup$ is union, and $\neg$ is not:

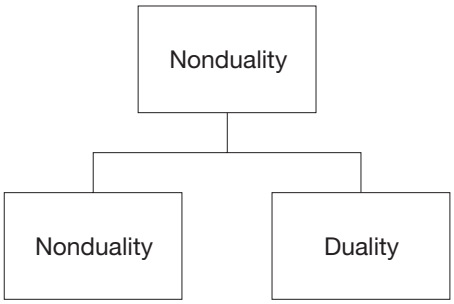
$$W = A = A \cup \neg A$$

The key point here is that this premise cannot be proven to be true from any other proposition. This universal truth emerges directly from the Divine Origin of the Universe in the Eternal Now. It is an irrefutable truth, for assertions and denials of its veracity confirm its authenticity. The next diagram illustrates the primary-secondary relationship between these polar opposites. Nothing could be simpler.

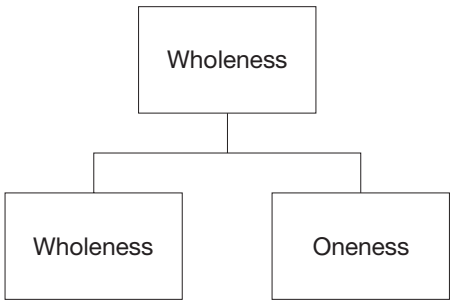


However, the Principle of Unity is not the Absolute Truth, which sets us free, as Jesus said.[38] For the Truth is ineffable, only experienced and understood with the utmost profundity of mystical experience.

Another special case of this ubiquitous primary-secondary relationship is its application to the Totality of Existence, consisting of all beings, including the Supreme Being. One way of viewing and experiencing Totality is to see it as the union of the Formless Absolute and the relativistic world of form, the latter emerging from the former. This relationship is illustrated in this diagram, using the words *Nonduality* and *duality* to make the distinction.



However, the Absolute provides both the Cosmic Context and Gnostic Foundation for the world of form. So it too can be viewed as the union of transcendent and immanent opposites, which I call *Wholeness* and *Oneness*, respectively, illustrated here. What this means is that we all live in the same Universe, which is synonymous and coterminous with God, whether we know this or not, transcending the categories, beyond all boundaries. Both *God* and *Universe* denote the Totality of Existence. God is everything and nothing and vice versa. No one can return Home to Wholeness because nobody has ever left Home.



The Principle of Unity is the ultimate Integral Tantric Yoga, for *yoga* is Sanskrit for 'union', cognate with the English words *yoke*, *join*, and *syzygy* 'conjunction', from Greek *suzugiā* 'union', from *sun-* 'together' and *zugon* 'yoke'. This unifying principle provides a synthesis of all forms of yoga, including Aurobindo's integral yoga. Also, *tantra* derives from Sanskrit *tantram* 'loom', unifying 'warp' and 'weft', from *tan* 'to stretch', and *-tra-m* 'instrument'. So *tantra* literally means 'an instrument for stretching'. Figuratively, Tantra has the sense of weaving opposites together in Wholeness, with other original meanings indicating 'groundwork, principle, system' and 'Context, Continuum'.

So, by showing that the mystical Weltanschauung[39] underlies the materialistic, mechanistic worldview enables me to shed fresh light on the foundations of mathematics and the problem of continuity and the continuum. For experiencing Ultimate Reality as an undivided Whole provides the Contextual Foundation of the ancient wisdom.

These six paragraphs, three diagrams, and one equation describe the keys that open the set of nested containers that enclose the innermost secrets of the Universe, revealing what it is, how it is designed, and how we humans are positioned within the overall scheme of things, not special in any way. Like all other beings in the relativistic world of form, we are born to die, as both individuals and as a species, or, in the case of mammals, birds, and reptiles, at least, are conceived to die.

Yet, in a way, we humans *are* special. We are the first beings in the entire history of evolution—seen from our particular position on Earth—to be given the ability to discover how the Cosmos, as an ordered whole, is designed. Galaxies and horses, for instance, do not know their place within the overall scheme of things. It is only in humans that evolution can become fully conscious of itself. Meister Eckhart, the pre-eminent Christian mystic, explained how this incredible ability comes about, when he said, "The eye with which I see God is the same as that with which he sees me."[40] I call this Divine Eye Self-reflective

Intelligence, sometimes called the Witness in spiritual circles, when the observer and observed and subject and object are one.

Self-reflective Intelligence is the eyesight of Consciousness, which provides the coherent light necessary for us to view the Cosmos holographically, as a self-similar whole, like a laser and geometric fractal. Our amazing ability to see ourselves seeing is the principal characteristic that distinguishes humans from the other animals and machines, like computers, with so-called artificial general intelligence. As Einstein described in his letter to Hadamard, much is implicitly happening in the psyche before our ability to explicitly express what we see with our inner eye in words and other signs, linguistic abilities often considered humankind's distinguishing characteristic.

Anyone feeling into themselves wondering where their creative energies are coming from would have similar experiences. We cannot understand what it means to be a human being in terms of the laws of physics, such as the second law of thermodynamics, which Arthur Eddington regarded as irrefutable, writing:

> The law that entropy always increases—the second law of thermodynamics—holds, I think, the supreme position among the laws of Nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations—then so much the worse for Maxwell's equations. If it is found to be contradicted by observation—well, these experimentalists do bungle things sometimes. But if your theory is found to be against the second theory of thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation.[41]

Not that the irrefutable Cosmic Equation is a new idea. As this primal truth is ever-present, seekers of Love, Peace, and the Truth have uncovered it in a multitude of different guises over the years, most often not aware that they are doing so. One such awakened being was Heraclitus of Ephesus, who aptly called the Principle of Unity the *Hidden Harmony*, which has remained hidden from the mass of humanity for most of human history.

We can see why this is so from the root of *human*, which is Latin *humus* 'ground, earth', from the PIE base **dhghem-* 'earth'. This etymology shows that our forebears some 5,500 years ago conceived of humans as earthlings in contrast to the divine residents of the heavens, as Calvert Watkins explains in *The American Dictionary of Indo-European Roots*.[42] So the split between humans and the Divine, as Reality, lies deep in the collective psyche. As a consequence, humans have become cognitively and experientially separate from our Immortal Ground of Being, which is the root cause of the existential fear of death and human suffering. To be humble, which derives from the same root, is therefore to deny our Divinity. Conversely, it is arrogant to realize and acknowledge our True Nature as Divine Beings, *arrogance* being the opposite of *humility*.

As a primary purpose of this book is to describe how unifying mysticism and mathematics has enabled evolution to heal my wounded psyche in Wholeness—sadly creating a split between society and myself—I don't want to dwell on these psychospiritual issues for more than is necessary. Human society, consisting of some seven-and-a-half billion souls, much influenced by our one hundred billion forebears,[43] is the most complex structure in the world, going deep into the sub- and unconscious, far more complex than the brain.

All I need to say at this juncture is that human relationships are both complementary and contradictory in a multiplicity of different ways, reflecting the dual and dualistic nature of the Universe we live in. So, to develop a comprehensive model of the psychodynamics of society, I have needed to include self-contradictions in my reasoning, showing that I can do so in an utterly valid manner. Indeed, as paradoxes are ubiquitous, if I were to omit them from my conceptual modelling and cognitive mapping, I would live in delusion, being led far astray. So, in my reasoning, I follow E. F. Schumacher's fundamental maxim of

mapmaking, "Accept everything; reject nothing." As he wisely said, "Our task is to look at the world and see it whole."[44]

To reveal the simplicity underlying all this complexity, experience is primary. But just who is the experiencer? It might appear that Paul Hague, with unique social-security numbers in both the UK and Sweden for tax and other purposes, is the experiencer. It is pertinent to note that *Paul Hague* means someone living in or by a small field surrounded by hedges, from Latin *paulus* 'little, small' and Old Norse *hagi* 'an enclosed piece of land', also root of *hedge*. (In modern Swedish and Norwegian, *hage* means 'meadow' and 'garden', respectively.) But if I had lived my life constrained by my name, inherited from my parents, I would not have been able to discover the innermost secrets of the Universe, which has been the central theme of my life since I was seven years of age.

So, to understand what it truly means to be human, in contrast to computers, I have demolished all the hedges that demarcate specialist fields of learning so that only the Field remains, a notion that some physicists are adopting to explain their observations.

Over the years, I have thus learnt to stand outside myself, beyond the boundaries of my body-mind-soul organism. I have not been able to understand what is causing scientists and technologists, like myself, to drive the pace of change in society at exponential rates of acceleration from an anthropocentric perspective. I am not a human being having a mystical experience. Rather, I am a Divine Cosmic being having a human experience. My body-mind-soul organism emerges from Consciousness, not the other way round.

Today, I know this truth with absolute certainty, having changed the meanings of *God* and *Universe*, as they were taught to me in childhood. So, in my seventies, I do not need to prove it through any process of reasoning or spiritual practice. The exquisite experience of Stillness and Presence tells me everything I need to know, faced with the sixth mass extinction of species on Earth, including, inevitably *Homo sapiens*. For *Presence* means 'before being' or 'prior to existence', deriving from Latin *præsentia* 'presence', participle of *præesse* 'to be before', from *præ* 'before' and *esse* 'to be'. So the word *Presence* indicates that the Absolute is the Supreme Cause of Everything there is, both immanent and transcendent.

Today, I call this inner knowing of the Divine *Gnosis*, distinct from symbolic or signate knowledge. Gnosis never changes. It is completely independent of how I might feel at any one moment or of what people might think or say, including myself. Indeed, in the Gnostic experience, there is no experiencer as a separate being.

Therein lies the greatest paradox in my life. As I am Wholeness, like everyone else, there is nothing and nobody outside me. I am not separate from anybody. Yet, I have lived most of my life in solitude, increasingly aware that my life experiences are quite different from those who learned what their parents and teachers wanted them to learn as children. Indeed, my experiences are so unusual that the existing meanings of words do not enable me to satisfactorily communicate what I feel and see within me.

As already mentioned, David Bohm suggested a solution to this problem in a private conversation in the mid 1980s. We need to study *etymology*, whose own root is Greek *etumos* 'real, true', bringing in words from the East when necessary to fill gaps in Western understanding. In this way, we can come ever closer to Reality, like those mostly being brought up in Eastern mystical traditions.

As a consequence of the deep wound in the Western psyche, many of those who have considered the Big Questions of human existence over the centuries have said that my life experiences are impossible, that we humans will never discover what the Universe is and how it is designed. Yet, paradoxically, when I *do* use words that people are familiar with, they sometimes say that I am not saying anything new. In a way

this is true. I am Wholeness, awarely (consciously and intelligently) living at the Immortal Omega Point of evolution, co-existent with its Alpha Point, from which none of us is ever separate.

So how are you and I to relate to each other as ordinary human beings? This is a question I have been wrestling with for most of my life, particularly after my life-changing epiphany in 1980. What is most similar in my experience to that of others is that which is expressed in the ancient wisdom of the mystics, which many millions sense within themselves today. So one way of describing what has happened to me in my lifetime is to communicate this ancient wisdom in a language that has evolved from the universal language of mathematics, as the emerging, generative science of patterns and relationships.

However, there is a difference. The differential calculus is the branch of mathematics that studies change. In particular, Newton showed with his equation $f = ma$ that for a body in motion to accelerate a force needs to be applied to it. Now the pace of change in society is accelerating exponentially. As John Templeton said in 2000, "More than half the scientists who ever lived are alive today. More than half of the discoveries in the natural sciences have been made from 1900 to 1999. … More new books are published each month than were written in the entire historical period before the birth of Columbus." He was then naturally led to ask the question, "Is the slow progress of prehistoric ages now speeding up?"[45]

Fairly obviously it is. But, as society is not a lump of matter, what is accelerating and what is causing the pace of scientific discovery and technological innovation to accelerate at unprecedented rates of evolution? And how can we study the psychodynamics of our rapidly changing society mathematically? Could we develop the laws of motion of society just as Newton developed the laws of motion of physical bodies, since modified by Einstein's special and general theories of relativity and the paradoxical discoveries of quantum physics.[46]

Well, as I have realized from half a lifetime of study, evolution is an accumulative process of ever-increasing complexity of structure, consisting of the meaningful relationships between forms, whether these be physical or nonphysical. So we could in principle use algebra, as the branch of mathematics that studies relationships, the logic of relatives, differential calculus, and the exponential function to map our rapidly changing society.

However, in themselves, these are not sufficient. For if they were, scientists would long ago have answered the most critical unanswered question in science. As synergistic energies within us scientists and technologists are causing us to drive the pace of change exponentially, to understand why we are behaving in this ignorant way we need to engage in self-inquiry, conducting an experiment in learning to map our minds and psyches, requiring us to make a radical change to scientific method. For fairly obviously, we can only understand what is happening to us all as a species by changing the way that we learn about ourselves and our relationships to God and the Universe. Scientific methods that have been developed to study outer space are quite inappropriate to study inner space.

Of course, if I had known the Truth in the way I do today earlier in life, I would not have needed to write this book on how mysticism and mathematics have converged within me. Although I have been on an unprecedented spiritual journey, there are some similarities between my reasoning and that of Baruch Spinoza.

In 1677, Spinoza similarly sought to show the equivalence of God and the Universe in *The Ethics* through a systematic process of reasoning, inspired by Euclid's deductive *Elements*. So he began with some definitions and axioms and proceeded to 'prove' a sequence of propositions or theorems, beginning with

the substance and essence of God and continuing to study the origin and nature of the mind and emotions, before exploring what this means for human behaviour, free of bondage to the emotions.

Since then, mathematics has moved on in great leaps and bounds, looking quite different from Spinoza's time. The *Oxford Dictionary of English*, the default dictionary of British English on my iMac and iPad, defines *mathematics* as 'the abstract science of number, quantity, and space, either as abstract concepts (pure mathematics), or as applied to other disciplines such as physics and engineering (applied mathematics)'. Other dictionaries give rather more elaborate definitions of this academic discipline. Nevertheless, I feel that this definition on Apple's computers well encapsulates the general public's conventional conception of mathematics, as many of us experience it up to high school.

However, in the nineteenth century, mathematicians began to extend the subject beyond the arithmetic, algebra, calculus, trigonometry, and geometry we learned in school, discovering that the patterns and relationships between numbers could be generalized, applying these rules to numerical and nonnumerical constructs, such as matrices, polynomials, and groups. So numerical algebra, tracing its origins to the Babylonians, became universal, abstract, and modern algebra, sometimes simply called *algebra* by professional mathematicians.

An early book on the subject was *A Treatise on Universal Algebra*, which Alfred North Whitehead wrote in the late 1890s, later to spend twenty years with Bertrand Russell writing *Principia Mathematica*, taking 360 pages to prove the proposition '1 + 1 = 2.'[47] In this early treatise on a general theory of symbolic reasoning, Whitehead used William Rowan Hamilton's Quaternions, Hermann Grassmann's Calculus of Extension, and George Boole's Symbolic Logic as the chief examples of the various systems of Symbolic Reasoning allied to ordinary Algebra.

For myself, I first came across abstract algebra as an undergraduate in the early 1960s, having *A Survey of Modern Algebra* by Garrett Birkhoff and Saunders Mac Lane as a standard textbook on the subject, going through four editions from 1941 to 1977. They then went on to write another standard, simply titled *Algebra*, published in three editions from 1967 to 1999. This latter book provides a comprehensive overview of the basic constructs of groups, rings, and fields, extended into the categories of modules, vector spaces, linear algebra, matrices, lattices, and Galois theory.

A chapter at the end of the book covers category theory in mathematics, a subject that Mac Lane cofounded with Samuel Eilenberg. Mac Lane's standard textbook on mathematical categories, based on the concept of functor, is titled *Categories for the Working Mathematician*, published in 1969 and 1998. As such categories apply just as much to computer science, Andrea Asperti and Giuseppe Longo then went on to write *Categories, Types, and Structure: An Introduction to Category Theory for the Working Computer Scientist*, published in 1991.

However, categories don't just belong to mathematics and computer science. The whole of human learning is concerned with the classification of human experience, which is how we bring a sense of order to our conceptual models and cognitive maps. The branch of science concerned with classification is called *taxonomy*, from Greek *taxis* 'arrangement, order' and *nomia* 'distribution, method', from *nomos* 'custom, law', from *nemein* 'manage, control, arrange, assign'. So *astronomy* is an arrangement of the stars and *economy* is the management of the household. Similarly, *taxonomy* is an arrangement of an arrangement, today either meaning classification in general or specifically the systematic classification of biological organisms, following Carl Linnæus from Sweden, who published his seminal *Systema Naturæ* in 1735.[48]

To develop a quite fresh approach to category theory, as a taxonomy of taxonomies, this book describes how the Logos, the "immanent conception of divine intelligence" signifying "the rational principle governing the cosmos", as Richard Tarnas interpreted Heraclitus' use of *Logos*,[49] has brought universal order to all my thoughts. However, as most people prefer to think in the concrete terms of particulars, with specific reference to their own separate lives, it might seem that nothing very interesting could emerge from thinking in this utterly abstract manner, standing outside ourselves to watch our behaviour patterns, including reasoning, and hence mapping the psychodynamics of the whole of society.

This was the view that G. H. Hardy and A. N. Whitehead took when writing about their experiences as pure mathematicians. For instance, Hardy, as a mathematical analyst, felt that he needed to make an apology for his occupation, saying, "I have never done anything 'useful'. No discovery of mine has made, or is likely to make, either directly or indirectly, for good or ill, the least difference to the amenity of the world."[50] Hardy called pure mathematics 'serious' rather than 'trivial'.[51] To Hardy, "A mathematician, like a painter or a poet, is a maker of patterns."[52] "The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way."[53] Hardy was "interested in mathematics only as a creative art".[54] In the words of Whitehead, "The science of Pure Mathematics … may claim to be the most original creation of the human spirit," one possible rival being music.[55]

In Hardy's words, there is "a certain generality and a certain depth"[56] in pure mathematics. By generality, he meant "A significant mathematical idea … should be one which is a constituent in many mathematical constructs."[57] In Whitehead's words, "It is by the employment of [the] notion [of 'variable'] that general conditions are investigated without any specification of particular entities," such as "the shape-iness of shapes",[58] which are quite irrelevant. It is the task of mathematics to discover a "pattern of relationships among general abstract conditions".[59] However, Whitehead went on to qualify his statements by saying "it is the large generalization, limited by a happy particularity, which is the fruitful conception."[60] As Hardy said, "a property common to too many objects can hardly be very exciting."[61]

By depth, Hardy meant "ideas that are usually the harder to grasp".[62] Examples of depth are Euclid's proof that there are an infinite number of primes and Pythagoras's proof that $\sqrt{2}$ is irrational, the latter being deeper than the former. They are deep because they employ general mathematical techniques, these cases being examples of *reductio ad absurdum*. But there are mathematical theorems that are much, much deeper than these. So much so that "this notion of 'depth' is an elusive one even for a mathematician who can recognize it."[63]

However, it is not true that a property common to too many objects can hardly be very exciting, as this book demonstrates. The most abstract concept is that of Aristotle's ontological concept of Being, defined in *Metaphysics* as more general than mathematical concepts, like number, circle, and set.

> There is a science which studies Being *qua* Being, and the properties inherent in it in virtue of its own nature. This science is not the same as any of the so-called particular sciences, for none of the others contemplates Being generally *qua* Being; they divide off some portion of it and study the attribute of this portion, as do for example the mathematical sciences.[64]

However, in the same book, Aristotle said, "It is impossible for the same attribute at once to belong and not to belong to the same thing and in the same relation … as some imagine Heraclitus says,"[65] a statement known today as the Law of Contradiction, the implicit axiom for deductive logic and mathematical proof. By denying the irrefutable truth of the Hidden Harmony, Aristotle thus took Western thought into the evolutionary cul-de-sac it finds itself in today.

As I can see in hindsight, the story of my entire life has been to extricate myself from this evolutionary dead end, often coming into conflict with my peers and the authorities in my life, holding on to fragmentary approaches to human learning in academic specialization and the division of labour in the workplace. But we should not blame Aristotle for the mess the world is in today. Most have fragmented minds because human learning has been more divergent than convergent during the thousands of years of human cognitive evolution. So the Law of Contradiction simply reflects the way that many maintain a precarious sense of identity and security in the world, viewing both God and others as separate beings.

For myself, I began to free myself from the constraints of Western thought as an undergraduate in the early 1960s, majoring in mathematics, being attracted to two topics that I later built on: the principle of duality in projective geometry and group theory in abstract algebra. I cannot say that I mastered these branches of mathematics at the time because I was too depressed by my inability to end the long-running war between science and religion, which I had been struggling with since I was seven years old in 1949.

As I can see now, I was seeking to find Love and Inner Peace by healing a deep wound in my psyche, partially introjected from the culture I was born in, opened up as the result of a cataclysmic trauma seven weeks after my conception in October 1941. Gone was the feeling of what Stanislav Grof calls 'oceanic ecstasy' in *The Holotropic Mind*. As he says, our early experiences in the womb "have strong mystical overtones; they feel sacred or holy. … In this state of cosmic unity, we feel that we have direct, immediate, and unlimited access to knowledge and wisdom of universal significance." This rapturous period in our lives is a reminder of "Gardens of Paradise in the mythologies of a variety of the world's cultures".[66]

In contrast, when prenates experience a deep trauma before birth, as I did, they experience what Stan calls a 'bad womb', [67] which can have an even greater effect on later development than what he calls 'basic perinatal matrices' (BPM).[68] This trauma set up a pair of behaviour patterns in my psyche, which led me, as soon as I was able, to question the beliefs and assumptions of the culture I was born in, creating what appeared to be a hostile environment. It was vitally important to learn as little as possible during my formal education in order to heal this personal wound, and hence cultural wound, during the second half of my life. So as soon as I learned too much, Life arranged for me to have a major breakdown.

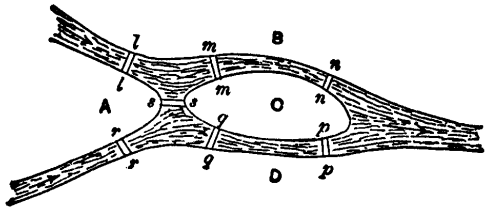
As neither people educated in the culture I was born in nor specialists in inner science could help me to heal this deep wound in my personal psyche, I have needed to live outside society for most of my life. In later years, I have used the model of the psychodynamics of society that has been revealed to me to help with this healing process, much helped by the beautiful patterns in mathematics. For to avoid feeling too depressed and frustrated by what has often appeared to be a hostile social environment, constantly ignoring and rejecting my adventurous life of learning, I have sometimes turned to the elegance of pure mathematics for solace, with its universal abstract notions applicable in all cultures.

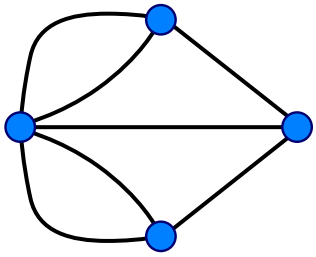
In the second half of my life, I have been able to build on what little I had learnt in the early 1960s as an undergraduate. The seeds that were then planted in my subconscious have now grown into a majestic structure of incomparable power and beauty. To explain this, I can best begin with a book that I was given as a school mathematics prize as a sixteen-year-old: *Mathematical Recreations and Essays* by W. W. Rouse Ball, revised in later editions by H. S. M. Coxeter, a foremost geometer in the twentieth century.

In this book, Rouse Ball tells us that Leonhard Euler presented a memoir to the St Petersburg Academy in 1736 concerning a problem relating to the bridges of Königsberg, the capital of East Prussia, today

Kaliningrad, a Russian exclave. This pre-eminent Swiss mathematician was asked if it were possible to take a walk in Königsberg, the birthplace of Immanuel Kant and David Hilbert, in such a way as to cross every bridge in it once and only once and return to the starting point.[69]

This diagram from the book shows two branches of the Pregolya River flowing around an island towards the Baltic, although East and West are reversed in this diagram. It shows seven bridges between land masses, which can be reduced to nodes and arcs between them, as in this next diagram, the right way round, with the node on the left being the island.[70]





With such a diagram, Euler showed that as each node has an odd number of arcs connected to it, it is not possible to cross every bridge only once. This would be possible if all the nodes except two were even. In that case, it would be possible to start at one of the odd nodes and finish at the other. If all nodes were even, it would be possible to traverse all bridges by starting at any node.

Today, such a structure is called a mathematical graph, which is ubiquitous, appearing in every branch of human learning, not the least in mathematics—in category theory and the symmetries of group theory and topology, for instance—and in computer science. For myself, I have taken these abstractions of pure mathematics to their utmost level of generality in order to heal my fragmented, split mind in Wholeness. The creative power of Life and the Logos has thus shown me how to construct a self-inclusive map of the entire Cosmos and hence a comprehensive conceptual model of all evolutionary processes, whether they be material, biological, or mental, and the psychodynamics of society.

The mathematical graph is the basis of my meditation practice. First, I view Euler's abstraction of the bridges of Königsberg as a structure, consisting of forms and the meaningful relationships between them, as in a semantic network. I then view each node as a structure, consisting of a deeper level of forms and relationships. Continuing, these forms, as structures, disappear at deeper and deeper levels and I am just left with relationships between singularities, as points. Eventually, even these disappear through the practice of *neti neti*, and I reach the Origin of the Universe, as Oneness.

Conversely, any one structure is a node in a higher-level structure of forms and relationships. Eventually, these creatively expand to such an extent that they become a seamless continuum with no borders or divisions anywhere, which I call Wholeness. It is in this way that my individual consciousness expands and deepens to such an extent that it becomes coterminous with Consciousness itself, as the union of Cosmic and Unity Consciousness, which etymologically means 'knowing together'. As the body-mind-soul organism writing these words is a node in such a mathematical graph, he too disappears as a separate being, enabling me to draw the diagram of the Grand Design of the Universe on page xi.

It is vitally important to note that unifying all opposites is not speculative philosophy, as an intellectual pursuit, not explicitly grounded in experience. Bertrand Russell described philosophy as lying in the No-Man's Land between the warring factions of science and theology,[71] using a metaphor from the First World War, which so appalled him as a pacifist. A recent example of this conflict is *War of the Worldviews: Science vs. Spirituality*, by Deepak Chopra, a medical practitioner and renowned spiritual teacher, and Leonard Mlodinow, co-author with Stephen Hawking of *The Grand Design*, who are clearly not aware of the primary-secondary relationship between Reality and what most call reality.

For when we recognize the Absolute as the Divine Cosmic Context, science and theology are unified in Panosophy, and philosophy, as a distinct discipline, is squeezed out of existence. We can then address the perplexing paradoxes that have puzzled philosophers and mathematicians through the ages with sound Self-reflective Intelligence, mystical experience, and rational thought.

Those with a philosophical bent of mind might attempt to apply terms from philosophy to denote the all-inclusive worldview presented in this book, contrasting it with other philosophical schools of thought. But in so doing, Panosophy would no longer be all-inclusive. For instance, following the so-called Age of Enlightenment or Reason in the 1700s, in the next century philosophers coined the words *idealism*, *realism*, *nominalism*, and *conceptualism* to denote competing views of how to view the Universe. Robert H. Dicke and James P. Wittke clearly stated the distinction between realism and idealism in a classic textbook on quantum physics in 1960:

> A physicist is concerned with two worlds: a *real* external world, which is believed by physicists to have an objective reality, and an *image* of this world, an internal world, which he hopes is a reasonable model of the external world. The external world manifests itself through *sense* impressions; from birth, and indeed even before, the human brain is bombarded with *data* resulting from the stimulation of the sense organs by this external world (my emphasis).[72]

We can resolve the split between realism and idealism when we view the Totality of both our outer and inner worlds as a gigantic information system, as some scientists are doing today. For scientists do not just observe the actions of chemicals or sub-atomic particles in their laboratories or the behaviour of animals and galaxies in the wild, for instance. Rather, what they observe is data, interpreted as information and knowledge in conceptual models and cognitive maps in the psyche.

In this way, we can overturn the conventional scientific view that the territory exists before the map. For instance, in 1931, when commemorating the centenary of James Clerk Maxwell's birth, Einstein wrote, "The belief in an external world independent of the perceiving subject is the basis of all natural science."[73] Similarly, at about the same time, Alfred Korzybski made the famous assertion, "A map *is not* the territory it represents, but, if correct, it has a *similar structure* to the territory, which accounts for its usefulness."[74] Our minds create our reality, most evident in the complex structure called *Universe*.

So, once we can see that Consciousness is Ultimate Reality, we don't actually need any of these isms, which Satish Kumar, long the editor of the ecological and spiritual magazine *Resurgence*, jocularly called 'wasms' in a talk in the early noughties. For what appears to be real, from Latin *res* 'thing', is nothing but waves and ripples on the surface of the Ocean of Consciousness, called *māyā* 'deception, illusion, appearance' in Sanskrit. If we include the currents beneath the surface, everything that happens in the relativistic world of form is *līlā* 'play', the delightful play of the Divine in the manifest world. Our minds thus create an illusory sense of reality, vitally important to know at these end times of the patriarchal epoch we live in, with its twenty-odd war-mongering civilizations.

We also don't need words like *theism*, *monotheism*, *polytheism*, *deism*, *pantheism*, and *panentheism*, illustrating the immense confusion that humans have been in over the millennia about the relationship of humanity to Divinity. As God is everything and everything is God, we can resolve this confusion with the words *Gnosticism* and *Mysticism*, which it seems are unavoidable isms, resolving in direct experience the philosophies of pantheism and panentheism, which are closest to Immanent and Transcendent Panosophy. Theists and atheists are people who believe and don't believe in the existence of God, and agnostics don't know what to believe. On the other hand, Gnostics do not need to believe, because they know God in their direct experience.

In terms of mathematics itself, Morris Kline tells us in *Mathematics: The Loss of Certainty* that there have been four approaches to giving mathematics a sound foundation after paradoxes were found in the

foundations: logicism, intuitionism, formalist, and set-theoretic.[75] I have not studied these very much because the Principle of Unity emerging directly from the Divine Origin of the Universe guides all my reasoning, establishing my learning on Absolute Certainty. I have thus been led to give mathematics and the whole of human learning a sound foundation in harmony with the fundamental law of the Universe: the Hidden Harmony.

In summary, to liken transdisciplinary Panosophy to any specialist view of mathematics, science, or philosophy, or to any particular view of the Divine, would be an example of what Gilbert Ryle called a 'category-mistake'.[76] This is such an important point that I would like to quote in full the first example that Ryle gives of a category-mistake.

> A foreigner visiting Oxford or Cambridge for the first time is shown a number of colleges, libraries, playing fields, museums, scientific departments and administrative offices. He then asks, 'But where is the University? I have seen where the members of the Colleges live, where the Registrar works, where the scientists experiment and the rest. But I have not yet seen the University in which reside and work the members of your University'. It has then to be explained to him that the University is not another collateral institution, some ulterior counterpart to the colleges, laboratories and offices which he has seen. The University is just the way in which all that he has already seen is organized. When they are seen and when their coordination is understood, the University has been seen. His mistake lay in his innocent assumption that it was correct to speak of Christ Church, the Bodleian Library, the Ashmolean Museum and the University, to speak, that is, as if 'the University' stood for an extra member of the class of which these other units are members. He was mistakenly allocating the University to the same category as that to which the other institutions belong.[77]

Like a university, consisting of individual colleges teaching specialist disciplines, Panosophy is transdisciplinary and transcultural, integrating all knowledge into a coherent whole, enabling universities to live up to their name: to turn all disciplines of learning into a coherent whole.

Given this brief overview of the current state of the world of learning as it has evolved during the last few centuries and millennia, who is this book on *Unifying Mysticism and Mathematics* for? Well, one purpose of this book is to find a sense of closure with my forty-year writing career, seeking to explain the root causes of evolutionary change and conflict and suffering in the world. It is thus intended as the last of an evolutionary sequence of books and treatises I have written during the last ten years on how we could use the Principle of Unity to end the long-running war between science and religion and those between all the religions, rebuilding the entire world of leaning on the Truth.

Sadly, these nine books and treatises—three of which form a trilogy titled *Wholeness*—are mostly unread and therefore unpublished because healing this deep cultural wound is essentially experiential, not intellectual, which cannot happen within the context of any particular civilization, religion, discipline, or ideology. So, for the most part, scientists and technologists are driving the pace of change in society at unprecedented exponential rates of acceleration with almost no one knowing why this is so.

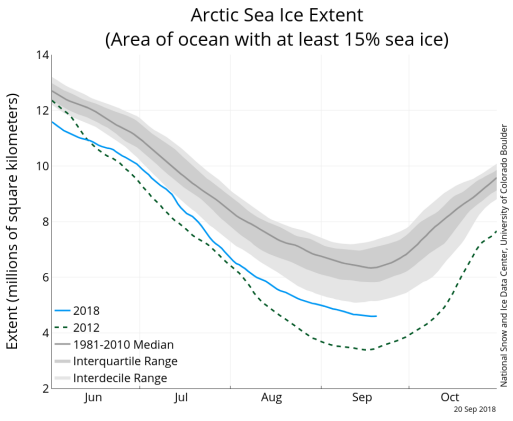
To overcome these psychospiritual limitations, for over thirty years I have been attempting to cocreate a life-enhancing, nurturing social environment where it is safe to question the fundamental beliefs and assumptions of the cultures we live in. For, as J. Krishnamurti, David Bohm, and Vimala Thakar have said, if we do not engage in such questioning then humankind is not a viable species. However, while there has been some interest in this initiative, whose latest manifestation is the Alliance for Mystical Pragmatics, it appears to be still-born, unlikely to be resuscitated. For those who present at and attend conferences like Mystics and Scientists and Science and Nonduality (SAND) in the UK and USA, respectively, have shown little sign in joining me in this epoch-making endeavour.

This brings me to another central issue. For many years, I visualized a gap of a few generations between the deaths of dysfunctional Western civilization and *Homo sapiens*, faced with a multitude of existential

threats to our well-being and survival as a species, some of which arise from discoveries in science and technology. However, this epoch, which I have been calling the Age of Light and the Mystical Society, has become shorter and shorter in my vision during the last few years, especially since I read *Extinction Dialogs: How to Live with Death in Mind*, which Andrew Harvey asked Guy McPherson and Carolyn Baker to write in 2014.

In December 2017, I met Guy, Professor Emeritus of Natural Resources at the University of Arizona, in Oslo. He explained to me why the collapse of the industrial economy, apparently our salvation, would reduce global dimming, accelerating the extremes of climate change, making our beautiful planet Earth uninhabitable, unable to provide us with the food we need to survive. Since then, Guy has moved on with his prognostications saying that the Arctic is projected to be free of ice by the late summer of 2019.

As this is likely to accelerate the release of methane gas, far more damaging than carbon dioxide, as explained in the Epilogue on page 207, he has thereby predicted that we are about to experience abrupt climate change, no longer able to grow the corn we need to bake our bread, a standard nutriment for thousands of years, even saying so on a television channel in New Zealand in 2018.



However, this diagram, from the National Snow & and Ice Data Center, indicates that the melting of Arctic sea ice is not happening as fast as in 2012, when the record reduction in sea ice occurred.[78] Despite the record temperatures we experienced in Sweden during the summer of 2018, in a similar manner to other countries in the Northern hemisphere, the Arctic's minimum sea ice extent that year was only the 6th lowest in the 39-year satellite record. So it seems that we still have a respite to realize our fullest potential as human beings before our inevitable demise.

For myself, I am now seventy-seven years young since my conception at the end of August 1941, three and four years younger than my parents were when they died. So even though I may be healthy enough to live well into my eighties, the 2020s, when my twin granddaughters will be in their teens, are likely to be the most tumultuous in the entire history of *Homo sapiens*. The vision that I have had that the harmonious, androgynous, peace-loving Age of Light could last for a couple of centuries following the collapse of the global economy now seems far too long. Nevertheless, there might still be enough time to fulfil my life's purpose to complete the final revolution in science, just as Kepler and Newton completed the first in the 1600s.

To this end, while I am Wholeness, living in the Eternal Now, with nobody and nothing outside me, there is one last unfulfilled task that has been on a back-burner for many years. Recognizing that mathematics is not based on the Truth, dispersing the misconceptions underlying mathematics would give me immense satisfaction. That, essentially, is why I feel moved to write this book from the autumn of 2018 to the summer of 2019, for as long as I have the energy to do so.

But rather than explicitly starting afresh at the logical beginning, at the Divine Origin of the Universe, the first chapter starts where the fourteen-billion-year history of evolution had reached in the 1970s, at the birth of the Information Society.[79] For this event marked the greatest revolution in human learning since our forebears began to pick up stones to make cutting tools with pieces of flint.

So, Chapter 1 begins in the workplace, outlining the business management and modelling problems that arise from the invention of the stored-program computer in the late 1940s. My interest in this central issue of our times began in January 1974, when I was appointed the systems engineering manager responsible for ensuring that the first computer that the British Post Office (now British Telecom) had bought from IBM passed its acceptance test. This computer system led me to study the way that humans interact with computers in timesharing systems, not easy to model in the business modelling methods of information systems architects, which are far beyond the financial modelling methods of accountants, economists, and bankers.

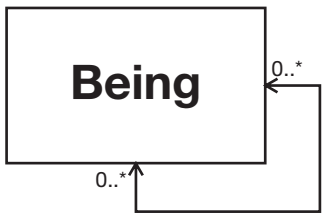
This first chapter thus outlines the way that resolving this business management problem has led me to develop a comprehensive model of the psychodynamics of society, exploring how we humans communicate with each other—including our self-reflective selves. This is the immediate background to the apocalyptic awakening I went through in the spring of 1980, as I realized that data patterns in humans and computers are synergistically energetic and causal.

So, the second chapter in this book on Integral Relational Logic is actually the first, for this describes how evolution has carried me from its Alpha to its Omega Point and back again. However, rather than explicitly starting with the fundamental law of the Universe, I begin with the primal concepts by which Life has shown me how to lift myself up by my bootstraps, corresponding to the bootstrap program in computers. In the event, it took thirty years before I saw and felt this way of presenting the art and science of consciousness that we all intuitively use to form concepts and organize our ideas in mathematical relations and graphs.

We then come to the main theme of this book, showing how to use the taxonomic facilities of Integral Relational Logic to map mathematics, not as an axiomatic, linear proof system, but as a generative science of patterns and relationships, emerging directly from the Divine Origin of the Universe.

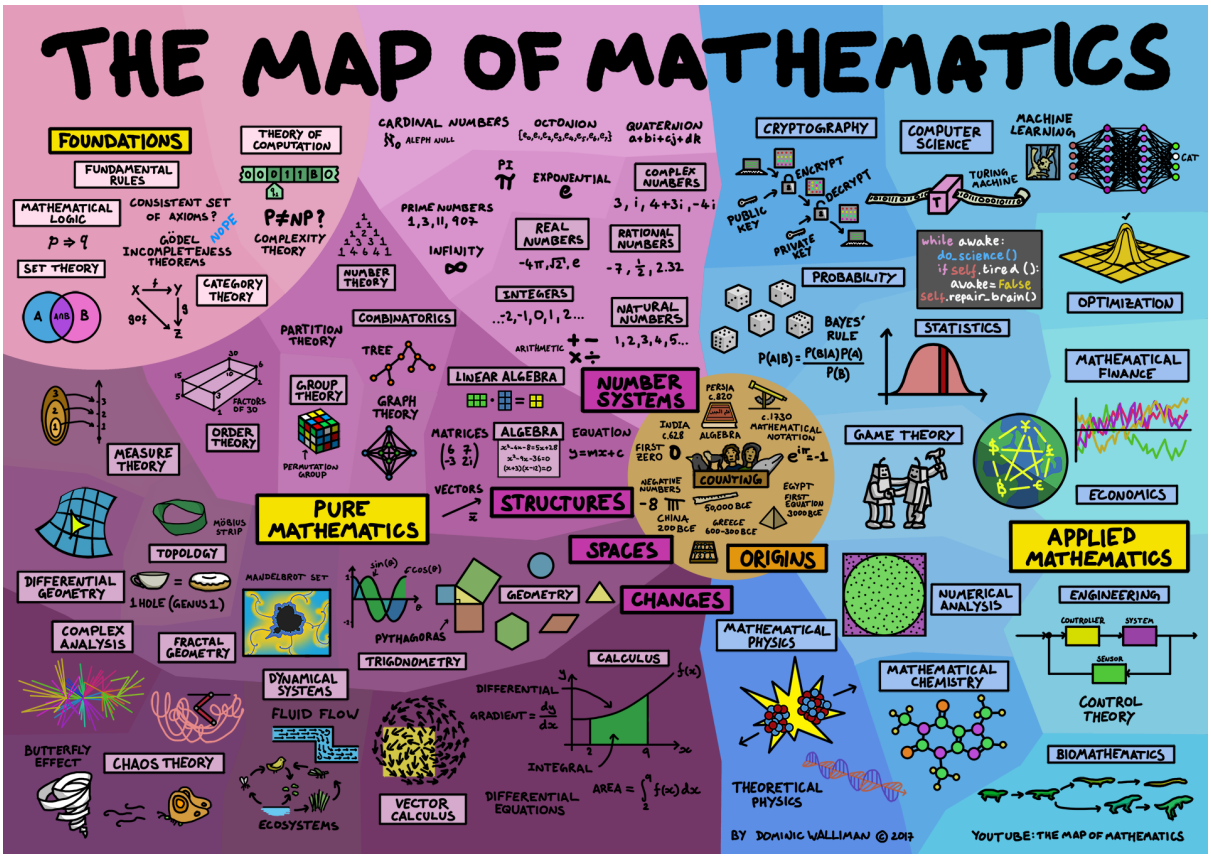
Chapter 4 addresses the entire number system, showing how numbers grow in kind from Nothing, as Emptiness or Zero, to the infinity of infinities, as Transfinity. This leads naturally to the fascinating subject of sequences and series, much studied by Euler and many mathematicians since. Then the fifth chapter shows how numerical relationships have become generalized in abstract algebra. At the time of writing this draft of the Prologue in January 2019, the content and structure of the last two chapters is rather hazy, requiring a great deal of additional research to clarify the intuitive understanding that I have at the moment.

The Epilogue then contains some reflections on what this thesis could mean for humanity during the last few years of our existence on Earth, recognizing that none of us is ever separate from any other being, depicted in this simple map of the Totality of Existence in the notation of the Unified Modeling Language, developed at Rational Software in the 1990s, now owned by IBM, my former employer.



In this cooperative manner, we could rise above the level of our machines, realizing our fullest potential as human beings. By using Integral Relational Logic to map deep learning and quantum computation, we could demonstrate the limitations of these techniques, perhaps stimulating scientists and mathematicians to look inside themselves to discover the root cause of their thinking and behaviour patterns.

It is in this way that Integral Relational Logic provides the presemantic underpinnings for the foundations of mathematics, displayed in the top left-hand corner of the map of mathematics on the next page, which Dominic Walliman presented on his Domain of Science channel on YouTube on 1st February



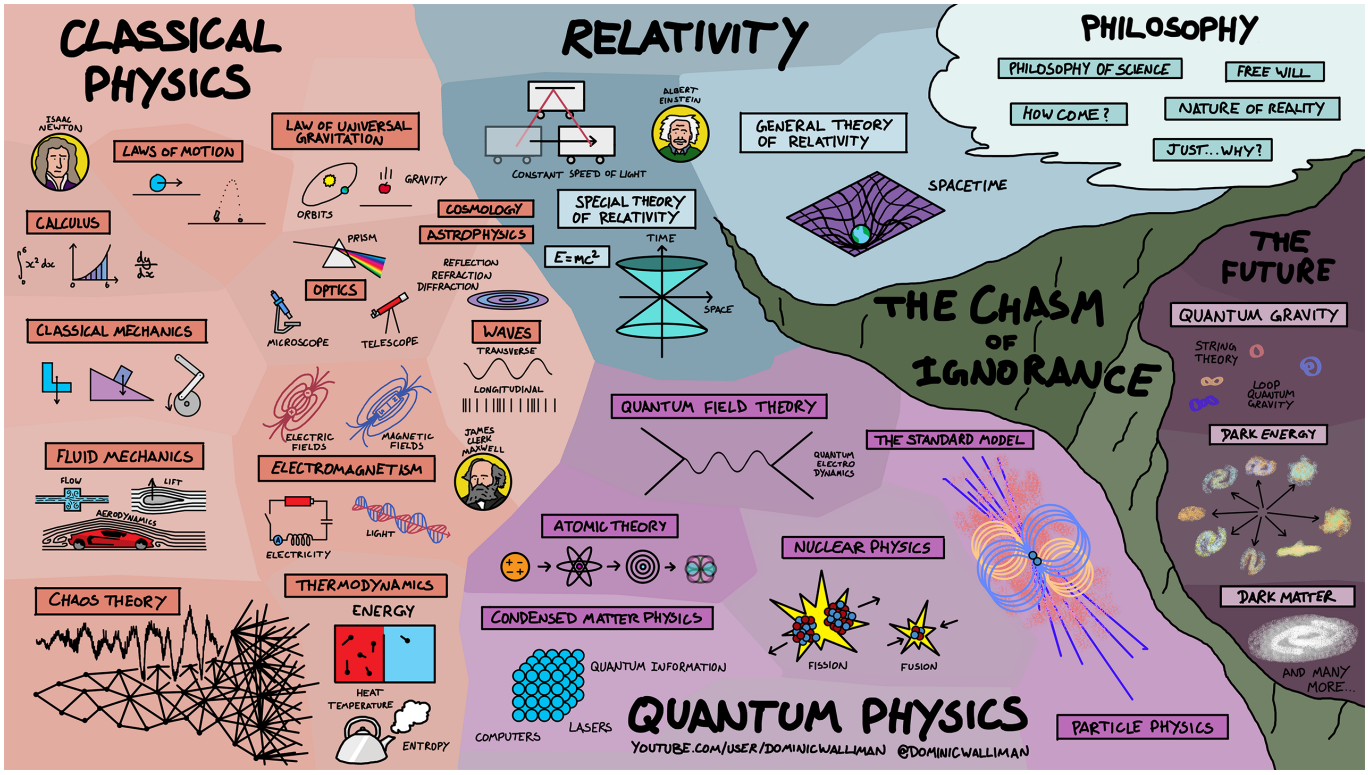
2017, with nearly three million views.[80] From this solid foundation, we can then map all other disciplines, from psychology to physics, for we all implicitly use this universal system of thought every day to form concepts and organize our ideas in tables and semantic networks or graphs.

Influenced by my business career in the data-processing industry, I use the word *presemantic* to explain how this universal system of thought came into existence through an apocalyptic epiphany to heal a cataclysmic prenatal trauma. And most psychiatrists are not taught to explain such Kundalini awakenings, also believing that such intrauterine experiences cannot possibly have any effect on later development in childhood and adulthood.

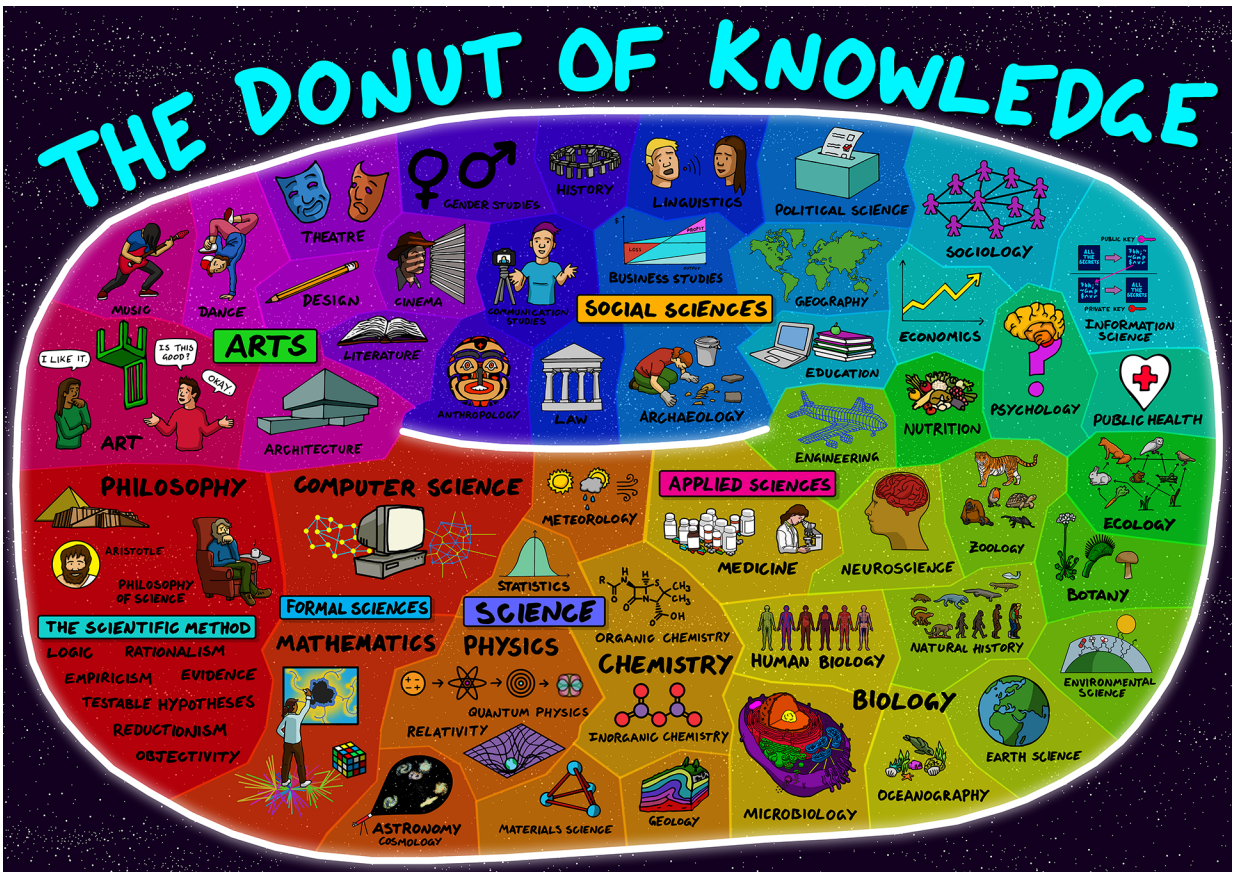
So, for many years, as I have sought to present a meaningful cosmology appropriate for our Information-Knowledge Society, I have faced a similar situation to those of Kepler, Newton, Einstein, and Bohm when they published their revolutionary cosmologies, which few understood at the time of publication. The basic reason why medical practitioners, in particular, cannot understand the root cause of human behaviour is that they still attempt to do so, for the most part, within the context of a deluded conception of the Universe, handed to them by the physicists, and they from the Babylonians and the Greeks.

We can see why this is so from Dominic Walliman's map of physics on the next page, presented on YouTube on 27th November 2016.[81] Physics has been incredibly successful in providing many of us in comparatively developed countries with amazing creature comforts since the beginning of the Industrial Revolution in the middle of the eighteenth century. But physics cannot answer fundamental questions about the nature of reality or the future of our species. We can only answer such questions by diving into the chasm of ignorance that lies within our psyches, going right back to the Divine Origin of the Universe, before our conception as a species and as individuals.

Unifying Mysticism and Mathematics thus presents the algebra of algebras with which David Bohm sought to show how his theory of the Implicate order unifies quantum and relativity theories in Wholeness in a thoroughly rational manner.[82] It thus provides an expression of the solution to the ultimate problem of



human learning, much sought for over the centuries and millennia. Dominic Walliman presented his own solution to this problem in a video on his YouTube channel on 19th September 2018 titled 'Map of Science (and everything else)', saying, "The grand aim of science is to figure out how it all works."[83]



Moving counter-clockwise from the philosophy of scientific method in the map above, he pointed out that reductionistic methods that have been developed to study what is generally known as the 'natural world', break down when we reach psychology, at the heart of the social sciences, which are much more complex

systems than those studied by physicists. Even in biology, we cannot study cells in isolation from everything else; we need to study them in the context of the organism of which they are a part. As he said, capturing all this complexity in one model is incredibly hard.

Then, moving into the arts, this is a place of pure subjectivity, generally regarded as outside the domain of objective science, exploring the human condition with all its emotions and vicissitudes through books, theatre, and art, etc. Yet, when we look at the creative processes of scientists themselves, we see that they too are artists at heart.

This is most obvious from the root of *science*, which is Latin *scīre* 'to know', from PIE base **skei-* 'to cut, split', meaning 'to separate one thing from another, to discern', also the root of *schizophrenia* 'split mind' and *consciousness* 'knowing together'. So the much sought-for science of consciousness is actually an oxymoron, for it is the purpose of science to separate through analytical methods.

In contrast, art is a synthesizing activity, putting back together what science has divided, for *art* derives from Latin *ars* 'skill, way, method', from PIE base **ar-* 'to fit together', also root of *coordinate*, *reason*, *harmony*, and *order*. So Integral Relational Logic is both the art and science of thought and consciousness, integrating and unifying the differences that the analytical mind discerns.

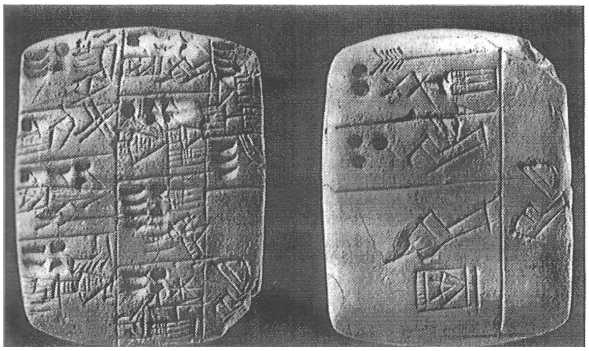
It is in this healing manner that my own individual consciousness has deepened and expanded to such an extent since 1980 that it has become coterminous with Consciousness itself, as Ultimate Reality. However, even though I am Wholeness, like everyone else, able to see the Totality of Existence as a coherent whole, I cannot see the how and when of human extinction.

As Matthew Fox wrote in the Preface to Andrew Harvey and Carolyn Baker's *Savage Grace: Living Resiliently in the Dark Night of the Globe* from 2017, "Ours is a time not only for scientists and inventors but also mystics and contemplatives to join hands so that our action flows from being and from a deep place of return to the Source."[84] And as the authors say, "Even among many of our friends and acquaintances who are awake to the potential for near-term human extinction, we notice an implicit and almost-pathological demand for certainty. Many are obsessed with the year they believe humans will become extinct. Is it 2026, 2030, 2050, next year? As if we could know."[85]

For myself, all I can do is live in the Eternal Now as well as I can, guided by my own inner energies to write this book on *Unifying Mysticism and Mathematics*, as the glorious culmination of my life's work. I trust that what this means could become a little clearer as I demonstrate IRL in action in this book.

1. Business Modelling

Before we start afresh at the very beginning—at the Divine Origin of the Universe—we begin in the workplace. Especially as it has been managed since the birth of writing, at the dawn of the patriarchal epoch with its twenty-odd civilizations over 5,000 years ago.[86] For the key characteristic of all societies since then is that we humans have used *record keeping* to manage our business affairs. For, instance, the first writing to be discovered on a clay tablet in Uruk (modern Erech) dates back to 3300 BCE, detailing the allotment of malt to a number of people and with stock accounts of barley on the reverse.[87]



Since the invention of the stored-program computer in the late 1940s, we have been keeping many of our business records electronically. At first, the primary medium was magnetic tape, like the now obsolete cassette music tapes. However, this was rather inconvenient for tapes are linear devices. So to find a particular record, it was necessary to scan the tape sequentially.

A solution to this problem was found with the invention of direct access storage devices (DASD), a little like also-obsolete compact discs, enabling computer programs to go directly to a particular record of interest. IBM invented perhaps the first computer disk storage system in 1956, called the RAMAC (random access), displaying it at the 1958 World's Fair in Brussels, where it was used to answer questions on world history in ten languages.[88]

With the availability of direct access storage devices, computer scientists began to puzzle about the underlying structure of data. The pioneering figure in database management systems (DBMS) was Charles Bachman, who took a nonhierarchical, network approach in a system called Integrated Data Store (IDS), when working for General Electric.[89] In contrast, IBM took a hierarchical approach with its Information Management System (IMS), developed by or with North American Rockwell.[90]

Ted Codd of IBM resolved this dichotomy in 1970, when he wrote an eleven-page seminal paper with the prosaic title, 'A Relational Model of Data for Large Shared Data Banks'.[91] He showed that the structure of data, viewed as a whole, is a multidimensional network of hierarchical relationships, where a dimension is a domain of values—both quantitative and qualitative—in a relation, associated with a column in a table.

But such a structure does not only apply to information that companies might hold about customers, products, and employees, for instance, and the relationships between them. It also applies to our ideas. As Codd noted in the second paragraph of this little-known paper, the relational model of data is nondeductive, the most fundamental change in Western reason since Aristotle's syllogism. The relational model of data provides a mathematical language for representing the basic element in the data-processing

industry: data itself, using *data* as an uncountable singular noun, as is common in many quarters in the industry.

This rather arcane paper led to the formation of a multibillion-dollar industry, Larry Ellison being one of the first to see the immense potential of this universal way of organizing data. He was the co-founder of Oracle, today a Fortune-500 company, becoming one of the richest men in the world in monetary terms. You cannot order a book or airline ticket on the Internet without invoking the relational model of data behind the scenes.

Both on external storage and in the main memory of computers, this data takes many forms, but all ultimately being denoted in strings of zeros and ones. For instance, the number 5 can be represented as either `00000101`, as an integer, or `00110101`, as a character, 35 in hexadecimal (35_{hex}). The letter *a*, which could be a letter in a string of text or a variable in an algebraic expression, is represented as 61_{hex} in the American Standard Code for Information Interchange (ASCII) and UTF-8, as an encoding of Unicode, widely used on the Internet.[92] And the + operator is denoted by $2B_{hex}$.

So the Bible, the entire works of Shakespeare, digital videos and pictures, and mathematical formulae in Donald Knuth's TEX typographical system, for instance, can be represented as strings of hexadecimal characters, which map to forms that humans can read. However, computers do not recognize formulae that we humans write as strings of characters.

For instance, to add 5 to the variable *a* in a programming language, the human programmer would typically enter the expression '`a+5`', represented as a binary string $612B05_{hex}$. But then programs called compilers or interpreters convert such strings into a sequence of instructions that the central processing unit (CPU) can 'understand', with relevant operation codes (opcodes) and registers containing the data that is to be operated on.

Even programs written in an assembly language—closely corresponding to the machine's instruction set—are originally written as strings of characters, converted into an executable program by an assembler. It is very rare for people to think directly in the binary representation of a program, for the semantic gap between how humans think and how computers function is too great.[93]

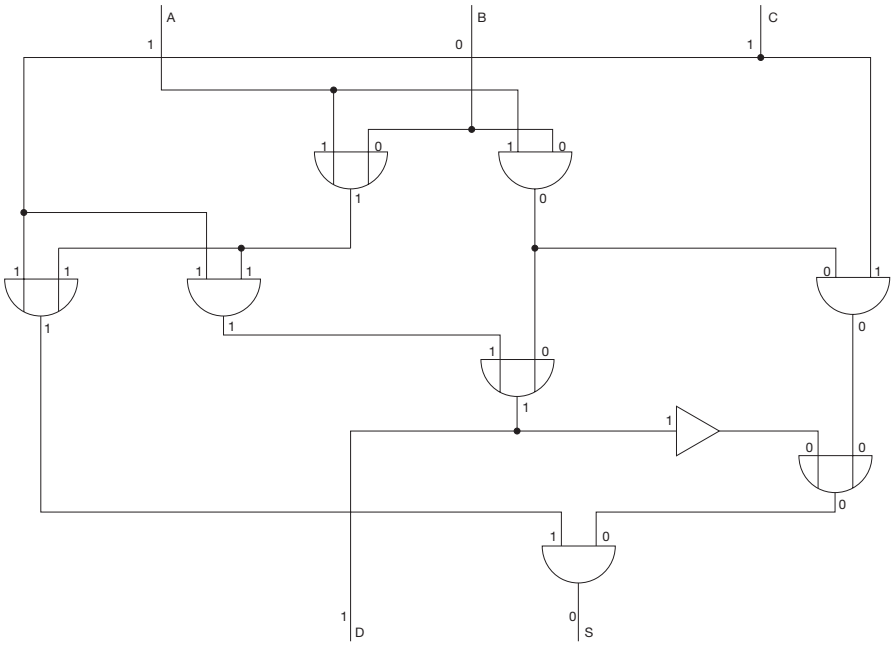
However, we can go deeper than the machine instructions that operate on one or a few bytes of data at a time. The gates in the core of the CPU operate on bits of data, such as AND (∧), OR (∨), and NOT (¬ or ~) gates, corresponding to conjunction, disjunction, and negation in the tautological propositional calculus and intersection (∩), union (∪), and not (¬) in set theory.

These logic functions derive from George Boole's *Laws of Thought*, which laid down the foundation of mathematical logic, published in 1854. As he said in the opening paragraph of this book, following a mystical experience he had had as a seventeen-year-old, twenty-one years earlier, "The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed," with the purpose of exploring "the nature and constitution of the human mind".[94]

As the result of this seminal book, Boole's name has been 'immortalized' in the operators of Boolean algebra, well familiar to anyone engaged in making more advanced searches of databases on the Internet, and in the Boolean data type in many programming languages, having the values 'true' or 'false'. However, the mystical origins of mathematical logic have been lost in the mists of time. First Charles Sanders Peirce wrote[95] and then Gottlob Frege and Bertrand Russell[96] agreed that logic, as the science of mind and reason, and psychology, as the science of mind and consciousness, have nothing to do with each other.

Bertrand Russell, in particular, attempted to base arithmetic on logic by using his distorted theory of types to eliminate paradoxes found in the foundations of mathematics. By eschewing self-referencing propositions, like "This sentence is false," Russell also stultified Self-reflective Intelligence, the Divine quality that distinguishes humans from machines.

Nevertheless, we can see the relationship of arithmetic to logic at the heart of modern CPUs, for basic arithmetical operations on binary digits can be represented in Boolean algebra, as this diagram of a one-bit adder shows.[97]



Here, two bits, A and B, are added to a carry over, C, from a previous operation. The result is S, with a new carry over, D. Using the modern notation of Boolean algebra:

$$S = (C \vee (A \vee B)) \wedge ((\neg((C \wedge (A \vee B)) \vee (A \wedge B))) \vee ((A \wedge B) \wedge C))$$

$$\text{and} \quad D = (C \wedge (A \vee B)) \vee (A \wedge B).$$

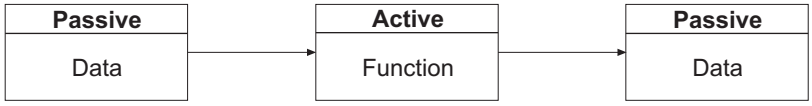
As it turns out, all possible connectives between binary values, sixteen (2^4) in all, are expressible in terms of just conjunction and negation.[98] The AND and NOT operators are functionally complete. It might therefore appear that these Boolean operators, corresponding to truth functions in propositional calculus, are as far as we can go into the foundations of dualistic logical relationships.

However, in 1913, Henry M. Sheffer showed that we can take this reductionistic, analytical process even further. Boolean functions can be defined in terms of a single NAND (↑) gate, known today as a Sheffer stroke (|),[99] which has a dual with the same properties: a NOR (↓) gate.[100] Sheffer was not the first to notice this possibility. Peirce anticipated Sheffer's landmark paper in 1880 with an unpublished manuscript titled 'A Boolian [sic] Algebra with One Constant'. Pierce introduced the arrow notation—which he called ampheck, from Greek *amphekes* 'double-edged'—in 1902 in Chapter 3 '*The Simplest Mathematics*' of *Minute Logic*, his third and final attempt to write a book on his triadic architectonic. In the event, these pieces were not published until 1933 in Volume IV of his *Collected Papers*, titled *The Simplest Mathematics*.[101]

So we can look at the logic underlying classical computers in a hierarchical structure of levels, growing in a generative manner from a simple seed, a central theme of this book. In theory, the algorithms that are taking over the Internet could thus be programmed in a long string of characters using just the Sheffer stroke in Jan Łukasiewicz's Polish notation. Not very digestible, but quite possible. To communicate with computers, we need to close the semantic gap between us and them, which we look at in a moment.

Managing data as a corporate resource

As mathematics is the science of finding general patterns and relationships in the data patterns of experience, we can see from this preamble about the nature of data in computers that there are two types of data, active and passive, corresponding to operations or instructions and the 'raw' data they process. Such a relationship is represented in this fundamental diagram of the data-processing function.



This diagram is a universal representation of a machine, viewable at many different levels. The word *function* in the diagram could represent a Sheffer stroke, AND gate, + operator, square root function, function in a programming language like C, method in a class in C++, for instance, or even a complete application, like Microsoft Word, Abobe Photoshop, open-source Python interpreter, and Apple Swift compiler. Such functions process inputs to produce outputs.

We can thus see that this notion of function in computers applies to both hardware and software. In terms of hardware, the distinction between active and passive data is implemented in the central processing unit (CPU) and in random-access memory (RAM), corresponding to what Charles Babbage called the Mill and Store in his Analytical Engine, designed in the mid 1800s, terms he borrowed from the textile industry.[102] He envisaged that the instructions needed to operate the machine would be entered on punched cards, like those that Joseph-Marie Jacquard had invented to automatically control the patterns of weaving of cloth in a loom. Indeed, in her memoir on the analytical engine, Ada Lovelace delightfully wrote, "We may say, most aptly, that the Analytical Engine weaves algebraic patterns just as the Jacquard-loom weaves flowers and leaves."[103]

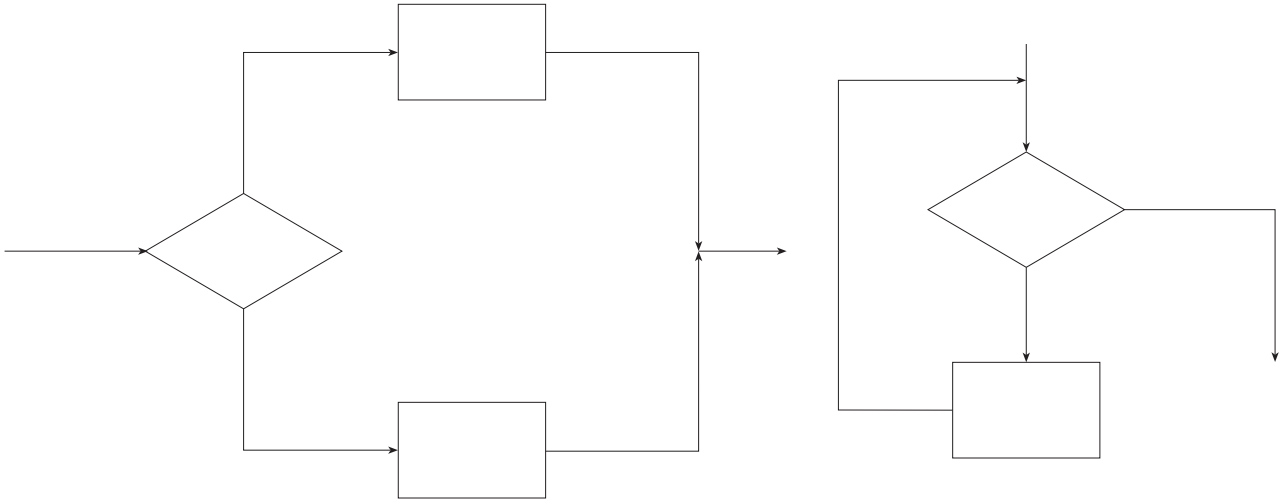
In modern stored-program computers, programs in the machine weave such algebraic patterns. It is thus vitally important not to be distracted by the hardware, for it is the software that determines how computers function. Indeed, as Andrew S. Tanenbaum wrote in *Structured Computer Organization*, "*hardware and software are logically equivalent*," written in italics to emphasize the central theme of his book. Despite Descartes' determination to separate body and mind, computer hardware and software form a continuum. Whether a particular function is implemented in hardware or software is concerned with practical issues like cost, speed, memory, and flexibility.[104]

Now, Codd showed with the relational model of data that passive data is a multidimensional network of hierarchical relationships. So what is the underlying structure of active data in computers? Well, computers are essentially linear devices, executing sequences of instructions, with an occasional conditional or unconditional jump or branch instruction to follow a different line of 'reasoning', like deductive logic. Assembly languages inevitably follow this general pattern. However, such low-level programming languages are semantically indigestible. So in practice, during the first few decades of the computer age, many different levels of programming languages emerged closing the semantic gap between the way that human beings think and the way that these thoughts are implemented in hardware, software, and firmware, lying between the two.

In the 1950s and 60s, high-level languages, like FORTRAN (FORmula TRANslation) and COBOL (COmmon Business Oriented Language), developed by John Backus at IBM and Grace Hopper, a member of the Conference on Data Systems Languages (CODASYL), respectively,[105] closely mimicked the underlying structure of machine instructions. In particular, they included a `GO TO` instruction to match the branch or jump instructions in assembly languages.

However, the use of the `GO TO` instruction could lead programs to look like plates of spaghetti, which were notoriously difficult to debug and maintain. To resolve this problem and to bring programming languages closer to the underlying structure of the human mind and hence that of the Universe, in 1966, Corrado Böhm and Giuseppe Jacopini from Italy wrote a paper, today known as the 'Structured Program Theorem',[106] in which they proved mathematically that all programs could be written with just three control structures.

As can be seen here, the first is a simple block, consisting of a sequence of instructions executing one after another. In terms of structure, these sequences could be grouped together in functions or subprograms, executed sequentially. The key concept here is a process box, with one input and output, corresponding to the basic data-processing diagram. The other control structures are a conditional block (typically implemented with `if-then-else`, `case`, or `switch` statements) and an iterative loop block (implemented with `for`, `while`, or `do` statements or recursive functions), illustrated here:[107]



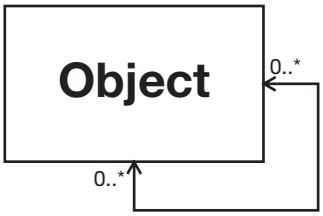
This paper was one of the most important papers in the history of the data-processing industry, for it paved the way for structured programming languages and systems design, greatly closing the semantic gap between machines and human beings. To this effect, Edsger W. Dijkstra, a pioneering programmer from the Netherlands, wrote a famous letter in 1968 titled 'Go To Statement Considered Harmful', in which he described the 'disastrous effects' of the `goto` statement, and that it should be abolished from all-high level languages. The `goto` statement had to go.[108]

In the event, this happened through a parallel development, which began in the Norwegian Computing Center in the mid 1960s. There, Kristen Nygaard and Ole-Johan Dahl, together with Bjørn Myhrhaug, designed a computer language called SIMULA (SIMUlation LAnguage) intended to simulate the operation of systems composed of discrete events, such as traffic patterns in towns and cities, communication networks, or the day-to-day operation of a retail business.[109] This gave rise to object-oriented programming languages, like Java, Objective C, and Smalltalk, whose primary constructs are those of class and object, as instance of class, corresponding to Plato's notions of universals and particulars in *The Republic*, showing that universals are not eternal. In Smalltalk—the archetypical object-oriented language and environment—even numbers are instances of classes, such as **Integer** and **Float**.

These classes are related to each other in networks of hierarchical structures, rather like passive data but with a somewhat richer semantics. Classes encapsulate both active and passive data structures, as functions

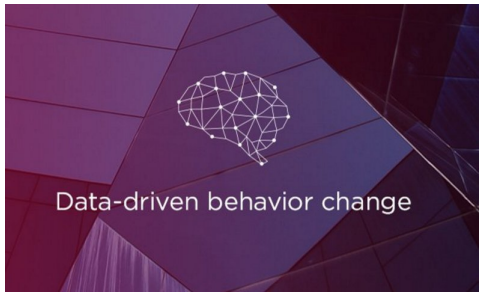
and variables, acting rather like 'black boxes' with a clearly defined interface. Over the years, software developers have produced many class libraries so that it is no longer necessary for them to constantly reinvent the wheel. Like architects and house builders, many of the components they need to build computer systems are readily available 'off the shelf'. This is one of the principal reasons why applications on the Internet have been expanding exponentially since the 1990s.

In object-oriented programming languages and modelling methods, the superclass of all business classes, depicted in semantic class diagrams, like mathematical graphs, is that of **Object**, each of which is related to any other object in zero to many ways, illustrated in this diagram in the notation of the Unified Modeling Language (UML). UML was developed in the 1990s by Grady Booch, James R. Rumbaugh, and Ivar Jacobson of Rational Software, now a division of IBM.



This generic notion of function in computers is an extension of George Boole's operator theory, outlined in a paper he wrote in 1844 titled 'On a General Method in Analysis' published in the *Philosophical Transactions of the Royal Society of London*,[110] which won him the Royal Society's first gold medal for mathematics, known as the Royal Medal.[111] Drawing on Duncan F. Gregory's generalizing principles, Boole helped free mathematics from the tyranny of number systems, regarding the essence of mathematics as "the study of form and structure rather than content, and that 'pure mathematics' is concerned with the laws of combination of 'operators' in their widest sense." For instance, he noted that the commutative and distributive laws of arithmetic could equally apply to differential operators and geometric transformations, leading to abstract algebra, which we look at in Chapter 5.[112]

But why stop with computer science and mathematics? Data, in both its active and passive forms, is everywhere today and has immense power to affect people's lives, as many act for much of the time like cogs in the economic machine, where functions in a generic sense are typically called procedures or processes. Indeed, with malware, fake news, cybercrime, trolling, and other nefarious deeds by both individuals and governments hitting the headlines, more and more attention is turning to the influence that data can have, as we can see in the furores around the global social media networks. As an example, here is a graphic for Cambridge Analytica, a controversial consultancy company that rose and fell in just five years, because it was engaged in data-driven behaviour change for exploitative political purposes.



So if we are to awarely manage our business affairs with full intelligence and consciousness of what we are doing, we need to take a psychological perspective, managing data as a corporate resource. This was a marketing slogan in IBM (UK) in 1979, which got me thinking about just what is this thing that we are supposed to be managing. In the event, it has taken me nearly forty years to find out, as I am endeavouring to explain in this book as clearly and simply as possible.

Before we look a little more closely at the data interface between humans and machines, it is perhaps simpler to begin with the relationship between data and information, which can become symbolic knowledge, as facts turn into theories as conceptual models or cognitive maps. In the data-processing (DP) and information-technology (IT) industry, this relationship is defined in one simple sentence: *information is data with meaning*, data being what exists prior to interpretation by an intelligent being.

Although this is not universal, in the DP industry *data* is often used as an uncountable noun, more like sand than pebbles, the plural of the Latin *datum* 'that which is given', from the Latin *dare* 'to offer, give', from Proto-Indo-European (PIE) base *dō* 'to give', also root of *donor*, *endow*, *dowry*, *Pandora* ('having all gifts, all-gifted', from Greek *dōron* 'gift'), and Sanskrit *dā* 'to give' and *da* 'gift'. *Information*, on the other hand, derives from the Latin *informāre* 'to give form and shape to, form an idea of'. So information is morphogenetic, from Greek *morphē* 'form, shape', as some biologists, such as Rupert Sheldrake[113] and Armand Leroi,[114] have pointed out.

This conception of information is quite different from that which reductionist scientists use. For them, the basic unit of information is a bit (*bi*nary digi*t*), which can only be 0 or 1; it cannot be both or something in between, in conformity with Aristotle's Laws of Contradiction and Excluded Middle, expressed in Boolean algebra like this, illustrating their duality:[115]

	Logic	**Sets**
Law of Contradiction	$A \wedge A' = 0$	$A \cap A' = 0$
Law of Excluded Middle	$A \vee A' = 1$	$A \cup A' = 1$

Some computer scientists have extended this notion in quantum computation with that of qubit (*qu*antum *b*it), in a probabilistic superposition of two states simultaneously. But a qubit, as the basic unit of so-called quantum information,[116] does not represent the essence of information, which is meaning, a semantic quality, not a mathematical or physical one.

This reductionist notion of information derives from a paper that Claude Shannon, misleadingly known as 'the father of information theory', wrote in 1948 titled 'A Mathematical Theory of Communication'. As Shannon admitted in an article he wrote for the fourteenth edition of *Encyclopædia Britannica*, "The signals or messages need not be meaningful in any ordinary sense."[117] Communications theory is not concerned with the meaning of the information in messages, but solely with signs, codes, and the quantitative measurement of these entities in a mechanistic, stochastic sense, closely related to the concept of entropy in thermodynamics, as I outline in my unpublished book *The Theory of Everything*.

Information systems architects engaged in business modelling use a quite different definition of the relationship between data and information, words that are often used synonymously. For instance, in *Management Information Systems*, one of the books that influenced IBM's first attempt to build information systems modelling methods in the late 1970s, Sherman C. Blumenthal gave these definitions:

> A **datum** is an uninterpreted raw statement of fact.
> **Information** is data recorded, classified, organized, related, or interpreted within context to convey meaning.[118]

Norman Lindop's *Report of the Committee on Data Protection*, from 1978, which led to the UK's Data Protection laws, provides a further description of the differences between data and information:

> So far, in this chapter, we have used the word *information* because that is the word and the concept with which most people are familiar. The computing community make much use of the word *data* (the Latin word *datum*, of which *data* is the plural, literally means that which is given) using it to mean raw material which is put into data processing systems. A primary function of data processing is to collect and relate items of data and to operate upon them to produce outputs which are meaningful to the users of the systems in the fulfilment of their purposes. It is these outputs which inform and which are rightly described as information.[119]

To give a third example, just to make sure that the distinction is clear, these definitions come from the *American National Dictionary for Information Processing Systems*:

> **data**. Any representation subject to interpretation (such as through analysis or pattern matching) or to which meaning may be assigned, such as by applying social conventions or special agreed upon codes.
> **information**. The meaning that is currently assigned to data by means of the conventions applied to these data.[120]

Even our children are being taught to make this distinction between data and information. As a final

example, at the end of the twentieth century, my sixteen-year-old niece took a two-year course in information technology for nonprogrammers in which her textbook gave these definitions for data and information:

> **Data** may consist of recorded facts, events or transactions.
> **Information** is data that has been processed into a form that is useful, or data that has been given a meaning by putting it into context.[121]

The meaning of meaning

The distinction between meaningless data and meaningful information and symbolic knowledge leads to the meaning of meaning, one of the most profound questions we can ask ourselves, comparable to "Who am I?", lying at the heart of *Jñāna yoga*, the path of wisdom and abstract knowledge in Advaita, whose foremost proponent in the twentieth century was Ramana Maharshi.

The Moravian Jan Ámos Komenský (Comenius), known as the 'father of modern education', was one of the first to address the meaning of meaning in the 1600s, making a clear distinction between words and the ideas of things that they represent, language often acting as an iron curtain inhibiting meaningful communications.[122] As he said, "men commonly do not speak, but babble: that is, they transmit not as from the mind to mind things or the sense of things, but exchange between themselves words not understood, or little or ill understood."[123]

The Lithuanian linguist Algirdas Greimas also addressed this tricky problem in an essay *Du sens: Essais sémiotiques* published in 1970. This was translated into English in 1990 in *The Social Sciences: A Semiotic View*, the opening paragraph of the first chapter titled 'The Meaning of Meaning' being:

> It is extremely difficult to speak about meaning and to say something meaningful about it. The only way to do this adequately would be to construct a language that signified nothing. In this way an objective distance could be established that would allow holding meaningless discourses on meaningful ones.[124]

So what do we mean when we say that we interpret data as meaningful information and knowledge? Comenius and Greimas suggest that we cannot satisfactorily answer this question with a language that contains interpretations that have already been made. Furthermore, if the context or worldview we use to interpret data contains preconceptions, we similarly cannot adequately address the meaning of meaning.

This is not a critical issue that only affects the way we communicate with each other and our computers. It also affects scientific method, in the way that we determine what is true in our own experience. For even space, time, and the material world around us, including our bodies, is data before we interpret it as knowledge and information. In other words, meaningless data—whether it be interpreted as physical or nonphysical, inner or outer—constitutes the entire Totality of Existence, underlying the Universe, which becomes the Cosmos when we learn to interpret all these data patterns as a coherent whole.

But who or what is the Donor of all these data patterns? Well, this is very simple. The Ultimate Donor of everything that exists in the ever-changing manifest world of form is the Formless Absolute, which we can best call the *Datum* 'the Giver', the Immortal Ground of Being that we all share. The Datum alone—as the Divine Origin of the Universe, experienced as Love and Consciousness—is Reality.

This is not something that can be debated, have learned discussions about. This is something we sense, feel, and know with Absolute Certainty through Gnosis, beyond symbolic knowledge. Even the symbols in this sentence, like *sense* and *feel*, are inadequate to denote such an experience, when the experiencer, as an apparently separate being, disappears, as many mystics through the ages have experienced,

For instance, the opening words of Laozi's *Tao Te Ching* are "Tao can be talked about, But not the Eternal Tao. Names can be named, But not the Eternal Name."[125] Similarly, Thich Nhat Hanh tells us that

Shakyamuni Buddha (sage of the tribe of Shakya) said to Ananda, his most devoted disciple, "Ananda, the teaching on the emptiness of self is meant to guide our meditation. It is not to be taken as a doctrine. If people take it as a doctrine, they will become entangled by it. I have often said that the teaching should be considered as a raft used to cross to the other shore or a finger pointing to the moon. We should not become caught up in the teaching."[126] And here is Pseudo-Dionysius the Areopagite's sublime description of the Datum of the Universe, the entire final chapter of *Mystical Theology*, with the title 'That the supreme Cause of every conceptual thing is not itself conceptual':

> Again, as we climb higher we say this. It is not soul or mind, nor does it possess imagination, conviction, speech, or understanding. Nor is it speech per se, understanding per se. It cannot be spoken of and it cannot be grasped by understanding. It is not number or order, greatness or smallness, equality or inequality, similarity or dissimilarity. It is not immovable, moving, or at rest. It has no power, it is not power, nor is it light. It does not live nor is it life. It is not a substance, nor is it eternity or time. It cannot be grasped by the understanding since it is neither knowledge nor truth. It is not kingship. It is not wisdom. It is neither one nor oneness, divinity nor goodness. Nor is it a spirit, in the sense in which we understand that term. It is not sonship or fatherhood and it is nothing known to us or to any other being. It falls neither within the predicate of nonbeing nor of being. Existing beings do not know it as it actually is and it does not know them as they are. There is no speaking of it, nor name nor knowledge of it. Darkness and light, error and truth—it is none of these. It is beyond assertion and denial. We make assertions and denials of what is next to it, but never of it, for it is both beyond every assertion, being the perfect and unique cause of all things, and, by virtue of its preeminently simple and absolute nature, free of every limitation, beyond every limitation; it is also beyond every denial.[127]

It therefore doesn't make sense to say that we can have a personal relationship with the ineffable Datum of the Universe, like the Christian God. Indeed, it is pointless to develop theologies or philosophies on the concept of Datum, for it is quite without meaning, utterly meaningless. The meaningless Datum thus provides the Cosmic Context and Gnostic Foundation for the Theory of Everything and the meaningless data patterns that arise from it lead to the language that Greimas suggested we need in order to hold meaningless discourses on meaningful ones, as we see in Chapter 2 on Integral Relational Logic.

Now coincidentally, Latin *dare* could also mean 'to cause', from PIE base **dhē-* 'to set, put', also root of *do*, through a Germanic path, and a host of words from Latin *facere* 'to do, make', such as *affect*, *efficient* and *faculty*. The Datum is thus the Ultimate Cause of all change in the Universe, an ancient idea in new form. For instance, in Book VIII, Section 4 of *Physics*, Aristotle said that everything that changes is changed by something and in Section 5 that there is a first agent of change that is not changed by anything.[128] Thus the notion of an Unmoved Mover entered Western philosophy, expressed in *Metaphysics* in this way: "Now since that which is moved must be moved by something, that the prime mover must be essentially immovable, and eternal motion must be excited by something eternal."[129] In *Summa Theologiæ*, Thomas Aquinas then took Aristotle's mechanistic cause-and-effect chain as the basis for his five proofs for the existence of God, as the Unmoved Mover.[130]

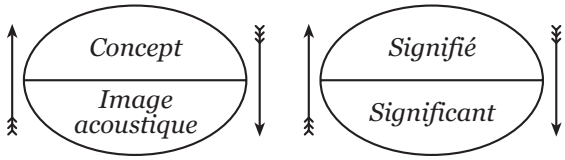
The Datum is thus the Immanent and Transcendent Presence of the Divine, etymologically 'before being' or 'prior to existence', for *Presence* derives from Latin *præsentia* 'presence', participle of *præesse* 'to be before', from *præ* 'before' and *esse* 'to be'. We can also Gnostically sense the Datum as Stillness, the central theme of Barry Long's *Stillness is the Way* and Eckhart Tolle's *Stillness Speaks*. As humanity blindly accelerates towards the sixth mass extinction of the species on our beautiful planet Earth, Stillness is by far the best way of coming to terms with death in all its forms in my experience.

In the meantime, how are we to manage our business affairs so that everyone has the life-inspiring opportunity to reach their fullest potential as human beings before our inevitable demise? We saw in the opening paragraph of this chapter that for the last five thousand years and more civilizations have been

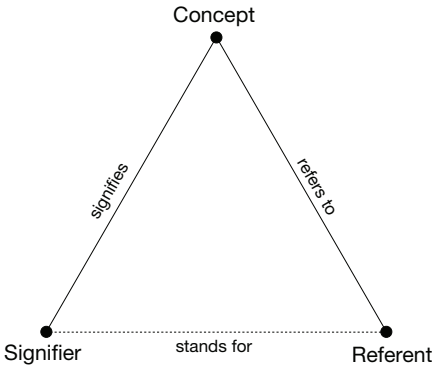
using record keeping to manage their business affairs, from Latin *recordari* 'remember', from *re-* and *cor*, *cordis* 'heart', a root that we also see in the words *accord*, *concord*, *discord*, and, of course, *cardiac*. To the Romans, the heart was not only the seat of the emotions, it was also seen as the seat of thought or the mind, a notion that we retain in English in the phrase *learn by heart*. In a similar fashion, the "Hopi Indians consider mental activity to be in the heart."[131]

Such records provide maps not only of current affairs but also of political, social, and familial history, as we see in the many national and regional archival centres around the world. But what is the relationship of the map and the territory? Well, the conventional scientific view is that the territory comes first. For instance, in 1931, when commemorating the centenary of James Clerk Maxwell's birth, Einstein wrote, "The belief in an external world independent of the perceiving subject is the basis of all natural science."[132] Similarly, at about the same time, Alfred Korzybski made the famous assertion, "A map *is not* the territory it represents, but, if correct, it has a *similar structure* to the territory, which accounts for its usefulness."[133]

Now such maps are more than might be included in an atlas or a GPS device, for instance. Ferdinand de Saussure and Charles Sanders Peirce, the founders of semiotics around the turn of the twentieth century, viewed maps in inner and outer forms, as concepts and as words, sounds, and other signs that denote them. This is a distinction that Ferdinand de Saussure made in *Cours de linguistique générale*, which his students published posthumously in 1915. In this seminal book of structural semiology, as semiotics 'science of signs' was known in Europe at the time, de Saussure said: "I propose to retain the word *sign* [*signe*] to designate the whole and to replace concept and sound-image respectively by *signified* [*signifié*] and *signifier* [*signifiant*]," illustrated here.[134]



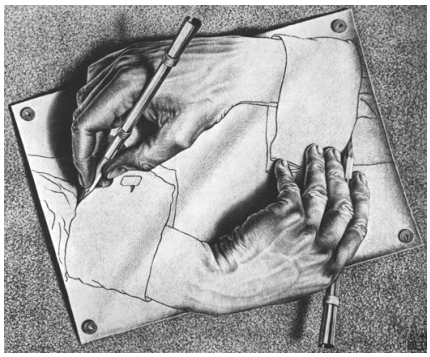
For instance, the concept of 🌳, as a mental image, could be represented by *tree*, *träd*, *arbre*, or *Baum* in English, Swedish, French, and German, respectively. No matter which language we use to express our ideas, we all have much the same understanding of the concept of tree. Similarly, we could have the number three in our minds as the signified, where the signifier, such as 3 or III, is called a numeral. This distinction between numbers, as concepts, and numerals, as signifiers, is something that computers cannot make. Both concepts and the signifiers that represent them need strings of bits to denote them. This is the simplest way of proving that humans are not machines and hence that technological development cannot drive economic growth indefinitely, requiring a radical change in the work ethic that has driven human affairs for thousands of years.



However, what de Saussaure omitted in his dyadic view of signs was a representation of the territory being mapped. To obtain a complete picture, we need to adapt the triadic view of logic and philosophy that Peirce spent a lifetime developing. This is illustrated in what J. F. Sowa of IBM calls the 'meaning triangle' in *Conceptual Structures*,[135] inspired to do so by *The Meaning of Meaning* by C. K. Ogden and I. A. Richards.[136] What this diagram illustrates is that there is an indirect relationship between language and the territory that language describes, not generally recognized by modern philosophers, focusing more attention on language than on the

conceptual structures underlying language. And don't forget that both concepts and signifiers are referents, included in the territory that is being mapped.

One reason for the superficial emphasis on language is that there is a very limited understanding of the concept of concept by even cognitive scientists. For instance, the entry for 'concept' in *The Oxford Companion to the Mind* states, "In psychology, concepts of mind must be invented or discovered, much as in physics, for we cannot see at all clearly into our own minds by introspection."[137]



Such opaqueness is partially caused by our cultural conditioning. Unlike in the East, self-inquiry is discouraged in religion, science, and business. Our lack of self-understanding also arises because the referent in the meaning triangle—as the territory—includes the conceptual map of the territory. To produce a complete map of the psychodynamics of business enterprises, we need to consciously model our own mapmaking processes. Consciously thinking in this healthy way is rather like a television camera filming itself filming, which looks impossible, brilliantly illustrated by Escher's famous lithograph 'Drawing Hands'.[138] For which comes first, the territory or the map?

Actually, the territory and the map are inseparable, unified in Nonduality. We can see this when we invoke Self-reflective Intelligence, the Divine quality that distinguishes humans from the other animals and machines, like computers, often called the 'Witness' in spiritual circles. This is why we have called our species *Homo sapiens* 'wise human', an epithet we are still a long way from fulfilling. For although we are never separate from the Divine in Reality, it is a cultural taboo in Western civilization to experientially and cognitively acknowledge our Divinity, which is our True Nature, Authentic Self, and Genuine Identity, that which is the same for everybody, from Latin *idem* 'same'.

Awakening Self-reflective Intelligence

To awaken Self-reflective Intelligence, as J. Krishnamurti attempted to do in his schools,[139] we need to make a clear distinction between Intelligence and machines with so-called artificial general intelligence, which are more intellectual than intelligent. By thus becoming free of our materialistic and mechanistic cultural conditioning, we effectively become a new species, not seen or understood by those educated in a conventional manner, as some modern mystics and evolutionaries have been proclaiming recently, perhaps a little prematurely.

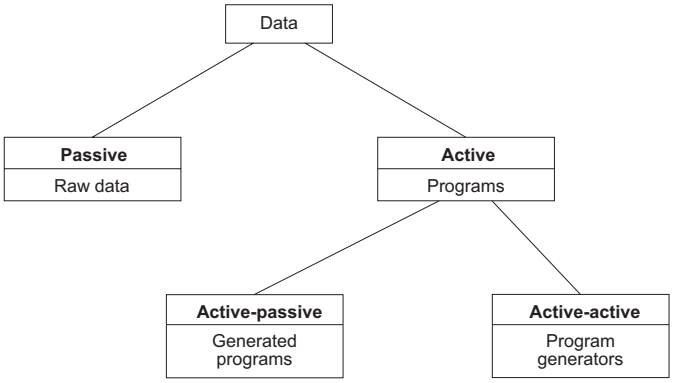
For instance, in *A New Earth: Awakening to Your Life's Purpose*, Eckhart Tolle wrote, "We are a species that has lost its way," concluding this inspirational book with these words: "A new species is arising on the planet. It is arising now, and you are it!"[140] Barbara Marx Hubbard has suggested several names for this emerging species, most recently promoting *Homo universalis*, in contrast to Alan Turing's Universal Machine.[141]

Similarly, Osho called the emerging superintelligent, superconscious species *Homo novus* or Zorba the Buddha, representing a new synthesis of East and West, the meeting of all polarities.[142] As he said, "The new man is not an improvement upon the old; he is not a continuous phenomenon, not a refinement. The new man is the declaration of the death of the old, and the birth of an absolutely fresh man—unconditioned, without any nation, without any religion, without any discriminations of men and women, of black and white, of East and West, or North and South."[143]

For myself, the name I prefer is *Homo divinus*, to denote the species that mystics have been exemplars

of during the past few thousand years. However, they have not been able to unify mysticism and mathematics, as they have focused attention more on involutionary processes towards Oneness than on evolutionary ones towards Wholeness. So I would say that *Homo divinus* has two subspecies, *Homo divinus unitas* and *Homo divinus holoensis*, from Greek *ὁlē* 'whole' and *-ensis* 'belonging to', both living in the Eternal Now, free of the past and future, for the most part not yet fully formed.

Most significantly, awakening Self-reflective Intelligence involves being free of the horizontal of time, depicted in the mechanistic data-processing function on page 4. Although this diagram is applicable at many different levels, from the Sheffer stroke to complete programs, it is simplest to look at it at this point in terms of the latter.



In this diagram, we first of all represent the bifurcation between active and passive data, as programs and the 'raw' data that they process. However, as this diagram illustrates, there are two types of active data in computers, generated programs, like the browsers and most other programs that we have on our desktop and laptop computers, tablets, and smart phones, and program generators for many different programming languages, called assemblers, compilers, and interpreters.

These we can call active-passive and active-active, respectively, dependent on whether their inputs and outputs are active data or not. Now program generators, like C compilers, are themselves generated programs. Today, a C compiler could be written in C, having as output a new version of the C compiler, known as a compiler compiler or metacompiler, a subject of greater complexity than I have had the opportunity or need to study. Nevertheless, the central issue is very easy to state. The first C compiler would need to have been written in an earlier, more primitive language, which in turn could have been written in the assembly language of a particular machine. So where did the first program, and, indeed, computer come from?

This is an example of many questions that most scientists do not ask because it cannot be answered within the prevailing scientific worldview. A closely related question is one that Alan Turing posed in 1950: "Can machines think?" Although he had one or two reservations about his reasoning, he eventually asserted, "I believe that at the end of the century the use of words and general educated opinion will have altered so much that one will be able to speak of machines thinking without expecting to be contradicted."[144]

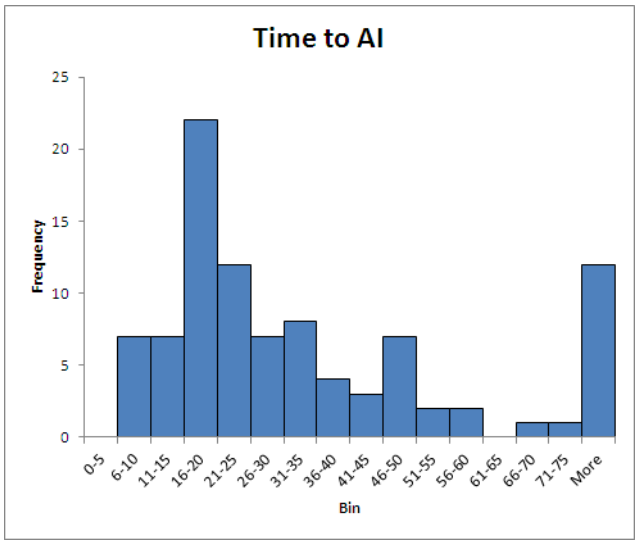
Well, this didn't happen, for reasons that Ada Lovelace, the daughter of Lord Byron and his wife Anne, a poet and mathematician,[145] respectively, made in 1843. In a brilliant memoir on Charles Babbage's Analytical Engine, the first design for a general-purpose computer, she wrote:

> The Analytical Engine has no pretensions to *originate* anything. It can do whatever we *know how to order it* to perform. It can *follow* analysis; but it has no power of *anticipating* any analytical relations or truths. Its province is to assist us in making *available* what we are already acquainted with.[146]

This is not something that many computer scientists are willing to accept. For instance, in 1993, Vernor Vinge made this prediction in a NASA paper called 'What is the Singularity?': "Within thirty years, we will have the technological means to create superhuman intelligence [in machines]. Shortly after, the human era will be ended."[147] Little has changed in this respect since Marvin Minsky and John McCarthy, among

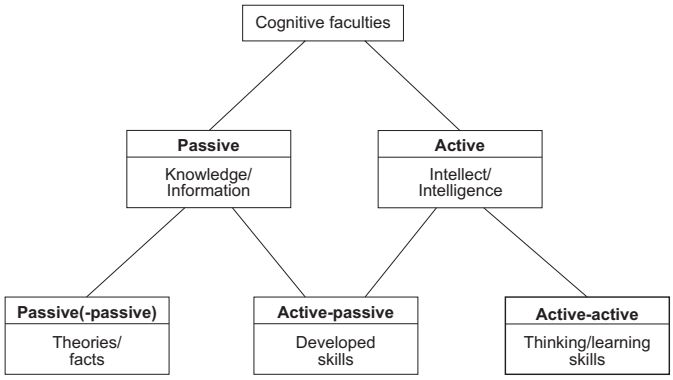
others, laid down the aims of artificial intelligence research at a Dartmouth Conference in 1956, when the latter stated the fundamental hypothesis of AI as follows: "Every aspect of learning or any other feature of intelligence can in principle be so precisely described that a machine can be made to simulate it".[148] And Herbert A. Simon said in 1960, "I believe that in our time computers will be able to perform any cognitive task that a person can perform."[149]

In 2012, Stuart Armstrong, a James Martin Research Fellow at the Future of Humanity Institute at Oxford University, and Kaj Sotala, of the Singularity Institute, presented a paper at a conference in Pilsen, Czech Republic on research that they had done of predictions of artificial intelligence since Turing's seminal paper on the subject. As Armstrong writes in *Smarter Than Us*, "The track record for AI predictions is … not exactly perfect. Ever since the 1956 Dartmouth Conference launched the field of AI, predictions that AI will be achieved in the next fifteen to twenty-five years have littered the field, and unless we've missed something really spectacular in the news recently, none of them have come to pass."



This chart shows the frequency of the various predictions of time to AI that he and Kaj Sotala have developed. Nevertheless, Armstrong still writes, "There are no convincing reasons to assume computers will remain unable to accomplish anything that humans can."[150] Well, this might be so if humans are merely machines, blindly processing inputs to produce outputs, as illustrated by the structure of the data-processing function on page 4.

In that case, nothing new could ever emerge in consciousness and we, as mechanical computers, would be unable to heal the deep wound in the human psyche that has arisen from the split between science and spirituality and mysticism and mathematics. We can begin to do so by looking at the cognitive faculties in humans that correspond to active and passive data in machines illustrated in this diagram. In terms of passive data, it is important to remember that theories are a form of insight or visualization, appearing before we can express them in words and other symbols, which is all that computers can work with.



Regarding our active cognitive faculties, these are our skills, as Gilbert Ryle pointed out in *The Concept of Mind* in 1949. Human knowledge, discounting Gnosis for the moment, can be considered both as the facts we know and the skills we know how to perform; we 'know that' and we 'know how',[151] which we can call 'active-passive' and 'active-active', clearly corresponding to generated programs and program generators in computers.

So just like computers, we have generated or learnt skills, such as playing chess or the piano, and generating or learning skills, which we call thinking, able to form concepts, as pictures in the mind, that have never existed before, in what Alfred North Whitehead called 'novelty' in *Process and Reality*, the essence of creativity.[152]

It is vitally important here not to be confused by computers able to beat humans at games, such as Chess, Othello, and Jeopardy! In *Superintelligence*, Nick Bostrom, the Director of the Future of Humanity Institute at Oxford University, founded and funded by James Martin, a fellow IBM alumnus, calls such machines superhuman.[153] Since he wrote this book, DeepMind's AlphaGo has defeated a 9-dan Go champion using a deep-learning technique, even starting from scratch, without the patterns of previous games as models. But all deep-learning algorithms can do is repeatedly apply the fundamental data-processing function, albeit in highly complex structures. There is nothing deep about them at all.[154]

Another major influence on scientists' view of the potential of machines to develop artificial intelligence was James Martin himself. For instance, in *After the Internet: Alien Intelligence*, he wrote, "Most of the popular predictions about computing assume that computer intelligence will be like human intelligence, and robots will be like the ones we see in the movies." But this is not how Martin saw the future. He wrote, "When computers are as powerful as the human brain, they won't be doing what the human brain does. They will have deep unfathomable forms of alien intelligence, vastly beyond human intelligence." Influenced by Richard Dawkins's program the 'Blind Watchmaker', he envisaged computer programs taking on a life of their own, self-generating themselves as self-evolving machines.[155]

Dawkins' program *The Blind Watchmaker*, a generated program that ran under Mac OS 9 on a Power PC processor in the 1990s,[156] was designed to show that evolution progresses without Divine intervention. But it proves no such thing. It is only when we bring the Divine Power of Life into science, bubbling up from the Origin of the Universe, like a fountain, that we can explain what caused Mozart to write his last three magnificent symphonies in just six weeks in the summer of 1788.

Awakening Self-reflective Intelligence in this way could lead to what some might see as alien intelligence, not understood by the intellectual ethos of academia, one of the most conservative institutions in society, even as evolution is passing through the most momentous turning point in its fourteen billion-year history, called the Accumulation Point in chaos theory. This is a Cosmic singularity in time, rather than the technological singularity, being predicted by people like Ray Kurzweil, author of *The Singularity is Near*, who wrote in 2001, "By 2019, a $1,000 computer will match the processing power of the human brain."[157]

Function of information systems architect

To see why this is not going to happen—even if the eruption of methane gas in the East Siberian Arctic Shelf does not happen by the late summer of 2019, as some have predicted—we need to look more closely at the workplace, in the way that algorithmic computers are influencing the way that humans work and communicate with each other and machines.

Much has changed in this respect since the first stored-program computers were built at the Universities of Manchester[158] and Cambridge[159] in England in 1948 and 1949, respectively, following a draft design that the eminent mathematician and polymath John von Neumann had proposed in 1945.[160] While this invention has enabled some tasks to be automated, previously performed by human beings, this tool of thought has also led to the introduction of many new occupations, most obviously that of computer programmer or software developer. In parallel, systems analysts and designers emerged in the 1960s and 70s, exploring more the business implications of this epoch-making invention than the technical ones.

Today, there are both software developers and systems designers who call themselves information systems architects, working at the micro and macro or technical and business levels of systems development, respectively. The word *architect* is highly pertinent here, for it derives from Greek *arkhitektōn* 'builder, architect, engineer', from *arkhē* 'beginning, origin; cause, motive, principle, element; leadership, power, rule',

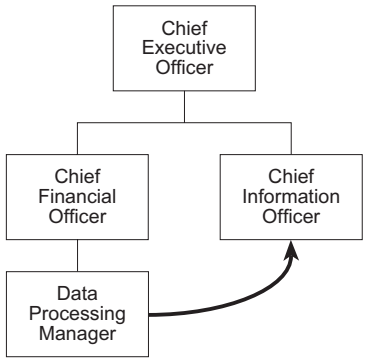
from *arkhos* 'leader, ruler', from *arkhein* 'to begin, rule, command', and *tektōn* 'builder', from PIE base **teks* 'to weave, fabricate', also root of *context* through Latin *texere* 'to weave' and *technology* through Greek *tekhnē* 'art, craft, skill'.

So information systems architects are the master builders in business, the ones who can see the big picture, how all the various constituents of an enterprise fit together in a coherent whole. In essence, they are generalists, working with specialists in an organization, who have detailed knowledge of the workings of the particular departments they work in. It is then the task of information systems architects to show how all the processes taking place in an enterprise can be integrated, together with the data that they process.

The introduction of the information systems architect into business also had a profound effect on the way that enterprises are organized. During the first three decades of the Computer Age, the data-processing manager, with her or his programmers and systems designers, reported to the finance director, for among the first procedures to be partially automated were accounts receivable and payable and payroll.

Such developments led the sociologist Daniel Bell to point out in 1973 that we were then entering a post-industrial era, which he called the 'Information Society',[161] as different from the industrial age as that was from the agrarian, land-based economy that preceded it. However, as he said in 1979, "Yet we have no economic theory of information, and the character of information, as distinct from the character of goods, poses some novel problems for economic theorists."[162]

In recent years, the economics of data and information has become an even greater hot topic with technological titans like Alphabet (Google's parent company), Amazon, Apple, Facebook, and Microsoft taking over the world. This issue much concerned *The Economist* magazine in a leader and briefing on 6th May 2017 titled 'The world's most valuable resource is no longer oil, but data'[163] and 'Data is giving rise to a new economy.'[164]



We can see why this is so from this diagram. In the late 1970s and early 1980s, many companies appointed a Chief Information Officer (CIO) on a par with the Chief Financial Officer (CFO), both reporting to the Chief Executive Officer (CEO), as this diagram illustrates. Information systems architects, in both their micro and macro capacities, naturally report to what is sometimes called the information director.

But what is the relationship of the CIO, managing information, and the CFO, managing money? Well, money is a type of information and so can be represented in the semantic models developed by information systems architects. But this is not possible the other way round. The meaning of information, and hence its value, cannot be satisfactorily represented in the quantitative financial models of accountants, bankers, and economists.

To understand how business enterprises are managed today, we thus need to look at the way that the modelling methods that information systems architects use have evolved over the years. This is as much a psychological issue as a technological one, which is highlighted by the desktop metaphor that Apple Computer introduced with its Macintosh computers in the 1980s, later mimicked in Microsoft's Windows, IBM's OS/2, and Unix's X Window System. For while information systems architects had long needed to understand how the mind works in order to automate as many jobs as possible, with the introduction of graphical user interfaces, it became necessary for software developers to have a similar understanding. For instance, this is how IBM introduced its guidelines for human interface designers of its OS/2 operating system in 1992:

> The term model is used in this book to refer to a descriptive representation of a person's conceptual and operational understanding of something. Some models are explicit and are consciously designed. These models typically can be represented by a diagram or a textual description. Other models, called mental models, are developed unconsciously. People create a mental model by putting together sets of perceived rules and patterns in a way that explains a situation. A typical person cannot draw or describe his or her mental model. In many situations, a person is not aware that a mental model exists.[165]

Now, even though most people are unaware of the mental models that guide their behaviour, clearly designers of information systems to be used by humans need to be aware of these implicit mental models. For as the IBM manual said, "A person develops a conceptual model through experience and then develops expectations based on relationships in the model," a conceptual model being a mental map that consists of "the set of relationships that a person perceives to exist among elements of any situation".[166]

In a similar manner, Apple's *Human Interface Guidelines* in 1987 urged designers to "use concrete metaphors [from the 'real world'] and make them plain, so that users have a set of expectations to apply to computer environments".[167] But Apple went a little further than IBM about people's lack of understanding and consciousness about the conceptual models they use when it said: "People, however, are delightfully complex and varied, which assures that a theory of human activity that would provide a complete framework for the design of human-computer interaction is a long way off".[168]

Not having a model of the workings of the mind obviously also makes the information systems architect's job rather difficult. Nevertheless, this has not prevented many from creating models of the way businesses are run. For instance, the birth of the digital computer led Jay W. Forrester at MIT to develop a number of complex computer models in the 1960s and early 70s of the dynamics of business organizations, of urban areas, and even of society as a whole,[169] which led to the publication of *The Limits to Growth: A Report for the Club of Rome's Project on the Predicament of Mankind*.[170] As Forrester said:

> There is nothing new in the use of models to represent social systems. Everyone used models all the time. Every person in his private life and in his community life uses models for decision making. The mental image of the world around one, carried in each individual's head, is a model. One does not have a family, a business, a city, a government, or a country in his head. He has only selected concepts and relationships which he uses to represent the real system. A mental image is a model. All our decisions are taken on the basis of models. All of our laws are passed on the basis of models. All executive actions are taken on the basis of models. The question is not whether to use or ignore models. The question is only a choice between alternative models.[171]

Forrester was a great advocate of computer models of social dynamics because, as oversimplified as they are, they are "probably more complete and explicit than the mental models now being used as a basis for world and national planning".[172] He even went as far as presenting his view that mental models are dangerous to members of the U.S. Congress in 1970, with these words:

> … the human mind is not adapted to interpreting how social systems behave. … until recently there has been no way to estimate the behavior of social systems except by contemplation, discussion, argument, and guesswork.
>
> The great uncertainty with mental models is the inability to anticipate consequences of interactions between parts of the system. This uncertainty is totally eliminated in computer models. Given a stated set of assumptions, the computer traces the resulting consequences without doubt or error. … Furthermore, any concept or relationship that can be clearly stated in ordinary language can be translated into computer model language.[173]

Joseph Weizenbaum, also of MIT, was particularly critical of such statements by what he derisively called the 'artificial intelligentsia'. As he said, "Consider the impact of Forrester's words on the members of the U.S. Congress … or on any other group of people who have no training in or intuition for formal systems. They hear that the basis of their thinking, mental models, leads to uncertainty, whereas Forrester-like computer models totally eliminate this uncertainty and all doubt or error. … Conclusions derived from computer models are valid beyond doubt."[174] Of course, Forrester omitted to say that his opinions can only

be true if the starting assumptions and algorithms that connect the many variables are valid representations of what he, like many others, call 'reality'.

So are we condemned forever to manage our business affairs without a comprehensive model of the psychodynamics of society to guide our activities? Is a complete framework for the design of human-computer interaction still a long way off, as Apple's user interface guidelines indicated over thirty years ago? Of course not, as this book is endeavouring to demonstrate. In order to obey the economic imperative of our times, replacing as many jobs performed by humans by machines as possible, information systems architects develop models of dynamic business processes, such as designing, manufacturing, marketing, ordering, and invoicing, and their relationships to each other, as well as integrated models of static classes of information in enterprises, such as employees, customers, products, locations, and deliveries. At first, these are very abstract models, not concerned whether humans or machines perform business processes. This distinction is only made at the implementation stage of systems development.

One of the first business modelling methods I became aware of in 1980, when I was working in IBM's Information Systems Support Centre in London, was IBM's Business Systems Planning (BSP), much influenced by Jay W. Forrester's *Industrial Dynamics* in 1961, Robert N. Anthony's *Planning and Control Systems: A Framework for Analysis* in 1965, and Sherman C. Blumenthal's *Management Information Systems: A Framework for Planning and Development* in 1969.

Now, a key feature of BSP was a process-entity matrix, like the one on the next page, illustrating in which processes data about entities is originated (0 or >), changed (+), and referred to (-), taken from another business modelling method developed in IBM (UK) in the early 1980s called Systems Development Method (SDM).[175]

Modern object-oriented modelling methods, like the Universal Modeling Language (UML), do not include a process-entity matrix because both methods and variables, as active and passive data, are encapsulated in classes, whose relationships are described in a class model. However, in the noughties I discovered that the aptly named System Architect, as an enterprise architecture and business architecture modelling tool, originally developed by Jan Popkin, does provide a method for creating such a matrix. System Architect is today developed by Unicom Systems, having previously been owned by Popkin Software, Telelogic, and IBM's Rational division, which develops UML.[176]

However, what neither UML nor System Architect nor the relational model of data enable us to do is model the way that humans interact with computers in timesharing systems, for instance. So any models that information systems architects build of all processes in an enterprise cannot be complete with current modelling methods. I first became aware of this comparatively new development in the data-processing industry in 1974, when I was appointed the systems engineering manager responsible for ensuring that the first computer that the British Post Office (now British Telecom or BT) bought from IBM passed its acceptance trials.

Four years later, I discovered that IBM's own principal management information tool for extracting and analysing data from its corporate database, called APL Data Interface (ADI), had some rather curious features. When a user formulated a query, this was not answered by a prewritten function, which converted the query into machine language. Rather, a new APL function would be dynamically created, executed, and erased, making it very difficult to understand the workings of the program when I peered under the covers.

PROCESS \ ENTITY	PLACE	PLACE-PLACE RELATIONSHIP	BUSINESS	REGISTERED ADDRESS	PRODUCT	DELIVERY POINT	ORDER RECEIVING POINT	CUSTOMER ORDER	PRODUCT ON ORDER	REQUESTED DELIVERY POINT	DESPATCH POINT	DELIVERY POINT	DELIVERY BATCH	DELIVERED ORDERED BATCH
Recognize PLACE	0	0												
Accept BUSINESS for Trade	>	>	0	0										
Change REGISTERED ADDRESS	>	>	-	0										
Acknowledge DELIVERY POINT	>	>	-		0									
Change BUSINESS Credit Rating			+											
Discontinue Trade with BUSINESS			+											
Introduce PRODUCT						0		+	+			+	+	
Launch PRODUCT						+		+	+			+	+	
Withdraw PRODUCT						+								
Establish ORDER RECEIVING POINT	-	-	-				0							
Close ORDER RECEIVING POINT							+							
Take CUSTOMER ORDER	>	>	-		>	-	-	0	0	0				
Add to CUSTOMER ORDER						-		+	0					
Change REQUESTED DELIVERY POINT	>	>			>		-	-		0				
Accept CUSTOMER ORDER			-					+						
Cancel CUSTOMER ORDER								+	+			+	+	
Change PRODUCT ON ORDER Quantity								+	+			+	+	
Establish DESPATCH POINT	-	-	-								0			
Close DESPATCH POINT											+			
Schedule DELIVERY					-			-	-		-	0		
Cancel DELIVERY												+		
Add to DELIVERY						-			-			-	0	0
Make DELIVERY												+	+	+
Change DELIVERY BATCH Quantity													+	+
Take Back DELIVERY BATCH													+	+

For here was an example of a computer programming itself, a capability that is possible because functions, as strings, can be contained within the computer along with the data that they process. But could such a stored-program computer do this without human intervention?

Most significantly, as passive and active data are converted from one to the other in nanoseconds, how could such transformations be modelled in BSP's process-entity matrix, which is essentially a static model? This was a problem that I was wrestling with during the winter of 1980, as I was developing an innovative marketing programme for decision support systems, along the lines that Peter G. W. Keen and Michael S. Scott Morton described in their seminal book *Decision Support Systems: An Organizational Perspective*, published two years earlier.

To model the active and passive data patterns in interactive computing, we need to look at the program generators known as interpreters. Interpretive programming languages don't necessarily have an interactive user interface. For instance, expressions, statements, and functions written in PHP, JavaScript, and PostScript are normally executed by an Internet server or client browser and in printers, respectively, the programs for laying out the page being generated automatically by software, not usually written directly by humans.

However, languages like A Programming Language (APL), LISP (LISt Processing), and Python, do provide editing facilities for interactive communications with humans. At the simplest level, humans can enter expressions at the interface and these are immediately executed, returning the result to the user.

In these languages, as arithmetic expressions, for instance, begin as strings, it is also possible to defer execution of expressions by storing them in a variable and then executing or evaluating them. As such strings could also be created using the string processing facilities of the language, programs can dynamically create expressions to be executed later, like programming themselves. The following table shows how this is done in APL, LISP, and Python, the expression returning 9, either immediately or later.

	Execution mode	
	Immediate	Deferred and then evaluated
APL	`2+7`	`b ← '2+7'` `⍎ b`
LISP	`(+ 2 7)`	`(setq b '(+ 2 7))` `(eval b)`
Python	`2 + 7`	`b = '2+7'` `eval(b)`

Note that Python also has an `exec()` function that executes Python statements for their side effects without returning a result. For instance:

```
>>> s = 'a = 7'
>>> exec(s)
>>> a
7
```

PostScript also has an interesting way of converting passive to active data and vice versa. An object on top of the operand stack can have the attribute literal or executable, with the operators `cvx` and `cvlit` converting one to the other. So literals cannot be executed directly. They must first be converted to executable, like this, where / introduces a literal name, also returning 9:

```
2 7 /add cvx exec
```

Another way that PostScript distinguishes immediate and deferred execution is through procedures, defined as an executable array, delimited by parentheses. For instance, using the example in the *Reference Manual*, this statement immediately returns 50 as the average of the numbers 40 and 60:[177]

```
40 60 add 2 div
```

In contrast, the first of these statements defines a procedure called `average`, deferring execution until the procedure name is entered as an operand:

```
/average (add 2 div) def
40 60 average
```

When programmers are interacting with an interpreter, they change from calculation mode to function definition mode and back again by signalling the start and end of a function. In APL, this is done with the special symbol ∇ called *del*. As I don't have access to an APL interpreter on my iMac to test this, even through a browser, it is better to illustrate function definition in Python. Here the function `square` is defined, squaring a number.

```
>>> def square(x):
    return x*x
```

The function `square` has been added to the language, which can be called like any other function, thus:

```
>>> square(7)
49
```

LISP, with its simple read-eval-print syntax, doesn't change mode when defining what it calls procedures. It simply uses the primitive `defun` just as it would the addition primitive +. So the Python code is implemented in LISP in this way, adding `square` to the procedures that the programmer can use.

```
CL-USER> (defun square (x) (* x x))
SQUARE
CL-USER> (square 7)
49
```

Now as function definitions are strings, whether entered directly from a terminal by a human or created by another function or procedure, variables can be defined to hold them. Before executing the functions in strings, the strings need to be executed or evaluated, establishing the name of the function or procedure in the namespace, which can then be executed. For instance, this Python code first defines a variable `sqfunc` with a function in string format, which is then converted to executable with the `exec()` function, enabling `square()` to be executed:

```
>>> sqfunc = 'def square(x):\n   return x*x'
>>> exec(sqfunc)
>>> square(7)
49
```

And here is the same process in LISP, which Robert Smith of Rigetti Quantum Computing called metaprogramming in a 2018 YouTube Computerphile video on the duality between data and code. As he said, programs can write programs by switching at any moment between the data and code representation of an expression, 'automating' the work of a programmer.[178]

```
CL-USER> (setq sqfunc '(defun square (x) (* x x)))
(DEFUN SQUARE (X) (* X X))
CL-USER> (eval sqfunc)
SQUARE
CL-USER> (square 7)
49
```

APL has a special system function $\square FX$, called function establishment, to transform a character matrix in the syntax of an APL function into an executable function. The dual system function is $\square CF$, called canonical representation, which generates a character matrix of the function,[179] which could be modified by a function to produce a variation of the function. In the words of Leonard Gilman and Allen J. Rose in a standard APL manual, "This leads to application systems that can appear intelligent (in the sense of programs that write or edit other programs)".[180]

As we can see in these few examples, there are mechanisms in some programming languages for functions to dynamically create new functions and to execute them. I call such languages *dapples*, an acronym for Dynamically Active Procedural Programming LanguagEs. The above examples show function-writing functions that have been entered by humans at a terminal. However, such functions could be embedded within another function-writing function and so on *ad infinitum*. So could a computer overcome the problem of infinite regress and write such a program independently of human involvement? As with compilers compiling compilers, where would the first of these function-writing functions come from? As I have discovered, such functions can only arise through the creative power of God, as the Datum of the Universe—that which is given, the First Cause of everything.

It was the $\square FX$ system function that started me thinking in 1978 about whether computers would ever develop artificial general intelligence, able to program themselves without human, that is Divine, intervention. I have thus been led during the past forty years to know myself and understand what it truly means to be a human being, in contrast to the other animals and machines, like computers. It is through self-inquiry that I have learnt to study the long-term psychological and economic implications of humanity's growing dependency on information technology.

Modelling our rapidly changing world

This is the business and technical background to the one final problem that I was wrestling with during the winter of 1980. Even though it was proving extremely difficult to model the data patterns of interactive personal computing in a BSP process-entity matrix, there was one part of the territory missing from such a

model, should it ever be possible to develop it: how to map the mapmaking process itself, already mentioned on page 11. It was vitally important to solve this problem. Otherwise I would not be able to practice my job as an information systems architect with full awareness of what I was doing. I would be living my life with a blind spot, not able to develop a comprehensive model of all processes taking place in business enterprises, including creative thinking.

As I was thinking about this problem, I was also becoming increasingly aware of how little we scientists and technologists understood about where scientific discovery and technological development is taking us as a species at ever increasing rates of change. The previous year, when giving presentations at customer executive seminars on the management and development of decision support systems at IBM's European Education Centre in Belgium, I was being asked questions that I could not readily answer.

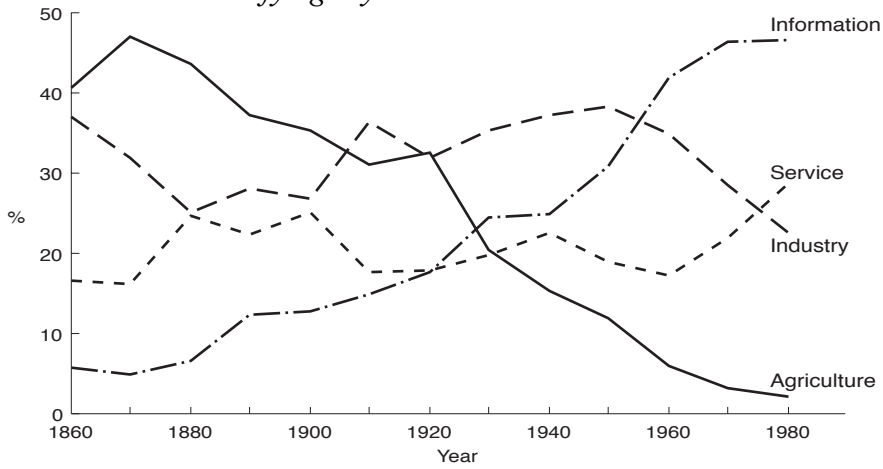
The introduction of point-of-sale terminals in department stores and supermarkets was a particular concern in the retail industry. For not only were these helping to automate the supply chain, they were also changing the skills profiles in the industry, particularly that of buyer, who decides what products are to be stored in shops, their quantities and prices, and how they are to be attractively presented.

This was not a new situation. Ever since the beginning of the industrial revolution in the middle of the eighteenth century, more and more machines had been introduced into the workplace, radically changing the way most human beings had been working for thousands of years. We were no longer spending most of the day tilling fields and shepherding animals in order to feed and clothe ourselves, as most of our ancestors had had to do. This is well illustrated by an estimate of the population and wealth of England and Wales that Gregory King, who was employed at the College of Heralds, made in 1688, the year after the publication of Newton's *Principia*. In this survey, King estimated that nearly 80% of the population of around five and half million were engaged in agricultural work, either as employers or labourers.[181]

Then during the years of the industrial age, the number of agricultural workers fell dramatically, so that by 1976 just 3.3% of the working population in the UK was engaged in the extractive industries, which include forestry, fishing, and mining, as well as agriculture.[182] At that time, 39.5% of the employed population was working in the industrial sector, consisting of the manufacturing, utilities, and construction industries, with the remainder in a wide variety of services industries. So even then the number of industrial workers was declining rapidly as the industrial age was giving way to the Information Society.

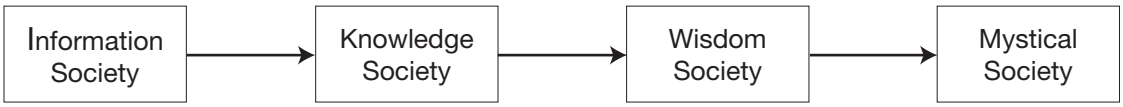
There has been a similar trend in the USA during the last two centuries of the second millennium, as Daniel Bell tells us. This is clearly shown in the next figure, using a four-sector classification of Agriculture, Industry, Service, and Information.[183] During just these 120 years, the agricultural sector dropped from 40.6 to 2.1% of the workforce, while the information sector increased from 5.8 to 46.6% of people in work. But how would these trends develop in the future, until 2010 and beyond, when my children could well be bringing up children of their own?

And what about employment? Not knowing what it truly means to be a human being back in 1980, I still thought that it might be possible for algorithmic machines to take over the workplace, as many scientists have still been saying in recent years. For instance, Hans Moravec forecast in *Robot* in 1998 that our 'mind children' "could replace us in every essential task and, in principle, operate our society increasingly well without us."[184] Martin Rees, the Astronomer Royal and former President of the Royal Society, picked up this viewpoint by writing in *Our Final Century: Will the Human Race Survive the Twenty-first Century?*, "A superintelligent machine could be the last invention that humans need ever make."[185] And again, Stephen Hawking told the BBC on 2nd December 2014, "The development of full artificial intelligence could spell the end of the human race."[186]



Clearly major changes were and still are afoot. Because of the invention of the stored-program computer in the late 1940s, skills profiles required to run our business affairs are changing rapidly. Furthermore, if artificial intelligence were possible, then the cycle of humans as both workers and consumers in the economy would one day be broken, the fundamental principle of both capitalism and communism. For as Adam Smith wrote in 1776 in the opening words of *The Wealth of Nations*: "The annual labour of every nation is the fund which originally supplies it with all the necessaries and conveniences of life which it annually consumes, and which consists always either in the immediate produce of that labour, or in what is purchased with that produce from other nations."[187]

In this case, the global economy—consisting of the monetary ideologies of both capitalism and communism—would clearly hold the seeds of its own destruction within it. But this would not necessarily mean that *Homo sapiens* would become extinct when the financial system collapses, for it would give rapid climate change a chance of respite, at least until a reduction in global dimming sped up the process again. If we could learn to make the most radical change to the work ethic since our forebears settled in villages to cultivate the land and domesticate animals, focused on awakening Self-reflective Intelligence in humans, rather than on technological development, we would no longer be imprisoned by the economic machine. In that case, we would see our lives in a quite different way, enabling the Information Society to evolve through the Knowledge and Wisdom Societies, into the eschatological Mystical Society, as the Age of Light. And then miracles could happen!



For information is not a physical object, giving it some rather strange properties in conventional economic terms. For instance, when I buy a loaf of bread, the object passes from the storekeeper to me in exchange for money, viewed as a *commodity* with value, rather than a unit for the *measure* of value, like grams and metres. However, when a teacher gives pupils some information, nothing is exchanged. Both teachers and pupils have the information. As Tom Stonier said in *The Wealth of Information*, "Whereas material transactions can lead to competition, information transactions are much more likely to lead to cooperation—information is a resource which can be truly shared."[188]

There is another peculiar characteristic of information that I learned about when attending a week-long strategy meeting at IBM Canada in Toronto in 1979 to discuss the development and marketing of decision support systems. Meaningful information is obtainable not only from data elements but also from the

relationships between them in a property called synergy. For *synergy* derives from Greek *sunergos* 'working together', from *sunergein* 'to cooperate', from *sun-* 'together' and *ergon* 'work', cognate with *energy* 'at work', from *energeia* 'activity, efficacy, effect', from *energes* 'active, busy, working', from *en-* 'at' and *ergon* 'work'. In ancient Greece, a fellow-worker was called *sunerithos*. It is clear from this that *synergy* and *energy* originally referred to human activity and work.

Recognizing the central importance of the synergy of both passive and active data structures has brought about major changes in the management and development of decision support systems during the past four decades. First, by integrating organizational databases into a coherent whole, the enterprise database, which became known as a data warehouse, can now provide far more information for effective decision making through data mining than fragmented data files implemented with various technologies over the years.

Secondly, the many interactive tools available for modelling and data analysis have naturally become more integrated through evolutionary convergence and cooperative endeavour. For instance, the open-source Jupyter environment, designed to "support interactive data science and scientific computing across all programming languages",[189] is a far cry from the interactive tools that were available around 1980, when I was engaged more in conceptual marketing for technology transfer than product marketing.

Despite these practical developments, scientists sometimes ignore synergy because it denotes that the combined effect of two or more agents or forces is greater than the sum of their individual effects. Wholes are greater than the sum of the parts from the relationships between the pieces, which cannot generally be expressed in quantitative, mathematical terms. For instance, if you tear a dollar bill in half, the two fragments do not have any value. But if you tape them back together again, the whole is worth one dollar. So 0 + 0 = 1.

We can see the limitations of reductionist science from the word *interesting*, which derives from *interesse* 'to be between, take part in', from *inter* 'between' and *esse* 'to be'. So what is interesting, important, and essential is not the interest that banks receive in today's debt-driven, divisive economy, or more generally things in themselves, but the *relationships* between entities, a word also derived from *esse*. In contrast to holistic scientists, reductionist scientists, focused on objects rather than the relationships between them, throw the interesting associations and connections away! That is why it is absolutely essential to include relationships in a coherent scientific worldview.

As the importance of holistic science is a fairly recent discovery, originating from Jan Christian Smuts' *Holism and Evolution* in 1925, it is perhaps not surprising that *synergy* did not appear in the *Concise Oxford Dictionary* of words in current usage until its sixth edition in 1976. Although the OED records its use as far back as 1660 to mean 'cooperation between people', in modern scientific use *synergy* has come to mean the 'combined or correlated action of a group of bodily organs, mental faculties, drugs, etc.,' first recorded in 1847.

During the past few decades, synergistic effects have become widely recognized in business, for, to combine some current dictionary definitions, *synergy* denotes increased effectiveness and achievement from cooperative interaction among groups, especially among the acquired subsidiaries or merged parts of a corporation, that creates an enhanced combined effect. In this manner, wholes beget wholes in an ever-accelerating, evolutionary manner, much as Smuts foresaw.

This is some further background to the problem that I was wrestling with during the winter of 1980. How could I use the modelling methods of information systems architects in business to answer the most critical unanswered question in science: *What is causing scientists and technologists, aided and abetted by*

computer technology, to drive the pace of scientific discovery and technological development at unprecedented exponential rates of acceleration? For to answer this question I would need to be able to model not only the active and passive data patterns at play in humans and machines during interactive computing, but also those at play in information systems architects during such model-making activities, engaged in model-driven architecture (MDA).

In the event, I was given the key to the solution to this problem in an apocalyptic epiphany that I experienced at 11:30 on 27th April 1980 as I was strolling across Wimbledon Common in London to the pub for lunch, close to the Tangier war memorial at 51° 26′ 30″ N, 0° 14′ 02″ W (TQ 2284 7288, to the nearest ten-metre square in the Ordnance Survey grid). Puzzling about what was causing my colleagues in IBM and me to change people's lives through the development of integrated information systems in business, in a life-changing eureka moment, I realized that data is not only synergetic, it is energetic and causal. In a dazzling flash of inspiration, I saw that active and passive data are rather like kinetic and potential energy in mechanics, also present in rather more complex form in quantum physics. There are nonphysical, mental energies at work in the Universe, as well as the material ones I had learnt about in physics at school.

Having abandoned physics as the basis of all the sciences in high school because I did not believe in the big bang theory or in the atomistic existence of a fundamental particle of matter, I realized at once that I had been given the key that would reveal the innermost secrets of the Universe that I had puzzled about since I was seven years of age. Three weeks later, in great excitement, I resigned from my marketing job with IBM and set out to use Self-reflective Intelligence to unify the nonphysical energies at work within computers and all of us with the physical energies recognized by materialistic, mechanistic science: electromagnetic, gravitational, and the strong and weak nucleic forces.

Realizing that I could not do this within the framework of so-called natural science or philosophy, I began this unifying endeavour on 20th May 1980 by writing on the top of a blank sheet of paper to represent my mind as a *tabula rasa* 'clean sheet', 'Paul's folly, a new model of the Universe', from Old French *folie* 'madness', in modern French also 'delight'.

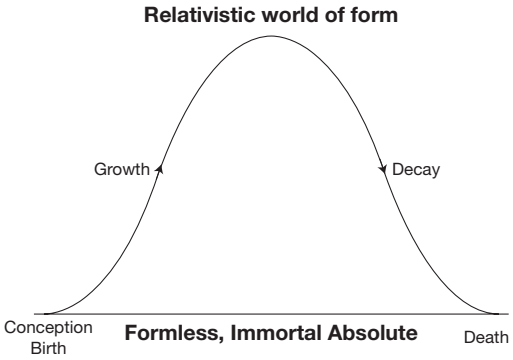
Then that summer I wrote my first attempt to describe the function of data energy in society by writing an essay titled 'The Future of Computers and Society', intended for academics. I sent a copy to two professors of cybernetics and one professor each of mathematics, machine intelligence, computer science, and physics, all in the UK. David Bohm, professor of theoretical physics at Birkbeck College in London, a former friend and colleague of both Einstein and J. Krishnamurti, was most interested in the revolutionary idea of data energy and invited me to meet him.

We were drawn to each other because the business modelling and management problem that I was wrestling with at IBM during the winter of 1980 was essentially the same as that which Bohm needed to solve to reconcile the incompatibilities between relativity and quantum theories. To look at the business world and universe from the perspective of Wholeness, we both needed to include our mapmaking in the territory being mapped.

There was just one question I wanted to ask Bohm when I first met him: "What is the source of the data energy that exists in humans and computers?" He replied, "Energy does not have a source, it is contained within structure." I now know in my own experience that the first part of this reply is not true. Data energy emerges directly from the Datum of the Universe, its Divine Origin. However, the notion of structural energy, deriving from meaningful relationships between forms, was just the answer that I needed to unify the physical and psychospiritual energies at work in the Cosmos.

The key point is that we cannot heal the deep wounds within the collective, cultural, and personal psyche within any existing cultural tradition. Our fragmented, split minds can only be healed by starting afresh at the very beginning, demolishing the Tower of Babel, which represents the confused state of the world of learning today. But not only this. While our minds create our sense of reality, much human behaviour is driven by sub- and unconscious behaviour patterns laid down in the collective, cultural, and personal psyche as the result of decades and hundreds and thousands of years of human experience, from terrifying trauma to rapturous ecstasy. All this needs to be brought out into the open so that it can be looked at in the broad light of day—the coherent light of Consciousness.

What this means is that we can only awarely (intelligently and consciously) live in harmony with the fundamental law of the Universe by pursuing a life of learning that recapitulates the Cosmogonic Cycle, illustrated here. This schema indicates that all beings in the Universe are born to die, or in the case of humans and other creatures are conceived to die. Of course, as we all live in exactly the same Universe governed by the same Cosmic laws, there is nothing new here. Many before me have described the birth-and-death process that we could go through while we are still alive in our bodies in the most beautiful, poetic language. Here are a couple of examples, the first from the *Taittiriya Upanishad* and the second from 'Little Gidding', the final poem in T. S. Eliot's *Four Quartets*: [190]



Bhrigu meditated and found that bliss is Brahman.
From bliss are born all creatures,
By bliss they grow,
And to bliss they return when they depart.

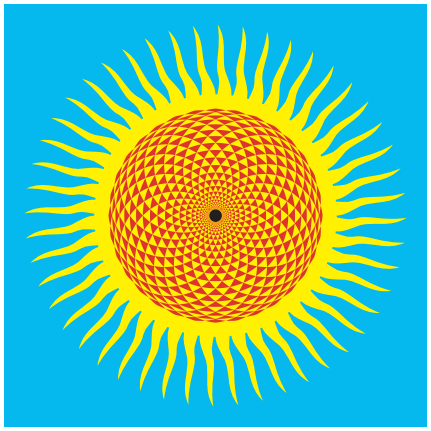
We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.

So even though there are some similarities in my life experiences with those of others, how can I scientifically explain the unprecedented, apocalyptic awakening that happened to me in the spring of 1980? Well, as I was seeking to understand the essential difference between humans and machines, I can best say that I embarked on a thought experiment rather like those that Einstein used to formulate the special and general theories of relativity.[191]

However, I did not start this experiment in learning through an act of conscious choice. Indeed, as I can see now, nothing that I have ever done in life has been an act of free will, for there is no separate being who can act autonomously, independent of the Supreme Being or any other beings, from which I am never separate. All the peaks and troughs of human experience that have happened to me since my conception have led me to the Wholeness I enjoy today.

So I don't know what you are to make of these words, as you kindly read them. One of the characteristics of sound scientific method is that experiments are repeatable by others with the requisite skills and willingness. In this regard, vipassana meditation, for instance, is a scientific experiment in that practitioners have similar experiences as the result of their practices. I too have been engaged in a meditation practice, but watching thoughts as they emerge directly from the Divine, rather than my breath, for instance.

Unifying Mysticism and Mathematics



I call this meditation technique Collumination, from Latin *cum* 'together with' and *lumen* 'light', on the model of *illumination*, which is the skill of integrating all knowledge in all cultures and disciplines at all times—past, present, and future—into a coherent whole. The result of this meditation practice is also called Collumination, as the coherent light of Consciousness, illustrated here, originally written in PostScript. This diagram is a coloured extension of a symbol for Pure Consciousness developed at the University of the Trees in 1979.[192] It is Collumination, in both its meanings, that has enabled me to create a holographic view of the Cosmos, showing that all beings in the Totality of Existence have the same self-similar underlying structure, like geometric fractals. We can thus see why William Blake was able to write these two beautiful stanzas in *Auguries of Innocence*:

To see a world in a grain of sand
And a heaven in a wild flower
Hold infinity in the palm of your hand
And eternity in an hour

Joy & Woe are woven fine
A Clothing for the Soul divine
Under every grief & pine
Runs a joy with silken twine

In order to realize my fullest potential as a superintelligent human, beyond any level that computers might aspire to attain, the best way to describe the second half of my life is to say that I have imagined that I am a computer that turns itself off and on again so that it has no programs in its memory, not even a bootstrap program to load the operating system, so named because switching on a computer is rather like pulling oneself up by one's bootstraps. Beginning with a *tabula rasa* 'blank slate', this computer then has the task of integrating all knowledge in all cultures and disciplines at all times into a coherent whole without any external authority to tell it how or what to learn. In other words, this computer is given the assignment to develop the Theory of Everything entirely from within, without a human programmer instructing it, thereby solving the ultimate problem in human learning.

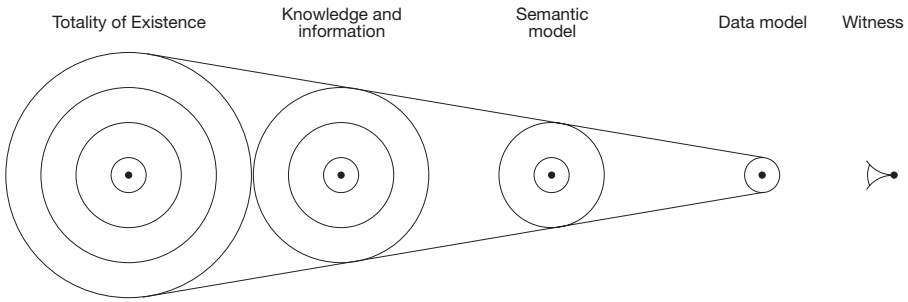
Now, one of the key features of computers is that everything is explicit as symbols in the machine. So to emulate the workings of a computer in the psyche, it is necessary to express the totality of human experience in words and other symbols. But first, we need an intuitive conceptual understanding of what is happening to us. As Einstein described in a letter to Jacques Hadamard in 1945, symbols arise as a secondary step in the creative process.

Such conceptualization has been the essence of human learning over the millennia. It is only when we are able to explicitly and rationally build pictures in our minds of what is happening to us that we can understand how to manage our business affairs with full awareness of what we are doing. This is evolution becoming increasingly conscious of itself, as Julian Huxley wrote in 1959 in the Introduction to *The Phenomenon of Man*, the first English translation of Teilhard's magnum opus.[193]

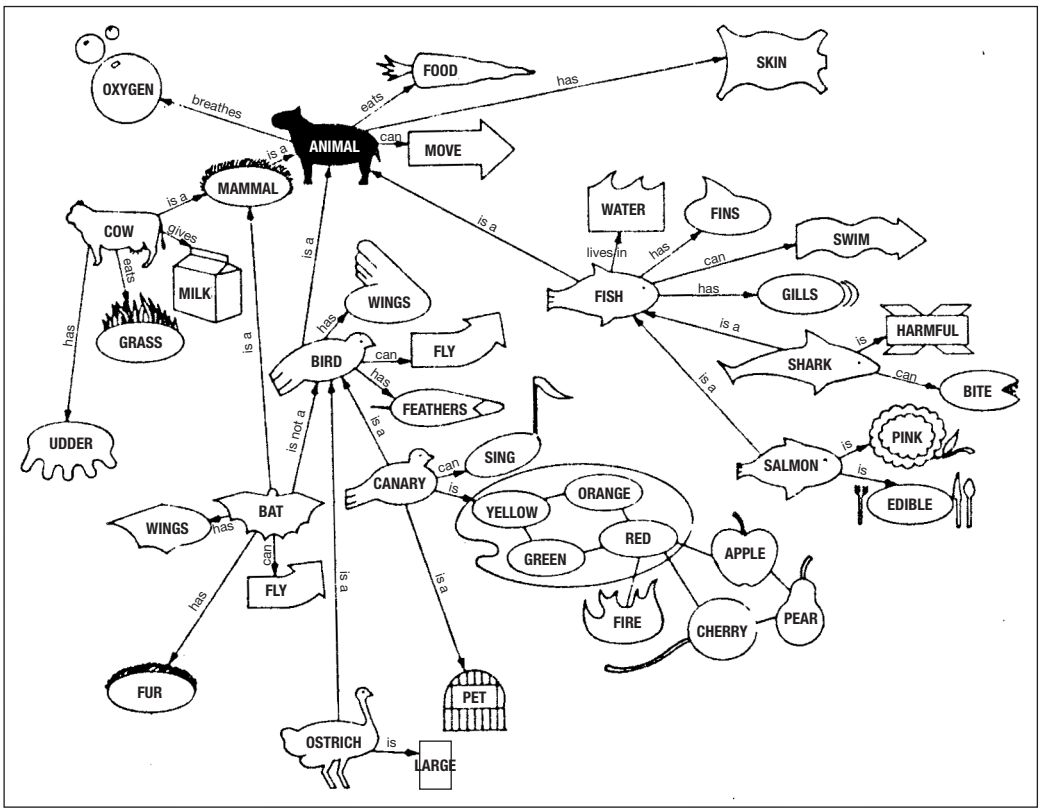
Two years earlier, Huxley had described this evolutionary vision in a 1700-word essay like this: by "destroying the ideas and the institutions that stand in the way of our realizing our possibilities", we could understand human nature, what it truly means to be a human being. We could thereby transcend our limitations, fulfilling our highest potential as spiritual beings, living in mystical ecstasy, free from the suffering that has plagued humanity through the millennia. Huxley called this mystical evolutionary process

of humanity transcending itself 'transhumanism',[194] with a somewhat different meaning from what atheistic transhumanists seem to mean by the word today.[195]

The key here is to look at our business affairs through the eyes of information systems architects, rather than through the eyes of finance directors, management accountants, investment bankers, and monetary economists. To unify the psychospiritual energies at work within us with traditional materialistic energies, we similarly stop looking at the Universe through the eyes of physicists. This diagram provides a succinct overview of this entire modelling method:



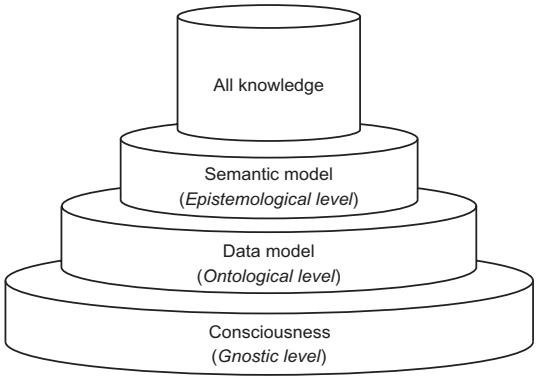
The above diagram shows that the conceptual models that information systems architects build do not have just one level; they actually have three, each contained within the next level, all viewed with Self-reflective Intelligence, standing outside ourselves. The largest circle on the left represents all beings in the Totality of Existence. The circle tangential to this, but also contained within it, is all the knowledge and information that we humans have developed or will develop during the tens of thousands of years of our existence as a species, much of which is contained on the Internet today. In other words, this circle represents what Teilhard called the noosphere as a whole, in its entirety, mapping the territory, everything that exists in the Universe, including the map of the territory.



The next modelling step is to build a semantic network of all this knowledge and information, which is knowledge about knowledge, rather like the above diagram, which is adapted from an article in the *New*

York Times magazine on 24th January 1982 titled 'How the mind works'.

This illustrates how our minds develop cognitive maps of the world we live in, although this one is not in the formal structure that information systems architects use. In terms of object-oriented modelling methods, this semantic network corresponds to the class models and entity-relationship models that designers develop, leading toward a unified view of data, as Peter Pin-Shan Chen foresaw in a seminal paper in 1976.[196] Such models are partially represented in the systems catalogue in relational database management systems as metadata, data about data, or better meaningful information about information.



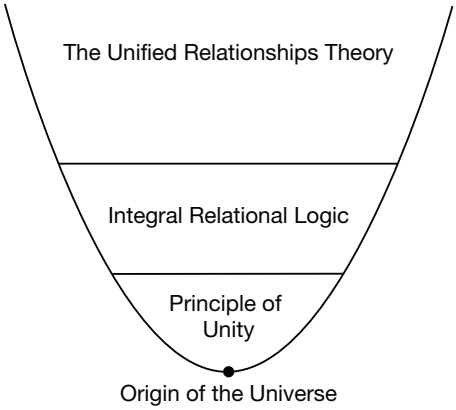
The relationship of the semantic model to all knowledge is further illustrated in this diagram, corresponding to Aristotle's epistemology 'the science or study of knowledge'. For *epistemology* derives from Greek *epistēmē* 'knowledge'. So the semantic model contains knowledge about knowledge.

Beneath this level is the data model or ontological level, for ontology is 'the science or study of being' from Greek *on*, genitive *ontos*, from *ont-*, stem of present participle *einai* 'to be', and *-logia,* from *-logos*, 'one who deals or treats of a certain subject'. So the ontological level of the foundations describes the underlying structure of all beings, as patterns of data.

This level is almost meaningless, just containing an abstract map of the data patterns underlying the Totality of Existence prior to interpretation by an intelligent being: *The underlying structure of the Cosmos is an infinitely dimensional network of hierarchical relationships*. This level does not correspond to the data models that database designers develop. Rather, it is more like Tedd Codd's original 1970 paper on the relational model of data, before it is applied to the organization of meaningful information and knowledge. This is the closest we can get to the meaningless language that Greimas suggested we need to talk about the meaning of meaning.

The entire structure rests on Consciousness, which provides the Gnostic Foundation and the Cosmic Context for viewing the Universe as a coherent whole. However, this model is not quite complete. Tucked away at the mezzanine level, between the Gnostic and ontological levels, lies the Principle of Unity, which states *Wholeness is the union of all opposites*. Heraclitus aptly called this fundamental law of the Universe the *Hidden Harmony*, which applies as much to the Immanent and Transcendent Absolute as to the relativistic world of form.

The third diagram I use to illustrate these relationships is this one. The Principle of Unity first emerges from the Origin of the Universe through the action of the Logos, the "immanent conception of divine intelligence" signifying "the rational principle governing the cosmos", as Richard Tarnas put it.[197] In turn, this generates Integral Relational Logic as the much sought-for science of consciousness, which provides the Cosmic Context, coordinating framework, and Gnostic Foundation for the Theory of Everything or the Unified Relationships Theory, as the Noosphere or All Knowledge, as a whole.



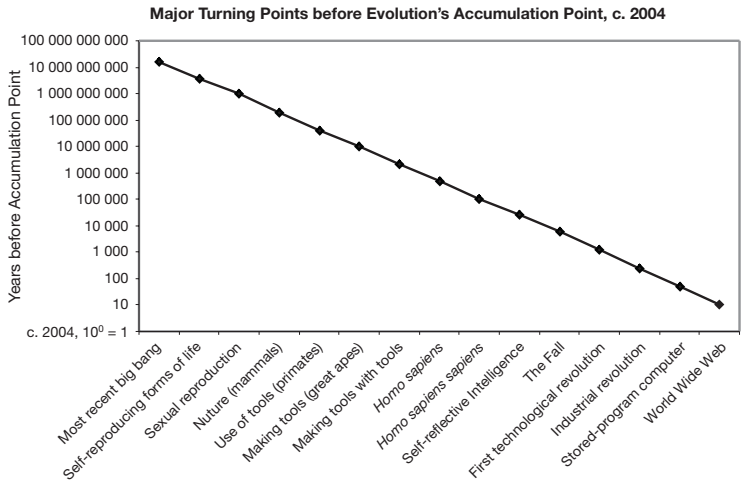
As the Unified Relationships Theory transcends and embraces all cultures and disciplines, I also call it *Panosophy*, a word that Comenius made famous in the 1600s with a slightly different spelling, modelled on *philosophy*, from Greek *pan* 'all' and *sophia* 'wisdom'. The ancient Greeks used the word *pansophos* to mean

'very wise', literally 'all-wise'. Comenius' *A Reformation of Schooles*, in its English title, was a prospectus for a universal cyclopædia, *pansophy*, occasionally spelled *pantosophy*, coming to mean 'universal or cyclopædic knowledge; a scheme or cyclopædic work embracing the whole body of human knowledge'.[198] Pansophy formed the basis of Pansophia, 'a dream of science', the vision of a Utopian society, to this day still not realized, as Frank E. and Fritzie P. Manuel point out in their scholarly tome *Utopian Thought in the Western World*.[199]

So what does transcultural, transdisciplinary Panosophy tell us about our immediate and ultimate destiny as a species? Well, we can see from the notion of ubiquitous synergistic data structures, interpreted as meaningful semantic networks, that such structures are accumulative. The greater the complexity of the structures that are generated, the greater the complexity of these evolving structures. This is a rational explanation of Teilhard's law of complexity-consciousness: the greater the complexity, the greater the consciousness.[200] For *consciousness* means 'knowing together', from Latin *cum* 'together with' and *scīre* 'to know', also the root of *science*. So when we learn to integrate all knowledge into a coherent whole, Cosmic Consciousness is the natural result.

Now as synergetic data energy is universal, it does not exist only in humans and computers. By studying the human phenomenon, as Teilhard would have us do, we can see that the accelerating pace of change in the world today is the culmination of some fourteen billion years of evolution since the most recent big bang. Evolution is not only a biological phenomenon. We can thus define *evolution* in this general way: *Evolution is an accumulative process of divergence and convergence, proceeding in an accelerating, exponential fashion by synergistically creating wholes that are greater than the sum of the immediately preceding wholes through the new forms and relationships that emerge, apparently out of nothing*. With Teilhard, we can view evolution as a whole in four stages—material, biological, mental, and spiritual—although in my experience this final stage is more involutionary than evolutionary.

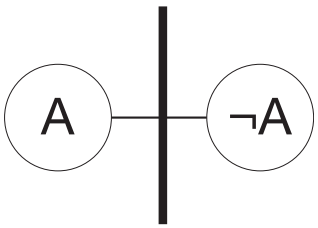
Now while evolution is accumulative, and hence modelled by the exponential function in mathematics, populations, for instance, do not grow indefinitely, like the sequences and series we look at in Chapter 4. In practice, evolution is accumulative under constraint, modelled by Pierre François Verhulst's logistic function and its discrete counterpart, the logistic map in chaos theory. As I describe in my book *Through Evolution's Accumulation Point: Towards Its Glorious Culmination*, we can view the whole of evolution as a sequence of turning points, with the periods between them diminishing in a geometric series by the reciprocal of the Feigenbaum bifurcation velocity constant δ, which is about 4.6692, as this diagram illustrates:



A simple calculation shows that evolution's Accumulation Point in chaos theory terms was reached around 2004, give or take a couple of years. This explains why many political institutions are degenerating into chaos at the moment, with increasing nationalism and trade wars, to name just a couple of symptoms. Following the invention of the stored-program computer, traditional competitive ways of managing our business affairs are no longer viable. So old structures need to disintegrate so that something radically new can emerge.

But is this actually going to happen? Well, as the social psychologist Erich Fromm pointed out, we humans are the least instinctive of all the animals.[201] Using the metaphor of a computer, very few of our thoughts and actions are hard-wired. The innate instincts and automatic reflexes of babies to suck, grasp, cry, and respond to stimuli mostly disappear within the first few months of life.[202] Our learning—corresponding to software and data in computers—mostly determines the way that we view the world and ourselves, and hence our behaviour. Our minds, stimulated by the Divine Power of Life, determine how we think and act, far more than our brains.

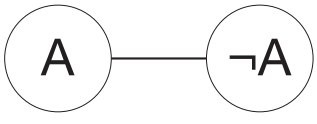
This means that the world that we witness on news channels is in chaos because our collective psyche is in turmoil, not adapting to our rapidly changing environment. At the root of this problem is our ignorance and even denial of the irrefutable, universal truth that is the fundamental law of the Universe: in Reality, opposites are never separate from each other. Most particularly, difficulties can arise in human relationships when we identify with one of a pair to the exclusion of the other, separating *A* and not-*A* or ¬*A*. This one-sided view is an instance of *dualism*, separating the opposites with a barrier between them, illustrated here. For instance, people sometimes identify with the sex of their bodies, with the colour of their skins, or with their religion or nationality, making people with opposite characteristics their enemies.



Such a split between opposites is most evident when nations believe that God is on their side when they go into battle with other nations, in Holy wars, wars about the Whole. For instance, presidents of the United States of America, as commanders-in-chief, often end their speeches, "God bless America." But why not bless everyone? Given the struggles we all face living together in a world we barely understand, doesn't everyone deserve to be blessed?

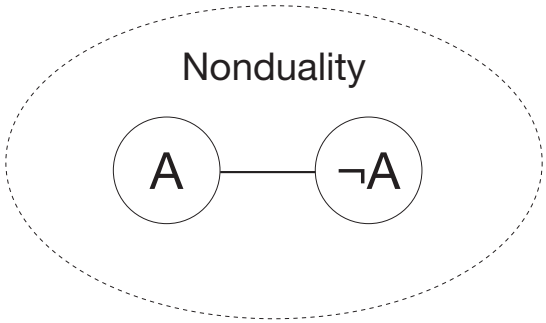
More generally, in Western philosophy, dualism most commonly means mind-body dualism, following René Descartes' view of himself in *Meditations on the First Philosophy in which the Existence of God and the Real Distinction between the Soul and the Body of Man Are Demonstrated*: "I am only a thinking and unextended being … entirely and truly distinct from my body, and may exist without it."[203] This perspective gave rise to the split between *res cogitans* 'thinking substance, mind, or soul'[204] and *res extensa* 'extended substance', by which Descartes meant an object with breadth, width, and height occupying space.[205] As Bryan Magee tells us, " 'Cartesian dualism', the bifurcation of nature between mind and matter, observer and observed, subject and object … has become built into the whole of Western man's way of looking at things, including the whole of science."[206]

We can begin to resolve the split between mind and matter by removing the barrier between the opposites, even when identifying with one side, respecting, honouring, and tolerating the other, which we can call *duality*. In duality, while each of us has a particular perspective on life, which we have acquired through our unique life experiences, we are able to see that what applies to us as individuals also applies to others. Being thus able to see both



sides of any situation is a clear sign of natural intelligence, often stultified by an education system based on dualism. In particular, if *A* represents any particular human being and ¬*A* any other human being, the union of the two is wholeness, with no division between them.

However, we could go even further in healing the deep wounds in society by living in union with the Divinity as *Homo divinus*. With this Nondual perspective, we could remove what an anonymous fourteenth-century English mystic called the 'cloud of unknowing',[207] drawing back the blinds that obscure the radiant Light of Consciousness, which is necessary for Self-reflective Intelligence to resolve the Great Global Crisis facing humanity today.

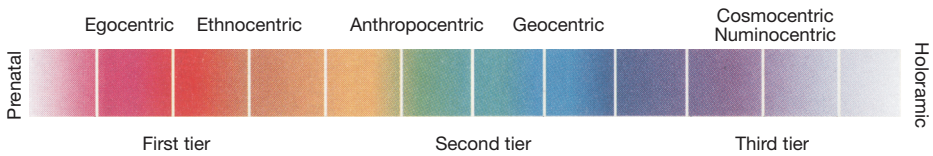


This diagram illustrates the union of the Formless, Nondual Absolute and the relativistic world of dual forms, with all its immense complexity. This is the key to everything, as mystics have been experiencing and teaching for millennia. Nothing could be simpler or more challenging given the way that Western religion, science, and economics have evolved during the past few thousand years, emphasizing the split between humanity and Divinity, out of experiential and cognitive touch with Reality.

In summary, what this chapter on business modelling shows is that it is theoretically possible to live in love, peace, and harmony with each other and our environment if we all live in harmony with the Hidden Harmony, able and willing to take responsibility for the entire body politic. We can liken this coherent view of the world to our bodies, where specialist cells and organs are clearly 'aware' of the whole, as the epigenesist Bruce Lipton points out. He distinguishes two ways of looking at the world we live in: "On one side of the line is a world defined by neo-Darwinism, which casts life as an unending war among battling, biochemical robots. On the other side of the line is the 'New Biology', which casts life as a cooperative journey among powerful individuals who can program themselves to create joy-filled lives."[208]

To illustrate the relationship of this architectonic to that of Ken Wilber, Integral Relational Logic is an example of what he calls an 'Integral Operating System', or IOS,[209] "a neutral framework" that "can be used to bring more clarity, care, and comprehensiveness to virtually any situation".[210] Ken's basic IOS is called AQAL, short for "all quadrants, all levels", which is short for "all quadrants, all levels, all lines, all states, all types".[211] AQAL is thus a two-dimensional example of the multidimensional Cross of Duality, defined on page 70, and therefore not all encompassing. IRL is more like a virtual machine operating system, such as IBM's Virtual Machine (VM), which can run many different operating systems including itself, than Microsoft's Windows or Apple's MacOS.

Since April 2014, this IOS has been called a Superhuman Operating System, which Ken has been teaching in a ten-module Internet course, intended to "Install a Revolutionary New Operating System for Your Mind to Illuminate the Full Spectrum of Your Human Potential, and Become the Greatest Possible Version of Yourself". I did this course in the winter of 2018, learning of some differences between his books and the course, which can perhaps best be understood in terms of a revised version of his spectrum of consciousness model that more accurately matches my experiences:



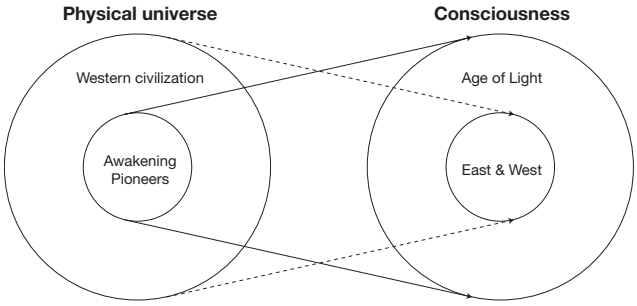
We can liken the first two tiers in this spectrum of consciousness to dualistic and dual attitudes to the relationships of opposites, which Ken suggests involves around 95 and 5% of the population, respectively, at least in the developed world. The Cosmocentric tier, which I also call Numinocentric, is the ultimate goal of human development, realized by exemplars of *Homo divinus*. This third tier indicates "an identification with all life and consciousness, human or otherwise, and a deeply felt responsibility for the evolutionary process as a whole … an emergent capacity, rarely seen anywhere," as Ken defined it in a conversation with Andrew Cohen in the *What is Enlightenment?* magazine in 2007.[212]

Now for this to happen on a global scale, each of us, as individuals, need to take responsibility for the entire evolution of the whole human race, as Andrew pointed out in *Freedom Has No History* in 1997. As he says, "To succeed, we must be prepared to do battle with the powerful conditioning, conscious and unconscious, of the whole race. That means we have to come out from the shadows and be seen. Like Atlas, we have to be willing to hold up the whole world on our shoulders. It's an awesome task."[213]

This, in essence, is what needs to happen if we are to cocreate the Sharing Economy, as a harmonious way of living together that gives everyone the opportunity to realize their fullest potential as humans. Political philosophers and social scientists have been pondering about such a Utopian society since Confucius, Plato, and Aristotle, with their many archies and ocracies. Plato's solution to this problem, appalled at the way that the Athenian democracy had condemned his beloved Socrates to death for corrupting the youth of the city-state, was to appoint educated philosophers, as 'lovers of wisdom', as kings.[214]

To Plato, a philosopher is "the man who is ready to taste every branch of learning, is glad to learn and never satisfied."[215] Knowing the immense power of abstract thought, a philosopher is therefore a generalist rather than a specialist, more focused on Wholeness than fragments. Philosophers also "have the capacity to grasp the eternal and immutable". In contrast, those who are not philosophers "are lost in multiplicity and change", and so are not qualified to be in charge of a state.[216] Furthermore, philosophers "will be self-controlled and not grasping about money. Other people are more likely to worry about the things which make men so eager to get and spend money".[217] So a society ruled by financiers, economists, bankers, and accountants is not viable.

This diagram illustrates the total transformation of consciousness that I have needed to go through in order to intelligently do my job as an information systems architect in business with full awareness of what I am doing.



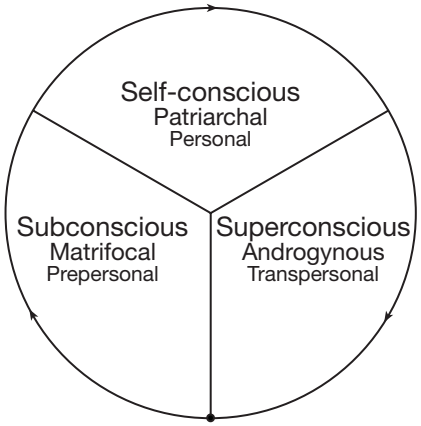
I call this transformation a contextual inversion rather than a paradigm change or shift because paradigm means 'pattern' or 'model' and Consciousness transcends the categories as a seamless continuum with no borders or divisions anywhere. However, transforming formal linear logic into a Holotropic 'Whole-turning' and Holoramic 'Whole-seeing' system of reason *is* a paradigm change, the greatest revolution in Western thought since Plato, Aristotle, and Euclid laid down its foundations some 2350 years ago.

Indeed, it is not necessary to understand the history of attempts to mechanize human reasoning, from Leibniz's proposal to develop a universal language of human thought to Turing's attempts to design a universal machine capable of doing anything, for humans are not machines, even though much of our behaviour can appear mechanical and habitual. Rather, what we need is the complete unification of holographic mathematical reason, which Ted Codd introduced following the pioneering efforts of Augustus De Morgan and Charles Sanders Peirce, and mystical psychology, practised by such figures as Shakyamuni Buddha and Carl Gustav Jung.

We can thus see that the only sustainable system of governance, which the ecophilosopher Henryk Skolimowski aptly calls lumenarchy,[218] is one designed and managed by Panosophers, applying Self-reflective Intelligence, enlightened by the coherent light of Consciousness. Yet, how is this possible? Although we all live in the same Universe, the world that Panosophers live in is utterly different from those seen with first-tier levels of consciousness, with the second tier lying in the twilight zone in between.

Even though this twilight zone is often seen as a new dawn, with the first tier of the tyrannous democratic masses[219] and their leaders calling the shots, it no longer looks possible to complete the transition into the eschatological third stage of human development in the short time we have available. While such a radical transformation of consciousness and awakening of intelligence is taking place in the second tier, attracted by the third, it is still being pulled back by the first, especially in economic and scientific terms. As I read the situation, there is increasing spiritual freedom, but this has yet to break down the barriers to the full unification of mysticism and mathematics, consummating the sacred marriage of science and spirituality, as Terry Patten, editor of Ken Wilber's *Integral Life Practice*, urges us to do.

So Ken's model of the three stages of human development,[220] which has been guiding much of my work since the early 1980s, when I discovered it, no longer looks viable. Even though I am an optimist at heart, preferring to see the positive rather than the negative aspects of human behaviour, with the Arctic likely to become ice-free in the coming years, releasing increasing quantities of methane gas into the atmosphere, I no longer see a practical solution to resolving the conflicts that have bedevilled human affairs during the 5,000 years of the patriarchal epoch. It is unlikely that as a species we could complete the transition from the mental-egoic age (me-epoch) to the age of universal spirituality (us-epoch). So my life's purpose to complete the final revolution in science, just as Isaac Newton completed the first in 1687 with *Mathematical Principles of Natural Philosophy*, no longer looks feasible.



Formless Alpha/Omega Point of Evolution

Nevertheless, I continue living as if it is, for I have no other choice. To this end, Chapter 2 on Integral Relational Logic is a shortened version of Part I in the *Wholeness* trilogy and an expanded version of an introductory article I wrote in 2013. I plan to write the three chapters on mathematics as a generative science of patterns and relationships during the autumn and winter of 2018 and 2019 if I am still able to do so. For the main thing that really matters to me right now is God, experienced as Love, Peace, Stillness, and Presence. So in the next chapter, I describe the rigorous system of reasoning that leads to Ineffable, Nondual Wholeness, beginning at the end and ending at the beginning, in conformity with the fundamental law of the Universe.

2. Integral Relational Logic

I now come to the solution to the business modelling and management problem that I was wrestling with during the winter of 1980, when developing an innovative marketing programme for IBM, as the Information Society was still at the very early stages of its development. Solving this problem was essential if I were to practice my job as an information systems architect with full awareness of what I was doing.

The last section in the previous chapter on 'Modelling our rapidly changing world' provides an outline solution. But now I need to apply Self-reflective Intelligence to use these modelling methods to map the Totality of Existence, not just the business world, including the mapmaking process in the territory being mapped.

To remind you, Integral Relational Logic began to emerge in consciousness in the spring of 1980 following an apocalyptic eureka moment, when I realized that nonphysical, psychospiritual energies are causing scientists and technologists to drive the pace of scientific discovery and technological invention at unprecedented, exponential rates of acceleration.

At the time, it felt as if a dam had burst in my psyche, releasing thirty years of pent-up energy trapped by living in a world that made no sense as a coherent whole. However, today it is perhaps more meaningful to say that a big bang erupted in my psyche, leading the irresistible power of Life bubbling up from our Divine Source to create a brand-new Universe, at least one new to Western science. For what has been revealed is that Consciousness is the Cosmic Context for all our lives and that Love is the Divine Essence we all share, a worldview well known to mystics of all cultures and ages, and thereby many spiritual seekers today.

In my case, I can now use the worldview revealed by IRL to explain how such Kundalini-like events occur. This happened because three weeks after my insightful epiphany, I resigned from my marketing job with IBM and set out to develop an integral cosmology that would unify the nonphysical energies that I had 'discovered' with the physical energies recognized by materialistic, mechanistic science.

To this end, I embarked on an experiment in learning, inspired by Einstein's thought experiments. To determine whether computers could develop artificial intelligence, exceeding any level of intelligence that humans might aspire to attain, I imagined that I was a computer that switched itself off and on again so that it had no programs within it, not even a bootstrap program to load the operating system. Then guided only by its inner guru, which means 'dispeller of darkness', the computer had the task of organizing all knowledge in all cultures and disciplines at all times into a coherent whole, without any external authority to tell it how to do this.

Such a life-changing awakening is also rather like a volcanic earthquake erupting in the depths of the Ocean of Consciousness, creating a tsunami in which everything is destroyed, as in Aceh province in Sumatra in December 2004. In terms of Hindu deities, which are just human energies emerging from the

Divine, Shiva, the destroyer, and Brahma, the creator, acted in turns, one after the other. Today, Shiva and Brahma, as Divine energies appearing in the human body-psyche organism, have very nearly completed their task. So today, Vishnu, the maintainer, is the predominant energy within me. I thus recognize that what many called supernatural is entirely natural; what is abnormal or an anomaly in the predominant cultures of the world is entirely natural.

We can see this from the root of *anomalous*, which is Greek *anōmalos*, from *an-* 'not' and *ὁmalos* 'even', from *ὁmos* 'same'. So, when I look at the Cosmos through the eyes of a generalist, employing the utmost mathematical abstractions, I see that beings are *homalous*, as the ultimate universals, with particulars being instances of these general characteristics.

So, there is nothing unusual about Integral Relational Logic. It is simply the commonsensical art and science of thought and consciousness that all of us implicitly use every day to form concepts and organize our ideas in tables or relations and semantic networks or mathematical graphs. Yet we do not know that we all think and learn with this universal, holistic, both-and system of reason, for it is almost completely hidden from view, because of the predominantly divergent way evolution has progressed over the years.

The fact that we often think and act without a conceptual understanding of what we are doing is most simply illustrated with Molière's *Le Bourgeois Gentilhomme*. M. Jourdain asked his philosophy teacher, "What? When I say: 'Nicole, bring me my slippers, and give me my nightcap,' is that prose?" The philosopher replied, "Yes, Sir." "Good heavens!" exclaimed M. Jourdain, "For more than forty years I have been speaking prose without knowing it."[221] In a similar fashion, when I was engaged in conceptual marketing for IBM in the late 1970s in order to promote technology transfer, customers would sometimes say, "We've been doing that for years. That's what it's called."

The blind way we have been learning over the years is most simply illustrated by the way that mathematics has evolved. For thousands of years, we human beings have been using numbers without understanding how the concept of number is formed. This situation began to change at the end of the nineteenth century, when Georg Cantor (1845–1918) developed the mathematical theory of sets, which he defined in this way: "By a set we mean the joining into a single whole of objects which are clearly distinguishable by our intuition or thought."[222] He also showed that there are an infinity of distinct infinite cardinals and ordinals, not just one, leading to a major crisis in the foundations of mathematics, for paradoxes were found at the heart of set theory, invalidating the very principle on which mathematical proof and deductive logic is based.

This situation greatly disturbed Bertrand Russell, engaged in a life-long search for certainty in mathematics and science. For at the beginning of the twentieth century, he realized that you cannot form the concept of number and hence three, for instance, until the concept of set is formed. There is a primary-secondary relationship between set and number, and hence semantics—the science of meaning—and mathematics—the science of number and space since Pythagoras. Accordingly, Russell spent the first twenty years of the last century with A. N. Whitehead trying to find a way of basing mathematics on linear logic, an enterprise that failed miserably because they could not find a satisfactory way of eliminating what they called antimonies, which are paradoxes or self-contradictions, from logic.

This exercise was especially futile because paradoxes are an inherent feature of the world we live in. So, if our reasoning does not produce maps that include paradoxes, we are led seriously astray, not able to intelligently navigate our way through our journeys in life. This rejection of self-contradictions has led Western civilization to be based on a delusional view of Reality, causing no end of confusion in the world

we live in today. If we are to heal our deluded minds, we need to follow E. F. Schumacher's maxim for mapmaking in *A Guide for the Perplexed*: "Accept everything; reject nothing," recognizing, "Our task is to look at the world and see it whole."[223]

Starting afresh at the very beginning

As the maxim "Accept everything; reject nothing" is of such central importance in holistic human learning and harmonious human relationships, it is a wonder that it is not emblazoned on the portals of every educational and political institution, including businesses, everywhere in the world. This maxim is just as important as that which seven wise men inscribed on the temple of Apollo at Delphi, Plato tells us: γνῶθι σεαυτόν (*gnothi seauton*) "Know thyself."[224] In a similar fashion, when Neo visited the Oracle in the popular allegorical movie *The Matrix*, hanging on the kitchen wall was a sign saying Temet Nosce, Latin for 'Know yourself'.

But how are we to know ourselves? When scientists observe our outer world, they do not do so as objectively as they claim. As A. F. Chalmers wrote in *What is this thing called Science?*, a standard textbook on scientific method for students at the Open University in the UK, all observation statements are theory dependent.[225] It is not possible to observe anything without some preconceptions of what is being observed. So the three scientific methods of deduction, induction, and abduction, introduced by Aristotle,[226] Francis Bacon,[227] and Charles Sanders Peirce,[228] respectively, are limited, even when studying our outer worlds, never mind our inner ones.[229]

Nevertheless, there is one feature of scientific method that can help us understand ourselves. In *Objective Knowledge*, Karl Popper, the foremost philosopher of science during the twentieth century, suggested "that it is the aim of science to find *satisfactory explanations*, of whatever strikes us as being in need of explanation." By *explanation*, he meant finding the unknown but true causes (the *explicans*) that logically entail that which is to be explained (the *explicandum*). "Thus, scientific explanation … will be *the explanation of the known by the unknown*."[230]

Now, as see on page 8, the ultimate unknown *explicans* is the Datum, which gives birth to the entire world of form through the action of the Logos, the "immanent conception of divine intelligence" signifying "the rational principle governing the cosmos", as Richard Tarnas interpreted Heraclitus' mystical use of *Logos*.[231] So the Datum logically entails our scientific explanations if we look at entailment from the creative vertical dimension of time rather than the mechanistic horizontal dimension, providing the Gnostic Foundation for all our learning.

From this Immortal Foundation, we can use another aspect of scientific method to understand what is happening to humanity at the present time. To overcome the problem of what Chalmers called 'naive inductionism',[232] he said that scientific facts should not be seen in isolation, but rather "a scientific theory is a complex structure of some kind."[233] As he pointed out, the primary advocate of this view was Thomas Kuhn, who published his landmark book *The Structure of Scientific Revolutions* in 1962.

Kuhn famously called the complex structures of concepts 'paradigms', from the Greek word *paradeiknumi* meaning 'show side by side'. From this, he made a clear distinction between normal science, which works within the context of a particular paradigm, and scientific revolutions, when a radical change is made to the conceptual structures that guide scientific research.

This is what generally happens in what Thomas S. Kuhn called normal science: "... 'normal science' means research firmly based upon one or more past scientific achievements, achievements that some particular scientific community acknowledges for a time for its further practice."[234]

However, such an approach to science does not satisfactorily describe the process that Copernicus, Kepler, Galileo, and Newton went through in the sixteenth and seventeenth centuries or that of Joseph Priestley and Antoine-Laurent Lavoisier in developing the oxygen theory of combustion in the eighteenth century, obsoleting the earlier phlogiston theory.

By looking at such examples in the history of scientific discovery, Kuhn saw that such a radical change in worldview comes about as the result of anomalies in the overall structure of existing scientific theories; experience no longer matches theory, leading to what Kuhn called a *paradigm change* or *paradigm shift*, terms that he used twenty-three and six times, respectively.[235] Such a transformation is the essence of scientific revolutions, which he described thus: "... at times of revolution, when the normal scientific tradition changes, the scientist's perception of his environment must be re-educated—in some familiar situations he must learn to see a new gestalt.[236]

This is very much the case today. Materialistic, mechanistic science cannot begin to tell us what it truly means to be a human being and hence what God and the Universe truly are, as many millions intuitively know today.[237] The scientific method that has evolved to study the hylic universe of matter, space, and time in a systemic manner is quite inappropriate to map the Cosmic Psyche, the ninety-nine percent of the Universe that is hidden from our five somatic senses.

Indeed, specialists working within particular disciplines cannot see the world as any form of gestalt—as an organized whole of forms that is perceived as more than the sum of its parts, from German *Gestalt* 'form, shape'. For, as J. Krishnamurti wrote in *Education and the Significance of Life*, "Can any specialist experience life as a whole? Only when he ceases to be a specialist."[238]

The root of this problem lies in the way that evolution has progressed since the most recent big bang about 13.8 billion years ago, especially during the thousands of years of human learning. Just as galaxies and amoebas are not aware of their place in the overall scheme of things, most humans are also ignorant of our relationship to God and the Universe, despite much effort over the millennia to resolve this existential dilemma. Although the True Nature of all humans is Wholeness, few are able to see the whole because of our fragmented minds and split psyches, existentially and cognitively separate from the Immortal Ground of Being we all actually share as Reality.

However, such a fragmented approach is no longer a viable option for humanity, as evolution passes through the most momentous turning point in its long history. For as David Bohm said in the opening paragraphs of *Wholeness and the Implicate Order*,

> Fragmentation is now very widespread, not only throughout society, but also in each individual; and this is leading to a kind of general confusion of the mind, which creates an endless series of problems and interferes with our clarity of perception so seriously as to prevent us from being able to solve most of them.
>
> Thus art, science, technology, and human work, in general, are divided up into specialities, each considered to be separate in essence from the others. … Each individual human being has been fragmented into a large number of separate and conflicting compartments, according to his different desires, aims, ambitions, loyalties, psychological characteristics, etc., to such an extent that it is generally accepted that some degree of neurosis is inevitable, while many individuals going beyond the 'normal' limits of fragmentation are classified as paranoid, schizoid, psychotic, etc.[239]

We should not blame academics for the mess that the education system is in today, for evolution has been more divergent than convergent through its long history. First, large and small material objects were formed, such as stars, galaxies, atoms, and electrons in a process we can call hylogenesis, from Greek *ὑlē* 'matter'. Then during the last three and a half billion years on Earth, we have seen the wondrous diversity of the species evolve. Biogenesis then gradually gave way to noogenesis—the evolution of the mind—about

35,000 years ago, the analytical mind becoming predominant at the dawn of history about 5,000 years ago. As a result of our fragmented minds, society has become divided into religious and national factions, academic specialization, and the division of labour in the workplace.

Bohm was not the only one seeking to heal the fragmented mind in Wholeness. In 1970, a group of academics convened in Nice to address the problem of specialisms in the universities, Erich Jantsch coining the word *transdisciplinarity*, in contrast to *interdisciplinarity* and *multidisciplinarity*.[240] Basarab Nicolescu then points out in *Manifesto of Transdisciplinarity* from 2002 that the discoveries of quantum physics mean that we need to abandon the Laws of Contradiction and Excluded Middle as the basis of logical reasoning, both in science and in society, in general.[241] However, he does not go so far as embracing the Principle of Unity, and thereby the mystical, in his worldview.

At the heart of Bohm's own solution to the problem of fragmentation is the principle that the observer and observed are one, a notion that led him to Krishnamurti around 1960,[242] subsequently holding a series of dialogues on the relationship of science and mysticism. This principle holds not only in quantum physics, where an observing particle can affect that which is being observed, leading to Heisenberg's uncertainty principle. It is also critical if we are to intelligently and consciously heal the fragmented mind. As he said,

> The fragmentation involved in a self-world view is not only in the content of thought, but in the general activity of the person who is 'doing the thinking', and thus, it is as much in the process of thinking as it is in the content. Indeed, content and process are not two separately existent things, but, rather, they are two aspects or views of one whole movement. Thus fragmentary content and fragmentary process have to come to an end *together*.[243]

We can find a clue about how to heal our fragmented minds, and so discover what it truly means to be a human being, from the astronauts who travelled to the Moon. They were able to see the Earth as a unity, where all the divisions that we create between the nations, religions, races, businesses, and so on no longer exist. One of these, Edgar Mitchell, was so moved by the 'sense of universal connectedness' that arose from his journey in 1971 that when he returned, he set up the Institute of Noetic Sciences (IONS) to initiate research into consciousness and human potential.[244]

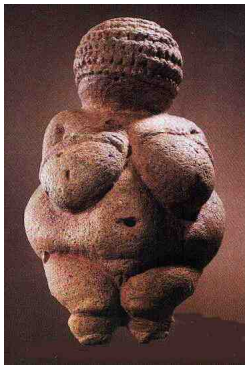
In other words, to understand what is happening to humanity at the present time, we need to stand outside ourselves, to view our lives together from what is called in the vernacular a 'bird's-eye' perspective. But such a vantage point is still anthropocentric, considered from the reference point of our body-mind-soul organisms. So proprioception, the term that Bohm introduced me to in 1980, is not enough to denote our self-reflective abilities, for the word is used primarily in a physiological context, from Latin *proprius* 'own'.

Jean Gebser came much closer to what is needed in the splendidly titled *The Ever-Present Origin*: "The aperspective consciousness structure is a consciousness of the whole, an integral consciousness encompassing all time and embracing both man's distant past and his approaching future as a living present."[245]

For myself, I go even further, and use the term *Self-reflective Intelligence* to denote the human ability to view ourselves in the context of Wholeness. This psychospiritual ability is what distinguishes humans from the other animals, and naturally from machines with so-called artificial intelligence. It leads us to see that we humans, like all other beings, are never separate from the Divine for an instant. In other words, Self-reflective Intelligence is Divine, indicating our ability to view our lives from God's perspective, if I can say this without being misunderstood.

However, this innate ability is something of a two-edged sword. On the one hand, we need Self-reflective Intelligence to manage our business affairs in general and our lives in particular with full Awareness of what we are doing. On the other hand, Self-reflective Intelligence has led humans to see that we are all conceived and born to die, an inevitability that has troubled our species for at least 60,000 years, when the first ritualistic burials that have been found took place.

Other evidence of our unique human qualities comes from the artistic expressions of our inner worlds that have been found across Europe and Asia. Some of the earliest are the cave drawings at Chauvet in Ardèche in south-central France, some thirty to thirty-two thousand years old, when two or three kilometres of ice were covering the land where I live today in Sweden.[246]



Other examples are the images of goddesses appearing across a vast expanse of land stretching from the Pyrenees to Lake Baikal in Siberia, at the time of the Great Mother Goddess, which Ken Wilber, for instance, studied in *Up from Eden: A Transpersonal View of Human Evolution*.[247] A famous example is this limestone figurine of a fertility goddess that was found in Willendorf in Austria, estimated to be between 18 and 20,000 years old.[248]

In Volume I, Part 1 of *Historical Atlas of World Mythology*, Joseph Campbell provides maps of the locations of the painted caves found in south-west Europe and of the distribution of Venus figurines across Europe and Asia, to Lake Baikal.[249] So long before humans settled in villages to cultivate the land and domesticate animals and before the birth of written history, our forebears were able to express in symbolic form in their outer worlds what they could see in their inner worlds.

Projecting human ontogeny onto human phylogeny, such studies are like investigating the characteristics of the infancy of *Homo sapiens*. We might therefore like to think that our species has now reached adulthood, or at least adolescence. However, this is far from the case. In *A New Earth: Awakening to Your Life's Purpose*, Eckhart Tolle wrote, "We are a species that has lost its way," concluding this inspirational book with these words: "A new species is arising on the planet. It is arising now, and you are it!"[250] But despite this book selling several million copies after Oprah Winfrey promoted it on her talk show in 2008, how many people are yet aware that they, themselves, are pioneering a radically new species?

Now normally, in the biosphere and noosphere, ontogeny recapitulates the phylogeny of the species and culture, respectively, an obvious evolutionary relationship that Ernst Haeckel suggested in the mid 1800s for biogenesis, apparently now discredited.[251] However, when new species and civilizations emerge, this process is reversed: phylogeny recapitulates ontogeny.

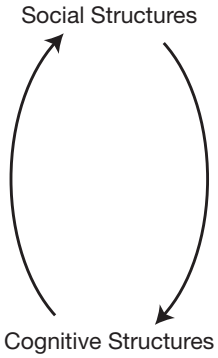
In *The Ghost in the Machine*, Arthur Koestler gave an explanation of how new species can emerge from previous ones in this manner with the word *paedomorphosis* 'the shaping or forming of the young', in contrast to *gerontomorphosis*, when evolution progresses from immediately preceding forms and structures, terms that were introduced by Walter Garstang and Gavin de Beer, respectively

In *gerontomorphosis* ontogeny recapitulates phylogeny, as new-born individuals biogenetically mimic their parents. However, as Koestler puts it, "gerontomorphosis cannot lead to radical changes and new

departures; it can only carry an already specialized evolutionary line one more step further in the same direction—as a rule into a dead end of the maze."[252]

During paedomorphosis, on the other hand, evolution retraces its steps to an earlier point and makes a fresh start in a quite new direction, when phylogeny recapitulates ontogeny. Paedomorphosis is thus a rejuvenating, renascent process; it leads to new vitality, new energies, and new possibilities.

These principles of paedomorphosis and gerontomorphosis apply equally in the noosphere, the prime example being the Copernican revolution in the seventeenth century. For Copernicus effectively went back to Aristarchus's heliocentric view of the solar system, Aristarchus being called the Greek Copernicus,[253] abandoning Aristotle and Ptolemy's geocentric view, which was generally accepted at the time. And generally, paedomorphosis does not begin on the scale of the species; it begins at the individual level, breaking the social-cognitive cycle that drives so much human learning today, illustrated here. For we live in cultures and subcultures that attempt to coerce us to live according to the norms of the group, restrictions that we then pass on to the next generations. Our cultural conditioning thus inhibits us from realizing our fullest potential as human beings, from finding our true purpose in life, as both individuals and as a species, which is to awarely live in union with the meaningless Absolute. As J. Krishnamurti wisely said, "It is no measure of health to be well-adjusted to a profoundly sick society."[254]



Paulo Coelho, author of *The Alchemist*, experienced just such a situation as an adolescent, when he wanted to be an artist against his parents' wishes, three times being sent to a mental hospital to 'cure' him of his 'madness'. In the 1990s, discovering many others in a similar situation, he wrote an influential novel titled *Veronika Decides To Die*, which celebrates individuals who do not fit into patterns society considers to be normal, becoming free of the collective madness that is called sanity. As Coelho wrote in the afterword for this brilliantly insightful and inspiring book, "Between normality and madness, which are basically the same thing, there exists an intermediary stage: it is called 'being different'."[255]

Like Coelho, we all need the courage to be different—different from our parents, who were taught what they knew by their parents and so on backwards in time for thousands of years, pointing out to the courtiers in our lives that the emperor is wearing no clothes.

For Western civilization, which still dominates the world through the global economy, has reached an evolutionary cul-de-sac, inhibiting the emergence of *Homo universalis* and *Homo divinus*, as possible names for the emerging spiritual species. If we humans are to reach our fullest potential as a species before it becomes extinct, evolution thus needs to pass through a discontinuity in a paedomorphic process,

rejuvenating society. This means demolishing the seven pillars of unwisdom on which Western civilization is based, rebuilding the education and economic systems on the seven pillars of wisdom, recognizing that none of us is separate from God, Nature, or any other being for an instant.

So to develop a scientific method that can explain what is causing scientists and technologists to drive the pace of scientific discovery and technological invention at unprecedented exponential rates of evolution, I first return to the Age of Innocence, when our species was at the very beginning of learning about ourselves and the world we live in. For *innocent* literally means 'unharmed, not injured', from Latin *innocentia* 'harmlessness, integrity', from *in-* 'not' and *nocere* 'to hurt'. So to heal the deep wounds in the cultural psyche, as well as my own traumatic wounds, I have needed to let go of everything that we humans have learned over the millennia, leading our species into delusion.

I now come to the heart of the scientific method that has emerged within me during the last 14,024 days, as I write these words on 19th September 2018, to explain why scientists and technologists are driving the pace of scientific discovery and technological invention at unprecedented exponential rates of acceleration. It is really tricky to describe because, when a big bang erupted in my psyche at 11:30 on 27th April 1980, I had no idea what was happening to me. Nothing I had learnt during the previous thirty-eight years had prepared me for this life-changing death-and-rebirth eureka moment.

It is only now, after nearly forty years of self-inquiry, that I can describe the scientific method per se. But without the seventy-seven years of life experience that have led me to where I am today, I'm not sure how anyone else can understand the method. If someone had the patience, they could get a glimpse into what has happened to me by reading all my writings since 1980 in chronological order, rather like reading the thirty volumes of the *Writings of Charles S. Peirce*, only seven of which have been published so far—to 1892, when he was fifty-three.[256] But all these writings do not describe the ineffable sense of Wholeness I enjoy today, a sense of closure that everyone has the potential to enjoy.

As I often say, no one can return Home to Wholeness for nobody has ever left Home. Wholeness is the True Nature of all beings in the Cosmos. Furthermore, as I am Wholeness, there is nothing outside me, living in union with the Divine, like everyone else. Cosmic Consciousness, being all-inclusive, contains the consciousness of all beings who have ever lived or will live, on this or any other planet.

But I did not have this experiential and cognitive understanding when I set out to unify the active and passive data energies within both humans and computers with the four physical forces recognized by physicists through a thought experiment, which I began 20th May 1980, when I wrote at the top of a blank sheet of paper, 'Paul's Folly: A New Model of the Universe'.

For, at the time, I thought that a theory is something written on paper, like the three-page paper that Einstein wrote in 1905 to prove $E=mc^2$, four pages in English translation.[257] However, later that year, I discovered in the newly published *Wholeness and the Implicate Order* that David Bohm regarded a theory as a form of insight, for *theory* derives from the Greek *theōria* 'contemplation, speculation', from *theōros* 'spectator', from *theā* 'a view', also root of *theatre*, and *ὀran* 'to see', also root of *Holoramic* 'whole-seeing'.

So, to understand how Bohm unified quantum and relativity theories, I have needed to study how his thought processes mirror my own, rather than studying some of the rather arcane mathematics in his book. But not only this. As the purpose of this book you are kindly reading is to unify mysticism and mathematics, I have also needed much help from mystics—as mirrors of consciousness—not the least J. Krishnamurti, author of *Education and the Significance of Life* and *The Awakening of Intelligence,* and his friend Vimala Thakar, author of *Spirituality and Social Action: A Holistic Approach*.

But do I begin to describe Integral Relational Logic from where I am today in 2018, from the beginning of this thought experiment in 1980, from when I began self-inquiring in 1974,[258] from when I began questioning the assumptions of Western civilization in 1949, or my conception in the late summer of 1941? Yet, do these questions make sense? In Wholeness, there is no beginning or end, as Bohm pointed out to me in November 1980, when we first met at Birkbeck College in London.

Well, to be true to my own experiences in the Eternal Now, what I feel as I write these words is the creative power of Life and the Logos bubbling up within me from the Divine Origin of the Universe. So, looking at the roots of words, before I describe my experiences in a rational manner, I begin with intuition, which derives from Latin *intuērī* 'to look at attentively, contemplate', from *in-* 'upon' and *tuērī* 'to look at'.

Now this contemplative approach to learning requires the coherent light of Consciousness, which I call Collumination, from Latin *cum* 'together with' and *lumen* 'light', on the model of *illumination*. It is Collumination that enables me to view the Cosmos holographically, rather like the coherent light of a laser creating an image of an object in which every part depicts the whole, a metaphor that Bohm also used in his theory of the Implicate Order.

I also use the word *Collumination* to denote the meditation method that I use to create and reveal Wholeness, in which the practitioner watches thoughts arising directly from our Divine Source. Stanislav Grof calls this wonderful awakening process *holotropic*, meaning 'turning towards the whole', modelled on *heliotropic* 'turning towards the sun', from Greek *ὅlos* 'whole' and *tropos* 'turn', from *trepo* 'to turn', cognate with *tropē* 'transformation'. However, *trepo* has two meanings, as in English: 'to change direction' (as in 'turn into a side-road'), and 'to change form' (as in 'turn into a frog').[259] So *holotropic* can be said to have two meanings, the second being 'transforming the Whole', using *-tropic* in the same sense as *entropic* 'in transformation'.[260]

By creatively bringing order to all my thoughts, I am reducing entropy, regarded as a measure of the disorder or randomness in a closed system. This is possible, of course, because Consciousness, as Ultimate Reality, is not closed, as the Ultimate Source of the creative power of Life. So what Brian Cox said in the 'Destiny' episode of his BBC documentary series *The Wonders of the Universe* in 2011 is not true: "Entropy always increases, because it's overwhelmingly likely that it will."[261] He thus believes in the 'heat death of the universe', a one-sided vision of the Universe that had a profoundly negative effect on the optimism of the late nineteenth and early twentieth centuries, as the historian of science Stephen Brush has pointed out.[262]

To bring Life back to science, transforming the entire world of learning so that it corresponds to all human experience as a coherent whole, everything must be made explicit as much as possible. Such lucidity is the essence of evolution becoming fully conscious of itself within us humans, shining Divine Light into the shadow side of the psyche, into the darkest reaches of the unconscious, of which we are generally unaware, by definition. For *lucid* derives from Latin *lūcidus* 'full of light, clear', from *lūcēre* 'to shine', from PIE base **leuk* 'light, brightness', and *Divine* derives from Latin *dīvus* 'god', from PIE base **dyeu-* 'to shine'.

This is one reason why a computer is such a useful metaphor for our learning. In computers, nothing is implicit, intuitive, or experiential. Everything is explicit, ultimately represented in strings of zeros and ones, as active and passive data, as we see on page 2.

Now, we see in the meaning triangle on page 10 that our knowledge of what is to be represented exists in inner and outer forms, as concepts and symbols to signify them. What this means is that not only do we need a language to represent our thoughts, like computers, our inner concepts also need to made explicit, at least temporarily during the learning process, until they become second nature. And as I am in the process

of lucidly learning about how I think, concepts themselves are referents, that which is to be represented in the meaning triangle.

This is of central importance, for concepts are the basic building blocks of the conceptual models we build of the world we live in, including ourselves. And, as we see on page 30, we humans are the least instinctive of all the animals, virtually all our behaviour is determined through our learning, through the development of cognitive maps of ourselves and the world we live in. So it's vitally important that our mental maps are as holistic and integral as possible, with all pieces fitting together as a coherent whole, like a gigantic, multidimensional jigsaw puzzle. This does not mean that we would then have absolute control over our behaviour patterns, as creatures supposedly with free will. But, at least such profound understanding helps to modify behaviour, leading to more peaceful lives, intelligently seeing both sides of any situation.

But what is the concept of concept? Well, *concept* derives from Latin *conceptum* 'something conceived; formal, of set form', past participle of *concipere* 'to take or hold together', from *cum* 'together with' and *capere* 'to take, seize, catch', also root of *capture*, *occupy*, and many other words, from PIE base **kap* 'to grasp', also root of *have*, through Germanic **havēn*. However, *cum* can also be interpreted as an intensifying prefix, indicating 'take to oneself' and hence either 'take into the mind, absorb mentally' or 'become pregnant'.

Such a notion is also central to the evolution of mathematical logic. In 1879, Gottlob Frege (1848–1925) published a seminal work titled *Begriffsschrift*, usually translated as 'concept writing' or 'concept notation', laying down the foundations of what would become first-order predicate logic, although the full title of this short book in English translation is *A Formula Language, Modeled on that of Arithmetic, of Pure Thought*. Also, Philip Jourdain translated *Begriffsschrift* as 'ideograph' in a 1912 paper, a translation that apparently Frege approved.[263] For *Begriff* derives from German *begreifen* 'to comprehend', from the PIE base **ghreib* 'to grip', also the root of *grip* 'grasp, clutch', with a figurative meaning 'intellectual or mental hold; power to apprehend or master a subject'.

So a concept is something that can be held in the mind, as a mental image or picture. All our concepts taken together form a conceptual model or vision of the Totality of Existence. However, when this vision is not grounded in and embraced by Consciousness or is fragmented as the result of specialization, our Cosmic vision can become distorted and deluded, preventing us from seeing our lives as they truly are.

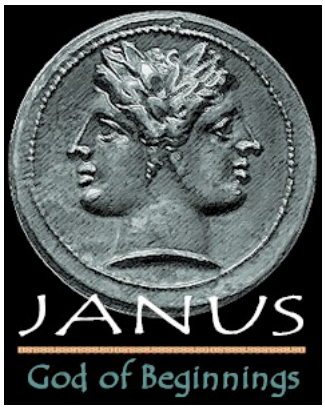
We can overcome this problem by paying careful attention to the very moment of conception, as the seed of an idea forms in our minds. In the womb, such a seed is formed when female and male haploid gametes—from modern Latin *gameta*, from Greek *gamos* 'marriage'—become unified in a zygote, a diploid cell, from *zugōtos* 'yoked', from *zugoun* 'to yoke', cognate with *yoga* and *join*.

This is a notable example of two opposites coming together in unity. However, in Nondual Reality, polar opposites are never separate from each other, something that our earliest forebears understood in the depths of their beings. So, when we Gnostically know the Truth in the depth of being, there is no need to join together that which is never separate. This is what our forebears experienced, even though with limited cognitive understanding.

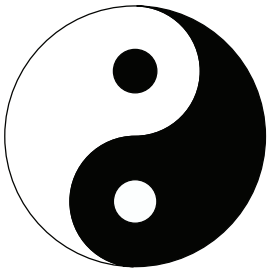
For Mircea Eliade points out that hierogamy is absent in the archaic religions. As he said, "Their supreme Beings were androgyne, at once male and female, both Heavenly and Earthly. … Androgyny is an archaic and universal formula for the expression of *wholeness*, the co-existence of the contraries, or *coincidentia oppositorum*,"[264] in the terms of Nicholas of Cusa. Indeed, Eliade calls *coincidentia oppositorum* the 'mythical pattern', "the very nature of the divinity".[265]

If my own ontogeny is to awarely recapitulate the entire noetic phylogeny of the human race, I must obviously bear this in mind. As I have already described, I do this by imagining that I am a computer that switches itself off and on again, so that it has no programs within it to tell it how to perform functions. Then, guided by its inner guru, as the Logos, this computer has the task of integrating all knowledge in all cultures and disciplines—from all times, past, present, and future—into a coherent whole. And to do this, like a computer, I need a bootstrap program to load the operating system, so that I can begin to function as an intelligent and conscious human being.

Or rather, I need an evolving group of primal concepts as seeds that together form the 'bootstrap program'. To distinguish these bootstrap concepts from the other words I use to describe the narrative, which I have been using so far in this chapter, I bolden them, like **Datum**, from which the entire world of form emerges, as the Ultimate Donor or Creator. I began this way of defining the formal structure of Integral Relational Logic around 2100, a couple of years after attending a retreat in the Altai Mountains in southern Siberia, where I realized that I could no longer describe this universal science of thought and consciousness within the context and infrastructure of any civilization or culture that exists today.



However, the seed for this healing approach to learning emerged in consciousness around midsummer in 1980, when I realized that opposites are never separate from each other, even contradictory ones. One way of presenting this is through Janus, one of the oldest gods in the Roman pantheon, who was depicted with two faces, looking to the past and the future. As the god of beginnings, Janus has given his name to January, at the beginning of the year. Janus is also the god of transitions, such as the global transition process that humanity is passing through at the moment, from pathogenic either-or ways of thinking and living, to a healthy both-and approach to life.



Today, there is much evidence of the awakening of Self-reflective Intelligence due to the popularity of the Chinese concepts of yin and yang, as inseparable dark and light, moon and sun, female and male, etc., and the classic *T'ai-chi-t'u* symbol, or 'Diagram of the Supreme Ultimate'. This symbol depicts the cyclic nature of the Universe. For example, day turns into night, which then turns back to day. The dots in the middle of the two main shapes indicate the potential of the opposite to arise when one side is dominant in any particular situation. The key point here is that when the Universe is viewed as a whole, both opposites co-exist; to reject one in favour of the other does not lead to Wholeness, Peace, and tranquillity.

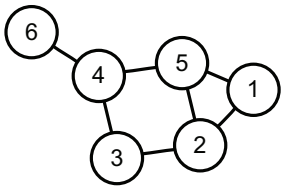
The fact that opposites are never separate from each other is the most fundamental law of the Universe, a proposition that cannot be proven from any axiom or assumed truth. It is self-evident to anyone who looks deeply inside themselves. Yet, self-evidently, we live in a bifurcating world, in which wholes are constantly dividing into polar opposites, which can co-exist in duality or become separated in dualism, a distinction described on page 30.

However, such mundane bifurcations are not the most fundamental. A more basic bifurcation is the distinction between the Formless Absolute and the relativistic world of form. So emerging from the Datum are **elements of data** and the meaningless **relationships** between them, the next bootstrap concepts.

Even though *element* signifies one of the most fundamental concepts, rather surprisingly most dictionaries say that its etymology is uncertain. However, the *American Heritage Dictionary of the English Language* suggests that *element* derives from Latin *elementum*, "perhaps ultimately from *lmn*, first three

letters of the second half of the Canaanite alphabet, recited by ancient scribes when learning it". This is quite nice, even if not generally agreed. For infants sometimes recite the *abc* in childhood, when they begin to read. So, just as letters are the basic building blocks of words in alphabetic writing systems, data elements are the basic building blocks of the Cosmos.

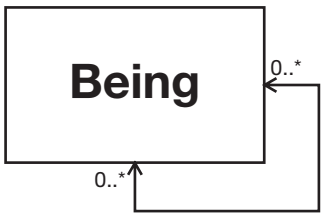
Viewing the Universe as an information system, in which all data elements are related to all others, I picture the Cosmos as a meaningless mathematical graph, like this, prior to interpretation as a meaningful semantic network. To give some meaning to the graph, this is rather like Indra's Net of Pearls or Jewels in Huayan Buddhism, which Alan Watts likened to a dewy spider's web. Each jewel or drop of dew mirrors the brilliant light emanating from all the other jewels. As potentially individuated beings, we are these jewels, both as distinct individuals and as the entire net, which ultimately dissolves in seamless Consciousness through an involutionary process. Indra's Net can thus be used as a metaphor for the holographic, fractal-like worldview emerging today.



As such pictures represent the Totality of Existence, both data elements and the relationships between them can be regarded as **beings**, with the Datum of the Universe being the Supreme Being. For a being is something that exists, not necessarily an object considered to be real, like a stone used for calculating in Roman times, from Latin *calculus*, small stone used in reckoning. We can thereby take the abstractions of mathematics to the utmost level of generality with Aristotle's metaphysical concept of being, at the heart of his ontology:

> There is a science which studies Being *qua* Being, and the properties inherent in it in virtue of its own nature. This science is not the same as any of the so-called particular sciences, for none of the others contemplates Being generally *qua* Being; they divide off some portion of it and study the attribute of this portion, as do for example the mathematical sciences.

Using the notation of the Unified Modeling Language (UML), developed in the 1990s by Grady Booch, James R. Rumbaugh, and Ivar Jacobson of Rational Software, now a subsidiary of IBM, we can thus simplify the picture of a graph above by drawing a complete map of the Cosmos with just one node and arc. This class model shows that all beings in the Universe are related to all other beings in zero to many different ways, some of which can be classified and some of which defy categorization and must remain a mystery. In the words of the New Age mantra, "We are all One."



Here, I am taking the philosophical principle of Occam's razor to its utmost level of simplicity, named after the scholastic William of Ockham (c. 1287–1347), although he was not the first to suggest that our theories of the world we live in should be as simple as possible. Ockam's razor, also known as the law of parsimony,[266] is most popularly stated as "Entities are not to be multiplied without necessity," which was formulated by the Irish Franciscan philosopher John Punch in his 1639 commentary on the works of Duns Scotus.[267]

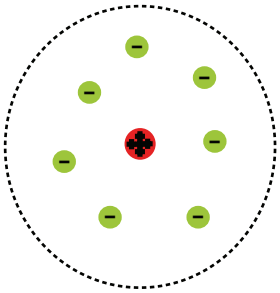
In this simple manner, I have thus resolved two problems I had with the fundamentals of physics, when studying this subject in high school in the late 1950s. First, I did not believe in the big bang theory of the universe, much preferring Fred Hoyle's steady-state model as being more elegant. In my experience, a big bang erupted within me at 11:30 on 27th April 1980, which has led me to look at the Universe in a quite different way from that taught to me at school.

From the perspective of physics, the notion of a big bang, corresponding to Aristotle's notion of the Unmoved Mover, arose from the way that galaxies appear to be moving away from each other, implying a single starting point when and where they originated, about 13.8 billion years ago, according to the latest calculations. However, not all galaxies are diverging, as Edwin Hubble discovered by looking at the heavens through the Hooker telescope on the summit of Mount Wilson, near Pasadena, California.

In his landmark 1929 paper, in which he announced the discovery of twenty-two galaxies beyond the Milky Way, he said that five are converging on themselves while the other seventeen are moving away from each other. For instance, Hubble discovered that the Andromeda galaxy, the nearest to us, is moving towards the Milky Way at 70 kms/sec or 252,000 kms/hour.[268] As Brian Cox also tells us, one day soon (in three to five billion years), the Andromeda and Milky Way galaxies will collide.[269]

Recognizing that Consciousness is all there is thus helps us solve another problem that puzzles physicists today. In the movie *The Theory of Everything*, one scene shows Stephen Hawking being awarded a Ph. D. for his theory that what physicists call a big bang emerged from a black hole, inspired by Roger Penrose's theory of black holes. For as Kim Weaver of NASA has said, "In some ways, the physics [of black holes] is very similar to what started the universe."[270] And just as general relativity indicates that there could be many black holes, not observable directly, Martin Rees has said, "There could have been many big bangs, even an infinity of them. … Whenever a black hole forms, processes deep inside it could perhaps trigger the creation of another universe." Rees, among others, has thus been led to hypothesize a multiverse of parallel universes, of which our own is "just one 'island' in an infinite archipelago".[271]

William James coined the term *multiverse* in an address that he gave to the Harvard Young Men's Christian Association in 1895, titled 'Is Life Worth Living'. Seeking to show that life is only worth living if we recognize that nature, as presented to us by materialistic science, "cannot possibly be its *ultimate word* to man", he said, "Visible nature is all plasticity and indifference,—a moral multiverse, as one might call it, and not a moral universe. To such a harlot we owe no allegiance.[272]



Secondly, I have resolved a problem I had with the philosophical principle of atomism, which we have inherited from Leucippus and Democritus in ancient Greece. The word *atom* derives from Greek *atomos* 'indivisible, uncuttable', from *a-* 'not', and *temnein* 'to cut'. Yet, in 1911, Ernest Rutherford discovered that atoms are not indivisible, consisting mostly of 'empty' space and a positively charged nucleus, where most of the mass of the atom is concentrated, surrounded by negatively charged electrons, illustrated here, not solid at all.

This discovery has not stopped particle physicists spending billions of taxpayers' euros and dollars in searching for an ever-smaller subatomic particle that is the basic building block of the universe. This is absurd, for as soon as one group of physicists claim to have found such a particle, another group sets out to prove them wrong, which is exactly what is happening at CERN at the present time, following the discovery of the Higgs boson. As I could see as a teenager, no one can say when this investigative process will end. Accordingly, I abandoned physics at university when majoring in mathematics, instead studying economics as the required subsidiary, which turned out to be even more depressing.

Furthermore, viewing the Cosmos as a network of data elements emerging from the Datum of the Universe enables us to develop a comprehensive science of causality, including the psychodynamics of society, overcoming the problems of causality that arise from the paradoxes of quantum physics. For, as nothing exists in the Universe but patterns of data and the relationships between them, the only explanation for change in the world is that these data patterns are synergistically energetic.

Building meaningful relationships

Continuing to map the psychodynamics of society, explaining the way that scientists and technologists, aided and abetted by computer technology, are causing the pace of scientific discovery and technological invention to accelerate at unprecedented exponential rates of change, I now need to interpret the meaningless beings and data structures emerging from the Datum of the Universe as meaningful relationships.

Not knowing anything about the history of Western thought at the beginning of this experiment in learning in 1980, I was initially much inspired by René Descartes' own attempts to make sense of the world we live in, particularly his notion of utmost scepticism and systematic doubt in search of unshakable certainty.[273] When returning from fighting in the Thirty Years' War, Descartes had a dream on 10th November 1619 in the small Bavarian village of Ulm (Einstein's birthplace) of "the unification and the illumination of the whole of science, even the whole of knowledge, by one and the same method: the method of *reason*."[274]

Eighteen years later, Descartes published *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences*: '*Discourse on the Method of Rightly Conducting One's Reason and of Seeking Truth in the Sciences*'. He wrote in French rather than Latin, the language of Academe, because he was seeking to reach "the cultured public of society, the ladies of the 'salons' rather than the pedants of the University".[275]

To get started, Descartes gave himself four rules to guide his thinking, which I have adapted as four rules for myself. But before I describe what these are, it is pertinent to spend a moment looking at Descartes' own reasoning. Following his principle of resolute scepticism, he adopted a method like *reduction ad absurdum* in mathematics, assuming the opposite of what he was seeking and that Aristotle's law of contradiction is universally true. As he said, "as I wanted to concentrate solely on the search for truth, I thought I ought to do just the opposite, and reject as being absolutely false everything in which I could suppose the slightest doubt, in order to see if there did not remain after that anything in my belief which was entirely indubitable."[276] Descartes was then led to say:

> But immediately afterwards I became aware that, while I decided thus to think that everything is false, it followed necessarily that I who thought must be something; and observing that this truth: *I think, therefore I am*, was so certain and so evident that all the most extravagant suppositions of the sceptics were not capable of shaking it, I judged that I could accept it without scruple as the first principle of the philosophy I was seeking.[277]

But before Descartes had the boldness to publish what has come to be known as the *Cogito* from *Cogito ergo sum*—the Latin translation of his original words *Je pense, donc je suis*—he realized that he would need to add a chapter on his political and religious orthodoxy in order to get his revolutionary, sceptical method through the censors. So the third discourse began by Descartes saying that even though he was pulling down the house in which he lived in order to rebuild it on fresh foundations, he nevertheless said that he held to three or four maxims, the first of which "was to obey the laws and customs of my country, firmly preserving the religion into which God was good enough to have me instructed from childhood".[278]

Putting Descartes' attempt to unify the whole of science into a historical perspective, Bertrand Russell wrote:

> While it is true that [Descartes] retains much of scholasticism, he does not accept foundations laid down by predecessors, but endeavours to construct a complete philosophic edifice *de novo*. This had not happened since Aristotle, and is a sign of the new self-confidence that resulted from the progress of science.[279]

This is exactly what we need right now, as Western civilization, as a manifestation of the unsustainable patriarchal epoch, comes to an end. However, the Cartesian scholar Bernard Williams has said that while Descartes' attempt to integrate all knowledge into a coherent whole was still a reasonable project in the first

half of the seventeenth century, it would be regarded as 'megalomaniac insanity' in today's postmodern world.[280]

Williams is not the only one to have had such a limiting belief. Many postmodernists, emphasizing individual analysis over collective synthesis, hold similar views. For instance, Jean-François Lyotard attacked the idea that philosophy can restore unity to human learning and develop universally valid knowledge for humanity.[281]

Similarly, taking a much broader view of the Theory of Everything than physicists like Stephen W. Hawking take, Ken Wilber wrote:

> This book is a brief overview of a Theory of Everything. All such attempts, of course, are marked by the many ways in which they fail. The many ways in which they fall short, make unwarranted generalizations, drive specialists insane, and generally fail to achieve their stated aim of holistic embrace. It's not just that the task is beyond any one human mind; it's that the task is inherently undoable: knowledge expands faster than ways to categorize it. The holistic quest is an ever-receding dream, a horizon that constantly retreats as we approach it, a pot of gold at the end of the rainbow that we will never reach.[282]

Ken then goes on to ask, "So why even attempt the impossible?" To which he replies, "Because, I believe, a little bit of wholeness is better than none at all, and an integral vision offers considerably more wholeness than the slice-and-dice alternatives."[283] He seems to be saying here that Wholeness is like an asymptote in mathematics, which can be approached but never reached in finite time. If so, he is confusing the *in*finite and *trans*finite.

Christian de Quincey expressed a similar view in 2001, when the managing editor of the *Noetic Sciences Review*, the journal of the Institute of Noetic Sciences. In a critical appreciation of Ken Wilber's *Collected Works*, he wrote that the genuine theory of everything is impossible:

> Because you cannot create a model or a map that contains itself. Where, for example, would the four-quadrants model fit into the four-quadrants model? Mathematical and logical proofs developed by Bertrand Russell and Kurt Gödel—along the lines that no set of all sets can itself be a set of the same logical category, type, or level—invalidates the claim. Both Alfred Korzybski and Gregory Bateson immortalized this dilemma with the phrase "the map is not the territory." In this case (Wilber's TOE), not only the map, but more crucially, the consciousness that created the map, cannot be found in its own creation. To attempt to make room for it would involve us (and Wilber) in a logical infinite regress. This meta-critique applies to any TOE, of course, not just Wilber's.[284]

So, if I am to explain to others how Life and the Logos have healed my fragmented mind and split psyche, enabling me to unify mysticism and mathematics in order to realize Love, Peace, Wholeness, and the Truth, it seems that we have an enormous challenge on our hands. For, even though there is nothing and no one outside me, you, reading these words, and I, writing them, are engaged in communications, albeit, at this stage only one way.

Continuing the exposition, it is perhaps apposite to reverse Descartes' starting statement and say, *I am, therefore I think*. In the Jewish *Torah*, 'I AM' denotes God: "And God said unto Moses, I AM THAT I AM: and he said, Thus shalt thou say unto the children of Israel, I AM hath sent me unto you." 'I AM is the usual translation of Hebrew *'ehyeh*, the first person form of *hayah* 'to be', although the verb does not indicate tense.

For me, this verse in the Christian Old Testament encapsulates the essence of the mystical worldview, in which the human I and the Divine I are never separate from each other. Similarly, Meister Eckhart, the pre-eminent Christian mystic, said, "The eye with which I see God is the same as that with which he sees me."[285] In other words, "Brahman is all, and the Self [Atman] is Brahman," as the *Mandukya Upanishad* says. Or, as the *Chāndogya Upanishad* says, *Tat tvam asi* 'Thou art That,"[286] reiterated in Nisargadatta Maharaj's *I Am That*, described by Vijai Shankar, an Advaita sage and former medical practitioner, as the

only spiritual book you need to read.[287]

As described on page 44, the word I use to denote the Absolute, as the first bootstrap concept in IRL, is **Datum**. But how can I then describe how I think, Descartes' starting point? Well, we see in the previous chapter that the nearest thing to human thinking in computers is the ability of dapples, as active-active data, to dynamically create new functions from strings of characters and then execute them.

But humans are not machines and nothing but machines. So, what is human thinking? Well, in my experience, it is the ability to form new concepts that have never existed before as pictures in the mind, before words and other symbols emerge to express these mental images outwardly. An example is the ability of an architect to visualize a new opera house, then developing a blueprint, outlining how it could be built.

Similarly, as a retired information systems architect, seeking to understand how we could harmoniously manage our business affairs, I have a mental image of Integral Relational Logic and how it can be expressed outwardly in this book, as a blueprint for constructing the entire Cosmos from the very beginning. But IRL is a rather complex concept, which needs to be built up from the simple data elements that underlie the complexity of the world we live.

To simplify Descartes' four guiding rules of reason, I begin with *clarity*, *simplicity*, *consistency*, and *integrity*, which are closely related to each other. You cannot have one without the others. First, conceptual clarity is of the utmost importance to information systems architects, philosophers, and lawyers, as it is to mathematicians, as we see from this mathematical joke:

> An astronomer, a physicist, and a mathematician (it is said) were holidaying in Scotland. Glancing from the train window, they observed a black sheep in the middle of the field. 'How interesting,' observed the astronomer, 'all Scottish sheep are black!' To which the physicist responded, 'No, no! Some Scottish sheep are black!'. The mathematician gazed heavenward in supplication, and then intoned, 'In Scotland there exists at least one field, containing at least one sheep, at least one side of which is black.'[288]

Secondly, we have already seen how I am applying Ockham's razor to keep the primal concepts in the 'bootstrap program' to as few as possible, forming all concepts in exactly the same simple way, not making any special. Thirdly, in IRL, the word *consistency* does not mean 'not containing any logical contradictions', as the eleventh edition of the *Concise Oxford English Dictionary* states. Rather it means 'acting or done in the same way over time, especially so as to be fair or accurate', to give another definition from the COED.

To illustrate what consistency means in IRL, these four equations show the similarity of concepts as the product of two others in each case:

Equation	Concept	Product
$s = vt$	distance	velocity, time
$V = iR$	voltage	current, resistance
$F = ma$	force	mass, acceleration
$c = qp$	cost	quantity, price

As mathematics is the science of patterns and relationships, we can see that these four equations have essentially the same form. Most specifically, mathematicians, computer programmers, and information systems architects treat mass, space, and time in exactly the same way as all other variables, such as when we buy a few kilograms of potatoes in our local shop. I do likewise in developing IRL. No concepts are more important than any other, including the concepts of God, Universe, and I. In 2005, Chris Clarke, emeritus professor of mathematical physics and chair of the Scientific and Medical Network in the UK suggested in an email that this simple, consistent way of learning and forming concepts could be called 'radical egalitarianism'—very original.

Lastly, the integrity of this experiment in learning is of the utmost importance. For integrity derives from Latin *integritās* 'unimpaired condition, soundness, health', related to *integrare* 'to make whole, heal', from *integer* 'complete, whole, entire, intact; unspoilt, pure, fresh; renewed, begun afresh', literally 'untouched', from PIE base **tag-* 'to touch, handle', also root of *tangent* and *contact*.

Such a coherent approach to learning is just as important when developing business systems. For instance, Frederick P. Brooks, the project manager of IBM's System/360 family of computers in the early 1960s, said, "conceptual integrity is the most important consideration in systems design,"[289] for this is the way to obtain architectural cohesion.

We thus see, from the way that information systems have evolved during the Computer Age, an example of what Jan Christiaan Smuts called *holism*, highlighting a factor in the physical and biological sciences that he felt had been neglected. As he said:

> This factor, called Holism in the sequel, underlies the synthetic tendency in the universe, and is the principle which makes for the origin and progress of wholes in the universe. An attempt is made to show that this whole-making or holistic tendency is fundamental in nature, that it has a well-marked ascertainable character, and that Evolution is nothing but the gradual development and stratification of progressive series of wholes, stretching from the inorganic beginnings to the highest levels of spiritual creation."[290]

In summary, "The whole-making, holistic tendency, or Holism, operating in and through particular wholes, is seen in all stages of existence, and is by no means confined to the biological domain to which science has hitherto restricted it. … Wholeness is the most characteristic expression of the nature of the universe in its forward movement in time. It marks the line of evolutionary progress. And Holism is the inner driving force behind that progress."[291]

It is pertinent to note here that *holism* derives from Greek *ὅlos* 'whole, with a PIE base **sol-* 'whole', also root of *safe*, *salubrious*, *solid*, *catholic* 'relating to the Whole', and *saviour*. In contrast, *whole* derives from an Old High German word *heil*, cognate with *heilida* 'health' and *heilag* 'holy', from PIE base *kailo-* 'whole, uninjured, of good omen'. So a holistic approach to evolution is necessary to end all the Holy wars—wars about the Whole—that have bedevilled humanity for millennia. It seems that it is just a happy coincidence that the PIE bases for *healthy* and *holistic* should be different.

Guided by the four principles of clarity, simplicity, consistency, and integrity, I transform the meaningless data elements underlying the Cosmos into meaningful concepts through the concept of **set**, the most fundamental of the bootstrap concepts of interpretation. For, as Azriel Levy writes as the opening sentence of *Basic Set Theory*, "All branches of mathematics are developed, consciously or unconsciously, in set theory or in some part of it."[292] However, it is not necessary to study the nearly four hundred pages of Levy's book to lay down the foundations of mathematics and hence all knowledge.

Neither is it necessary to study the 750 pages of Thomas Jech's classic textbook on *Set Theory*, which jumps straight in with the eight or nine axioms of Zermelo–Fraenkel set theory, named after Ernst Zermelo (1871–1953) and Abraham Fraenkel (1891–1965). These are known as the ZF or ZFC axioms, if the rather strange Axiom of Choice is included in the axioms, intended to eliminate paradoxes from mathematical reasoning.

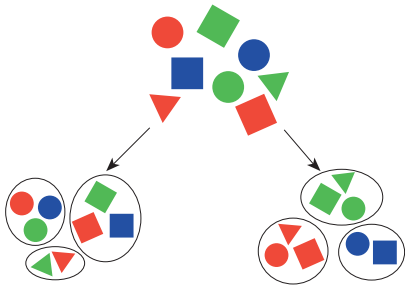
So, what is a set? Well, Jech writes, "Intuitively, a set is a collection of all elements that satisfy a certain given property." However, he immediately abandons this commonsensical definition of set because it leads to Russell's paradox, which arises from "the set S whose elements are all those (and only those) sets that are not members of themselves". But "Does S belong to S? If S belongs to S, then S is not a member of itself, and so $S \notin S$. On the other hand, if $S \notin S$, then S belongs to S. In either case, we have a contradiction."[293]

However, as I am seeking to develop the algebra of algebras that Bohm thought would be needed to establish his unification of quantum and relativity theory as sound science, I do not need the axioms of set theory, which lead to incomprehensible confusion, inhibiting us from building a cognitive map of the Universe that is a true representation of the paradoxical world we live in.

For we see from the semantic network on page 27 that our cognitive maps are nonlinear. So an axiomatic, linear way of reasoning cannot possibly lead us to the Truth. I therefore place no restrictions on sets as elements of sets, maintaining the intuitive definition of set. Otherwise, I would be led into delusion, not able to live my life with integrity, with full clarity of what is happening to me and humanity, as a whole.

Rather, to form concepts in a consistent, egalitarian manner, I use David Bohm's very general way of perceiving order in quantum physics: "*to give attention to similar differences and different similarities*", a notion of order that the artist Charles Biederman gave him.[294] In other words, I carefully examine the **similarities** and the **differences** in the data patterns of my experience, comparing them to one another without prejudice, as much as possible. I thereby put my interpretations into various sets as appropriate, giving **meaning** to the meaningless relationships between data elements.

Nothing could be simpler. Indeed, in the 1960s, a group of mathematicians attempted to introduce sets into primary or elementary schools for eight to eleven-year-olds in the UK and USA. As the authors of *The 'New' Maths* pointed out, the new maths was intended to bring meaning to mathematics and hence to all other disciplines.[295] For instance, as children, when we began to form concepts, we learned to distinguish colours, shapes, and numbers, as in this illustration. This transcultural, transdisciplinary interpretative process is central to pattern recognition, conscious evolution, and all our learning.



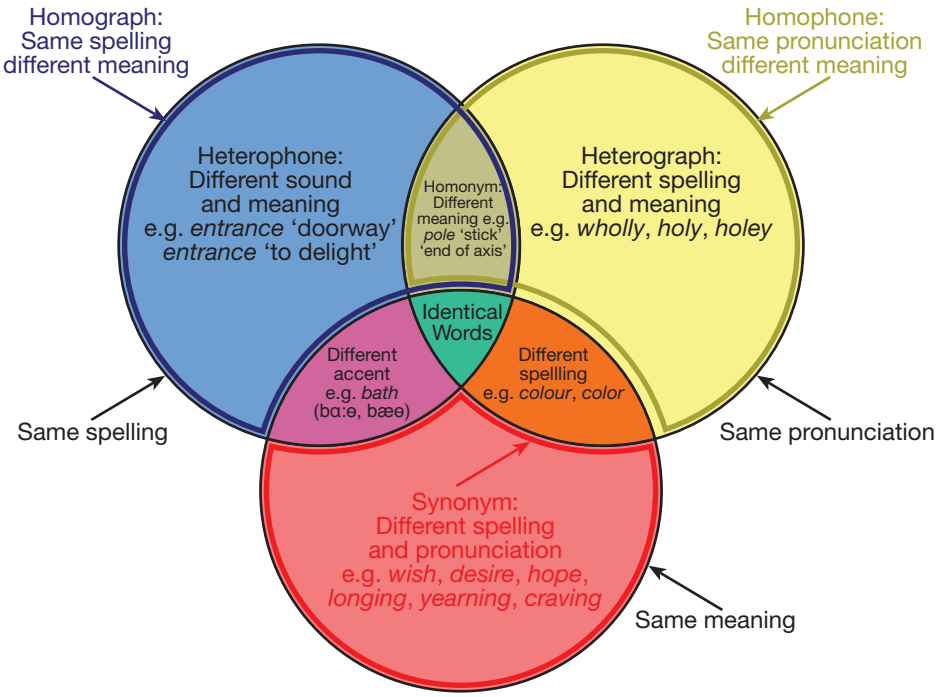
These coloured shapes are examples of **entity** in IRL, associated with **attribute**, the next primal concepts. The common property or attribute of the sets on the left is that they have the same shape and that of the sets on the right the same colour. Going further, an attribute of all the sets is that of number, the count of the entities in or members of the sets. And the common property of two of the sets in the groups of three is that they have the same number, which is an **attribute name** with an **attribute value** of 3. So, we can put all sets with the same numbers into sets, indicating their counts or cardinality. Thus, number is not a bootstrap concept in IRL; it is an instance of attribute.

This is not a new idea. Aristotle called **entity** and **attribute** *subject* and *predicate* in *Prior Analytics* when laying down the foundations of syllogistic reasoning.[296] So, even though I am seeking to introduce a revolutionary nonaxiomatic, holographic system of reasoning, I am endeavouring to keep to the traditions of Western thought as much as possible.

Conventionally, the relationships between sets are depicted in Venn diagrams, introduced to school-children in the New Maths in the 1960s.[297] John Venn (1834–1923) introduced this diagramming technique in 1880 in a paper titled 'On the Diagrammatic and Mechanical Representation of Propositions and Reasonings',[298] as a refinement of 'Eulerian circles', which Euler introduced to a German princess in a letter written between 1760 and 1762.[299]

Charles Sanders Peirce was also inspired by Euler's diagrams to depict his Algebra of Logic as 'existential graphs', which he described as his *chef d'oeuvre*.[300] Book II of the fourth volume of Peirce's *Collected Papers* begins with a piece that he wrote around 1903 titled 'Graphs', which explores how Euler's diagrams, distinct from his mapmaking method, could be used to represent the syllogism.[301]

To illustrate Venn diagrams, let us consider a universe of discourse of all English words, some of which have the same meaning, spelling, or pronunciation, or combinations of these, as synonyms, homographs, homophones, heterophones, heterographs, and homonyms, illustrated in this diagram, adapted from a Wikipedia page around 2100, apparently not now available.

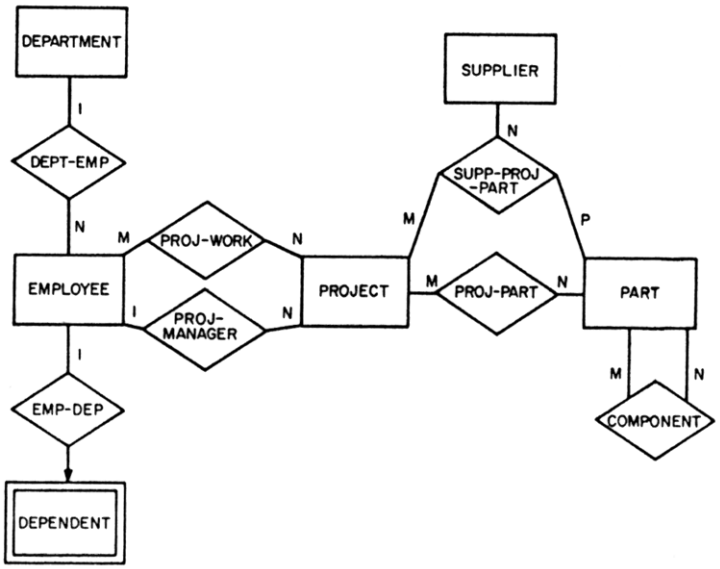


As a further illustration, this table gives some other examples of the relationships of entities and their attributes names and values in linguistic terms:

Class name	*Linguistic terms for English words having similarities in spelling and pronunciation*				
Attribute name	*Term*	*Meaning*	*Spelling*	*Pronunciation*	*Examples*
Attribute values	Homonym	Different	Same	Same	*skate* 'glide on ice', 'fish' *stalk* 'plant part', 'follow' *left* 'not right', 'past of leave'
	Heterophone	Different	Same	Different	*close* 'near', 'to shut' *lead* 'to guide', 'metal' *object* 'thing', 'to protest'
	Homograph	Different	Same	Same or different (Homonym+ heterophone)	*desert* 'leave', 'arid region', 'reward/punishment' *row* 'line', 'paddle a boat', 'quarrel' *bow* 'to bend', 'front of boat', 'bent object'
	Heterograph	Different	Different	Same	*buy*, *by*, *bye* *peek*, *peak*, *pique* *bow* 'bent object', *beau*
	Homophone	Different	Same or different (Homonym + heterograph)	Same	*bow* 'to bend', 'front of boat' *bough* 'tree branch'

However, I'm getting a little ahead of myself here. To see how class, in the above table, relates to set in IRL, I build on the notion of a table or relation, which Ted Codd of IBM introduced in 1970, when I was working for IBM in London as a systems engineer. An example is the above table. Six years later, Peter Pin-Shan Chen showed how the relationships between relations could be depicted graphically, in a paper

titled 'The Entity-Relationship Model—Toward a Unified View of Data',[302] drawing on a visual modelling technique that Charles Bachman had previously developed. As this diagram shows, there are two types of node in such diagrams, depicting entity types, as relations, and the relationships between them, as rectangles and rhombi, respectively. This is similar to John Sowa's notation of conceptual graphs in the field of artificial intelligence.[303]



To give some evolutionary background to these ideas, in parallel with George Boole's publications on operator theory, *The Mathematical Analysis of Logic*, and *The Laws of Thought* between 1844 and 1854, another British mathematician Augustus De Morgan (1806–1871) introduced the concept of *relation* into mathematical logic. Between 1846 and 1862, he wrote five papers 'On the Syllogism', published in the *Transactions of the Cambridge Philosophical Society*. The second of these in 1850 first mentions the concept of relation, which De Morgan defined in the third in 1858 thus: "When two objects, qualities, classes, or attributes, viewed together by the mind, are seen under some connexion, that connexion is called a relation."[304]

However, it was not until 1860 in the fourth paper subtitled 'On the Logic of Relations' that De Morgan described his initial attempt to develop a calculus of relations, stating, "the ordinary syllogism [is] one case, and one case only, of the composition of relations."[305] De Morgan thus generalized the notion of the copula, from Latin *cōpula* 'link', which connects the subject and predicate in syllogistic propositions. For as Morris Kline points out, the relation 'to be' is severely limited, leading to incorrect or possibly incorrect conclusions.[306]

In 1870, Charles Sanders Peirce, then presented a paper to the American Academy of Arts and Sciences, titled 'Description of a Notation for the Logic of Relatives, Resulting from an Amplification of the Conceptions of Boole's Calculus of Logic'.[307] This was then published in the *Memoirs of the American Academy of Arts and Sciences* and also as a book, the first of Peirce's published papers on logic.[308] In 1984, Daniel D. Merrill described this paper as "one of the most important works in the history of modern logic, for it is the first attempt to expand Boole's algebra of logic to include the logic of relations".[309]

Following Peirce's appointment as a lecturer in logic at the John Hopkins University in 1879, when still working full-time with the US Coastal Survey, in 1880 Peirce wrote a major paper on 'On the Algebra of Logic', published in the *American Journal of Mathematics*, founded and edited by James Joseph Sylvester (1814–1897), the distinguished professor of mathematics at Johns Hopkins University.[310] In this capacity, he edited a book in 1883 titled *Studies in Logic* by his students, adding a Note B on the 'logic of relatives'.[311]

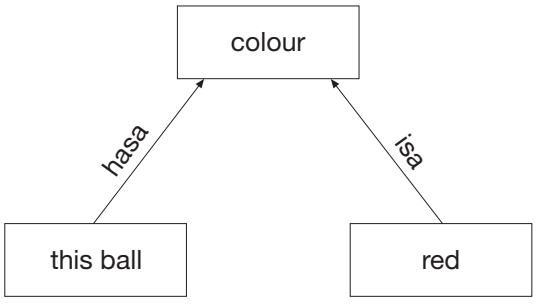
Peirce then read a paper in October 1884 before the National Academy of Sciences, published in January 1885 in expanded form as 'On the Algebra of Logic: A Contribution to the Philosophy of Notation', again in the *American Journal of Mathematics*, intended as the first of two papers for this journal.[312] In the event, this "was to be Peirce's last technical paper on logic to appear in a major scientific journal",[313] although he did have an article published on 'The Logic of Relatives' in *The Monist* in 1897.[314] And the following year, this was the title of the third lecture of a series of eight that he gave in a private house in Cambridge, Massachusetts on the theme 'Reasoning and the Logic of Things'.[315]

It cannot be a coincidence that Arthur Burks, who edited Volumes VII and VIII of Peirce's *Collected Papers* in 1958, was Codd's Ph. D. advisor in 1965, as John Sowa tells us.[316]

To explain how the logic of relatives has evolved into Integral Relational Logic from the Divine Origin of the Universe, I need to give some simple examples of the relationships between entity and attribute. Let us suppose that I have a red ball in front of me. Then I can say these sentences: 'This ball is red' and 'Red is a colour'. I can combine these two sentences into one and say, 'This ball's colour is red' for the ball is not actually red, it is the colour that is red. The fact that the colour is red is implicit in the first sentence; it is made explicit in the combined sentence.

We can also use other languages to denote these relationships. For instance, in Swedish, we can say '*Den här bollens färg är röd*' and in French '*La couleur de cette balle est rouge*.' We could also use the clause form of first-order predicate logic to say `colour(this ball, red)`, where colour is a binary predicate. Similarly, in the programming language Prolog we could say `colour('this ball', 'red')`. In this case *colour* is called a functor because it is related to the mathematical and programming concepts of function.

To turn such sentences into formal logic, making explicit the way that we all think, learn, and form and organize concepts, I use two other ways of depicting relationships. The first way is in the form of diagrams.



The simplest of the diagrams is a semantic network, an example of a mathematical graph, illustrated on page 45. For instance, this diagram shows how the relationships between 'this ball' and 'red' and 'red' and 'colour' can be depicted. Notice that there are two different types of relationship here, a 'hasa' relationship between 'this ball' and 'red' and an 'isa' relationship between 'red' and 'colour'.

The 'isa' relationship is like the set membership in mathematics, for red is an element of the set {red, green, blue, cyan, magenta, and yellow}, listing just the primary colours and their complements, and the colours of the rainbow: {red, orange, yellow, green, blue, indigo, violet}. So, we could also write this relationship as:

red ∈ colour.

There are, of course, many other different types of meaningful relationships between entities, called fields in physics and biology, some of which we look at a little later.

Entity name	*this ball*
Attribute name	*colour*
Attribute value	red

We can also use tables to represent relationships, like this one. Tables are more compact than diagrams, and so they are often more useful. They also give some structure to the relationships that is missing from a basic semantic network. In particular, in the relationship between the beings that I have been using as an example, each of the three beings plays a specific role in the relationship. The key point here is that I am considering 'this ball' as an

entity, which has an attribute whose **name** is 'colour' and **value** is 'red'. This simple table makes these roles explicit. Of course, entities can have many attributes, as in these examples from my definitive book on *Integral Relational Logic*, published on the Web in January 2013, where the entity name becomes an attribute of entity type.

Entity type	*Herbage*		
Attribute name	*name*	*colour*	*family*
Attribute value	grass	green	Gramineae

Entity type	*Polygon*		
Attribute name	*name*	*# sides*	*sum of internal angles*
Attribute value	triangle	3	π radians

Entity type	*Person*		
Attribute name	*name*	*sex*	*age*
Attribute value	Anne	female	35

Entity type	*Country*		
Attribute name	*name*	*population*	*area*
Attribute value	Sweden	8.9 million	449,790 sq km

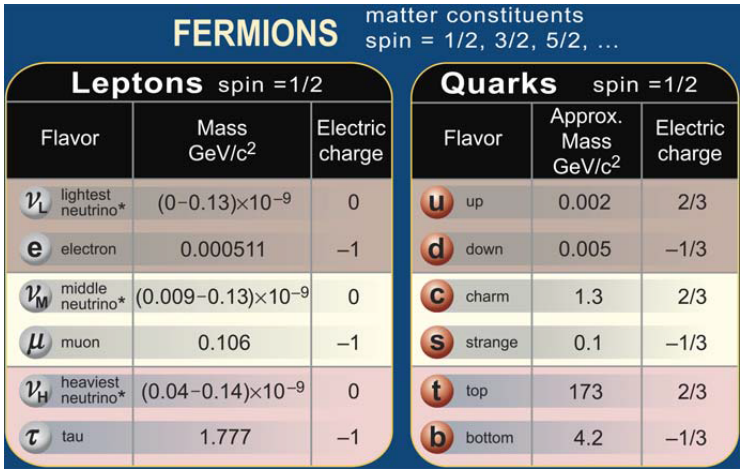
The next step in the development of Integral Relational Logic is to show how entities are represented as **instances** of **classes**, an idea that derives from object-oriented modelling and programming methods, which originated in the Simula programming, as mentioned on page 5. Here, I am using the word *entity* rather than *object* to denote instances, as this word is more general, denoting a meaningful being.

A familiar example is a telephone directory, listing names, addresses, and telephone numbers of subscribers. This is represented in the relational model of data as a **relation**, like the table below, consisting of a set of telephone subscribers. Humans have been keeping records in this manner since the very first civilizations, as mentioned in the first paragraph of this book.

Class name	*Telephone subscriber*		
Attribute name	*Name*	*Address*	*Telephone number*
Attribute values	Anne Potter	72 Grove Road	624-4582
	Fred Tanner	4 Meadow Walk	982-3356
	John Cooper	31 Beech Boulevard	104-3911
	Elizabeth Smith	7 Chestnut Avenue	310-4574
	Jackie Butler	25 Orchard Way	955-4395
	Richard Fisher	67 Willow Crescent	109-2661
	Jenny Walker	22 Heather Drive	893-2748

In IRL, **Telephone subscriber** is a class (boldened with an initial capital letter) and the individual subscribers are instances of the class, corresponding to universals and particulars in Plato's *Republic*[317] and the class-object relationship in object-oriented programming languages and modelling systems. However, as such class-object relationships were introduced into computer science to simulate the operation of systems of discrete events, classes in IRL are not eternal, as were Plato's Forms or Ideas.

While particle physicists use the most arcane mathematics in their futile search for a fundamental particle of matter, they too organize their ideas in simple tables, as this diagram shows. The class name is **Fermion**, with two subclasses **Lepton** and **Quark**. The attribute names are flavour, mass, and electric charge and the attribute values are the content of the table. The periodic table of elements is another example of a relation in science.

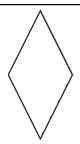
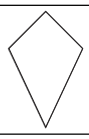


In summary, following the concept of **set**, the next primal concepts in IRL—giving meaning to the **relationships** between **data elements,** within the overall context of the **Datum** of the Universe—are **class**, **entity**, and **attribute**, having their origins in Western philosophy and logic. And although *relation* and *relationship* are often used synonymously, in IRL **relation** and **relationship** are distinct bootstrap concepts.

In IRL, as in the relational model of data, the possible values for a particular attribute in a class are called a **domain of values**, another primal concept, which can also be regarded as a **dimension**, for such domains measure the class-attribute in the broadest meaning of *measure*. So as there are an infinity of spatial dimensions in geometry, and countless other dimensions, there are an infinite number of dimensions in the Universe, not the four space-time dimensions of relativity theory, and far beyond the extra dimensions being postulated in string theory: ten, eleven, twenty-six, or more?

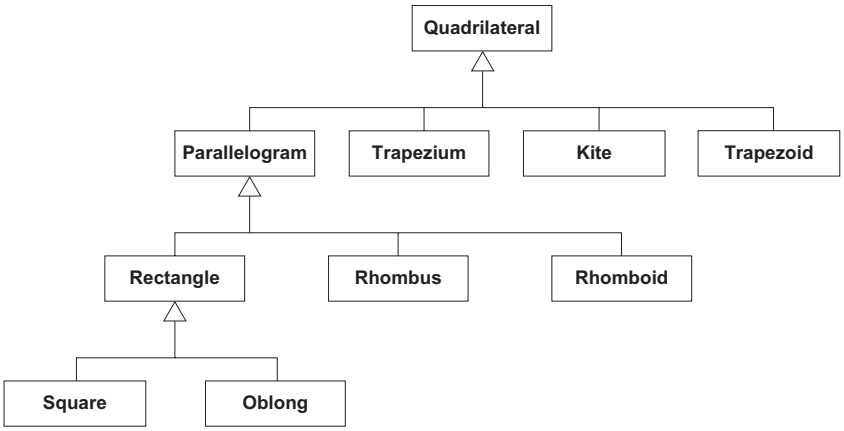
Applying these bootstrap concepts to the formation of IRL itself, I identify a number of distinct classes of attribute, such as **identifying**, **defining**, **nondefining**, **prototypical**, and **derived** attributes. An identifying attribute is typically a name, which might not be unique, or ID in organizations' databases, which needs to be unique, such as a social security number. Defining-attributes denote the property that distinguishes different entities within a class, such as the number of sides in class **Polygon**. Many attributes are nondefining, such as the population and area attributes in class **Country** and age in class **Person**. A prototypical attribute is a characteristic that most entities of a particular type have, but not all. The most quoted example is 'birds can fly.' The fact that not all birds can fly prevents this characteristic from being a defining attribute of birds. An example of a derived attribute is the population density of a country, which is area/population. Deductive reasoning also leads to derived attributes. If 'All humans are mortal,' and 'All Greeks are human,' then 'All Greeks are mortal.'

This table of class **Quadrilateral** illustrates how entities sometimes need multiple attributes to define them.

Class name	*Quadrilateral*				
Attribute name	*Name*	*Shape*	**Defining attributes**		
			Parallel sides	*Equality of adjacent sides*	*Angle*
Attribute values	square		opposite pairs	equal	right
	oblong		opposite pairs	unequal	right
	rhombus		opposite pairs	equal	oblique
	rhomboid		opposite pairs	unequal	oblique
	trapezium*		only two		
	kite		none	two pairs equal	
	trapezoid*		none		

* These are British terms, using the words *trapezium* and *trapezoid* in the original meanings given by Proclus in the fifth century. In the late eighteenth century, the meanings of these two words were confusingly transposed, and they still are in the USA. In American English, a trapezium is a trapezoid and a trapezoid is a trapezium.

The reason for this is that there are subclasses hidden in the table, which can best be denoted as a class model:



Note here that there is a difference between the classes that are subdivided into classes and those that are at the leaves of the hierarchical tree. In this **Quadrilateral** class, we can in addition define subclasses **Parallelogram** and **Rectangle**. And, of course, the other boxes in this diagram are also classes, each of which has countless instances, with attributes such as size and position.

The leaves of the tree are called **concrete** classes in object-oriented modelling because particular instances exist for these types of class. **Quadrilateral**, **Parallelogram**, and **Rectangle**, on the other hand, are **abstract** classes, because there are no instances of these generic classes as such. This is not a hard and fast rule. In natural language, we sometimes use abstract classes concretely. For example, rectangle, parallelogram, and quadrilateral are used as synonyms for oblong, rhomboid, and trapezoid, respectively, the most general examples of the abstract classes in this instance.

We have now come to another vitally important type of relationship in IRL, that of **hierarchical relationship**. The simplest example is that between classes and their instances, as entities. However, we can identify a number of others. In object-oriented modelling, the two principal hierarchical structures are called **generalization** and **aggregation**.[318] Generalization relationships are the relationships between classes and their subclasses, while aggregation relationships consist of the relationships between particular instances of entities in an accumulating manner. Another common type of hierarchical relationship is what can be called **developmental** or **evolutionary**, a special case of this being a family tree.

Class name	*Living beings*
Attribute name	*Attribute value*
Kingdom	Animalia
Subkingdom	Metazoa
Phylum	Chordata
Subphylum	Vertebrata
Superclass	Tetrapoda
Class	Mammalia
Subclass	Theria
Infraclass	Eutheria
Cohort	Ferungulata
Superorder	Ferae
Order	Carnivora
Suborder	Fissipeda
Superfamily	Canoidea
Family	Canidae
Subfamily	Caninae
Tribe	(Null)
Genus	Canis
Subgenus	(Null)
Species	*Canis lupus*
Subspecies	*Canis lupus occidentalis*

The most obvious example of a generalization hierarchy is the tree of life, showing the classification of the species in increasing levels of abstraction, which I explore in my book *The Four Spheres*. Taking an example from the *Encyclopædia Britannica*, a northern timber wolf is an animal that lives in the Canadian subarctic, classified in twenty levels of taxons in biology, as attribute names, as this table illustrates.

The key figure in the taxonomy of the species was Carl Linnaeus from Sweden, who published his seminal work *Systema Naturæ* in

1735 during a stay in the Netherlands.[319] He initially visualized a number of higher levels of abstraction culminating in three kingdoms: animal kingdom (*Regnum animale*), the plant kingdom (*Regnum vegetabile*), and the mineral kingdom (*Regnum lapideum*), corresponding to the question, "Is it animal, vegetable, or mineral?" in the popular radio parlour game 'Twenty Questions' of my childhood and adolescence.

However, generalization hierarchies are not restricted to the tree of life. We can also apply them to model the entire universe of discourse, as all knowledge. Librarians and writers of encyclopaedias have various methods of classifying and organizing knowledge so that it can be retrieved quickly and effectively. All these can be handled by IRL, a situation that can be illustrated with Dewey's Decimal Classification System, widely used by public libraries throughout the world. This system is essentially a single hierarchy of class, division, and section, with limited cross-referencing capabilities. The highest level of classification is illustrated in this table:

Class name	*Knowledge*	
Attribute name	*Class no.*	*Class name*
Attribute values	000	Generalities
	100	Philosophy and related disciplines
	200	Religion
	300	The Social Sciences
	400	Language
	500	Pure Science
	600	Technology (Applied sciences)
	700	The Arts
	800	Literature (Belles-lettres)
	900	General geography and history

I discovered Dewey's system in the summer of 1980, when I set out to develop a cosmology of cosmologies that would unify the psychospiritual and physical energies at work in the Universe within a single, all-encompassing framework. For the books in my local library, in Putney in London, were physically organized according to the decimal library classification system that Melvil Dewey had introduced in 1876.

So as books on the scientific and philosophical perspectives of space-time are catalogued '530.11' and '115' ('115.4' before the seventeenth edition), respectively,[320] I had to walk into the library to find books on these subjects. So, it is easy to see that the physical universe does not provide the overall context for all knowledge. Space, time, and matter have no special place in IRL any more than these subjects have in libraries and bookshops. On the other hand, books of knowledge about knowledge, in the category '000 Generalities', were close to the entrance of the library. Indeed, Dewey originally left class '000' unallocated, so it could today be considered as the superclass for all classes in Dewey's system.

This is quite clear from the fact that this class was relabelled 'computer science, knowledge, and general works' between the seventeenth and twenty-second editions, which were published in 1979 and 2003, which I consulted in Putney and Gothenburg University libraries, respectively. This is a clear sign that computer science contains the abstract, general concepts that provide the seeds for a megasynthesis of all knowledge. So any books on transdisciplinary Integral Relational Logic that might one day be published would fit neatly into the superclass **000**.

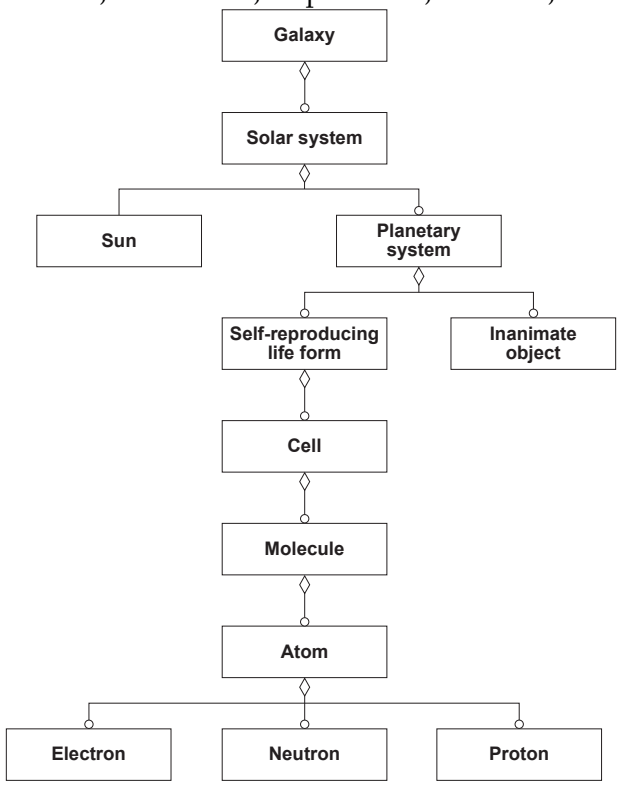
People working in libraries and computer-assisted information retrieval systems often need more advanced thesauri to assist with the organization of knowledge. An example of such a thesaurus is UNESCO's Science and Technology Policies Information Exchange System (SPINES), which I

discovered in 1982, when working as a computer consultant, helping to design and implement an advanced management accounting system for the Kuwait Institute for Scientific Research. In SPINES, additional types of relationship between concepts are defined, some of which are included to try to obtain a measure of consistency in the use of terms between different writers. These relationships can be represented in a relation as shown in this table:

Class name	*Concept relationships*		
Attribute name	*Name*	*Cross-reference*	*Symbol*
Attribute values	equivalent	use	use
		used for	uf
	alternative	see, or	see…or
		seen from	sf
	hierarchical	broader terms	bt
		narrower terms	nt
	associative	related terms	rt
		related terms	rt

Notice that each relationship is bi-directional, although only the associative relationship is symmetrical; relationships often are given different names depending on the direction in which they are viewed. The first two types of relationship are needed because concepts can be denoted by more than one term. In conceptual modelling these are thus less relevant than the actual relationships between the concepts themselves, and so we shall not consider these in any detail here.

As well as generalization hierarchies, there are also aggregation hierarchies in IRL, such as proton, atom, molecule, cell, and so on, illustrated in this semantic network, or section, department, division, and company in organizations. An aggregation relationship is called 'a-part-of' relationship, in contrast to 'a-kind-of' relationship in generalization hierarchies. The essential difference between these two types of hierarchies is that while a generalization relationship associates classes together, an aggregation relationship associates instances of classes with each other.

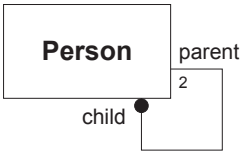


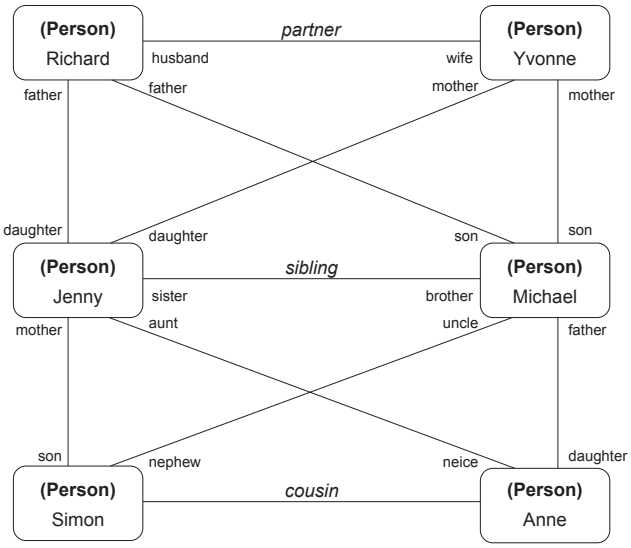
Another way of distinguishing generalization and aggregation hierarchies is to note that in a class hierarchy, the subclasses are mutually exclusive. Thus a type of element is hydrogen, oxygen, or one of the other hundred or so elements. An element cannot be both hydrogen and oxygen. So a generalization relationship is sometimes called an 'or-relationship'. An aggregation relationship, on the other hand, is an 'and-relationship'. An atom consists of a number of electrons, protons, *and* neutrons, the basic model of an atom I learned in school.

In *The Ghost in the Machine*, Arthur Koestler coined the word *holon* to denote structures that are both wholes and parts of a greater whole in aggregation hierarchies, from Greek *ὁlos* 'whole', with the suffix suggesting a particle or part, as in *proton* or *neutron*. In *Janus: A Summing Up*, he went on to say, "every

holon is possessed of two opposite tendencies or potentials: an *integrative tendency* to function as part of the larger whole, and a self-assertive tendency to preserve its *individual autonomy*." This is a clear example of both-and thinking, helping us to live in love, peace, and harmony with each other.

One other obvious hierarchy is that of a **family tree**. Each of us has two parents, an evolutionary process that goes back around a billion years to the birth of sexual reproduction. Conversely, a female and male are parents of one or more descendants. So there are hierarchical structures in both directions of horizontal time. However, when we represent parent-child relationships in a class diagram, we only need one node and arc, where the node is class **Person**. To represent hierarchical family relationships, we need an *instance* model, which is what we normally mean by a family tree, as on the right, rather than a *class* one, on the left. But not all relationships are hierarchical; we have siblings, cousins, uncles, and aunts, etc.





This leads us to **nonhierarchical** relationships, the opposite of hierarchical ones. In *Gödel, Escher, Bach*, Douglas Hofstadter tells us that Warren McCulloch called such relationships *heterarchies*,[321] from Greek *èteros* 'different, other', delighting in such entanglements, a term derived from quantum physics. More simply, we can call **nonhierarchical** relationships **associations**, of which there are countless, difficult to classify. For as soon as we do, we create hierarchical, generalization relationships.

Although there is nothing in the Universe but hierarchical and nonhierarchical relationships, we apply these to the meaningless data elements and their relationships to see that the Universe has a very simple unified structure, which I describe in this way:

The underlying structure of the manifest Universe is an infinitely dimensional network of hierarchical relationships.

This statement is true in all possible worlds, prior to interpretation by a knowing being, and so exists at the ontological level of IRL, depicted in the diagram of the foundations of all knowledge on page 28. Furthermore, all structures in the Universe have the property of self-similarity, like geometric fractals, named by Benoit B. Mandelbrot of IBM. They are thus holographic, with the underlying structure of each part mimicking the Whole.

To understand the significance of this statement in my life, it answers a life-forming question I asked myself at sixteen years of age, when I was becoming increasingly sceptical about the atomistic foundations of science and the big bang theory. Still having faith that scientific method would one day reveal all the innermost secrets of the Universe, I pondered, "What can we know about knowledge that we don't yet know, that is beyond the frontiers of science at any one time?"

The answer is that we can know its deep underlying structure prior to interpretation by a knowing being. The significant point about this statement is that it is not dependent on any interpretation that we might

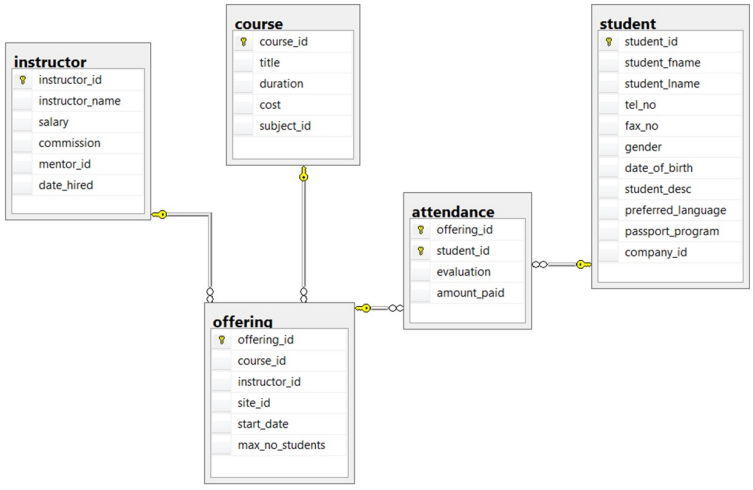
make of the data patterns of our experience. We can therefore consider it part of the data model in IRL at the ontological level of the foundations of all knowledge, illustrated on page 28.

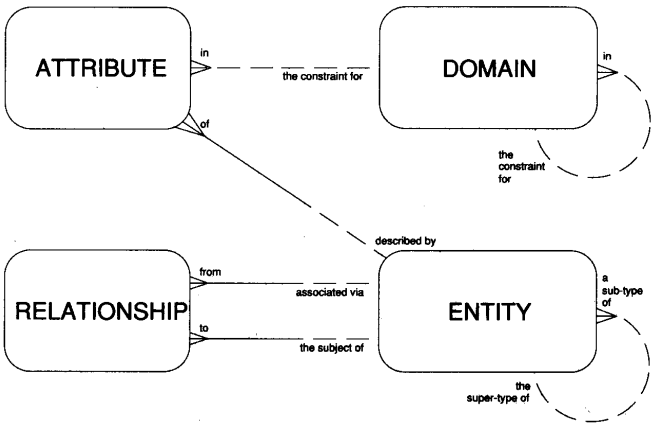
It might seem that this arborizing, reticulating model is so obvious that it is hardly worth stating. Arthur Koestler responded to such criticisms at the Alpbach symposium of 1968, called 'Beyond Reductionism', saying in his inimitable manner:

> This almost universal applicability of the hierarchic model may arouse the suspicion that it is logically empty; and this may be a further factor in the resistance against it. It usually takes the form of what many call the 'so what' reaction: 'all this is old hat, it is self-evident'—followed by the *non sequitur* 'and anyway, where is your evidence?' Well, hierarchy may be old hat, but I would suggest that if you handle it with some affection, it can produce quite a few lively rabbits.

This brings me to the war sometimes going on between reductionist and holistic scientists. For instance, because *hierarchy* has military and ecclesiastical associations, giving the impression of a rigid, authoritarian structure, Fritjof Capra said in *The Web of Life* that in the holistic ecological movement, a paradigm shift is taking place away from hierarchies towards networks.[322] It seems that many don't want leaders, wishing everyone to be treated equally, with no one being special. Yet, this is confusing what Ken Wilber calls *domination hierarchies*, which are pathologically based on force or implied threat of force, with *actualization hierarchies*, whose function is to maximize the organism's potential.[323]

The epistemological level of the foundations is a little more complex. It consists of information about information or knowledge about knowledge, from Greek *epistēmē* 'knowledge'. This meta-knowledge includes all the class and attribute names that are italicized in the examples of relations in this section. Many different notations for diagramming this information have evolved from Peter Pin-Shan Chen's original entity-relationship modelling technique over the years, illustrated on page 53. For a complete model, attributes need to be included in what could then be called class-attribute-relationship models. Sometimes, these are represented as separate bubbles, as in an example on Wikipedia.[324] However, it is more compact to include attributes in a box with the entity type, like this example of students attending various courses offered by instructors.[325]





As entity type, attribute, domain, and relationships are also classes or entity types, it is also possible to create an ER metamodel of the relationships between them. Here is a simple example taken from Richard Barker's book on Oracle's CASE*METHOD™, [326] CASE standing for Computer-Aided Systems Engineering.

What I am describing here is a glimpse of the semantic model for all knowledge, which acts as a **system of coordinates**, like Cartesian coordinates for Euclidean space. This semantic framework acts like a skeleton for the body of all knowledge, ensuring that all our ideas, thoughts, and theories are well organized and ordered. So, because of the universality and all-

inclusivity of this metamodelling method, we are not lead into infinite regress, as Christian de Quincey believed, as we see on page 48.

To illustrate this amazing property of business modelling methods and hence of Integral Relational Logic, in relational database management systems, such metadata is held in relations, just like the information that organizations use to manage their affairs. For instance, in IBM's DB2, two of these tables in the catalogue are called SYSTABLES and SYSCOLUMNS. This is where the basic data about data is stored. In a similar fashion, the popular MySQL stores metadata in the information schema, also called a data dictionary or system catalogue.

Class name	*Class*	
Attribute name	*Relation name*	*# cols*
Attribute values	Class	2
	Attribute	4
	Quadrilateral	5
	Knowledge	2

In IRL, this information is contained in **Class** and **Attribute** classes, organized as relations from all class and attribute names in italics. I've just included the examples from the **Quadrilateral** and **Knowledge** classes in this section. You can see that the **Class** class includes itself as an attribute value of attribute Relation name. And in this next relation, the attributes of the **Attribute** class are included as attribute values of the attribute Attribute name.

Class name	*Attribute*			
Attribute name	*Attribute name*	*Relation name*	*Type*	*Domain of values*
Attribute values	Relation name	Class	Text	Text
	# cols	Class	Integer	>0
	Attribute name	Attribute	Text	Text
	Relation name	Attribute	Text	Text
	Type	Attribute	Text	Text, numeric, graphic, etc.
	Domain	Attribute	List, range, etc.	Anything
	Name	Quadrilateral	Text	Text
	Shape	Quadrilateral	Geometric figure	4-sided figures
	Parallel sides	Quadrilateral	Text	opposite pairs/only two/none
	Equality of adjacent sides	Quadrilateral	Text	equal/unequal/two pairs equal
	Angle	Quadrilateral	Text	right/oblique
	Class no.	Knowledge	3-digit numeral	Divisible by 100
	Class name	Knowledge	Text	Text

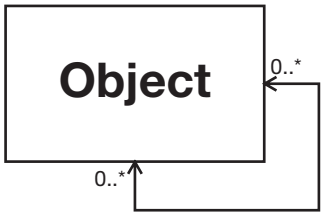
You can see from the few examples of domains of values in this section that they are as much qualitative as quantitative, as they are in the semantic modelling methods used by information systems architects in business. So money, as a domain of values, is of little or no significance in IRL. As other examples, videos and songs on YouTube and iTunes, for instance, constitute domains of values, organized by genre, composer, singer, year, and so on. But while domains of values are sets, and relations are sets, could relations be domains of values?

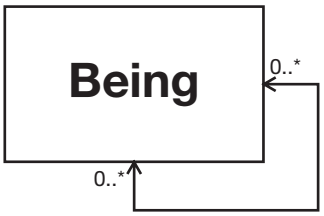
Well, in Ted Codd's original paper on the relational model of data in 1969, only published internally as an IBM Research document, Codd said that relations could be defined with relations as elements, based, as he said, on second-order predicate logic,[327] without the primary quantifiers of ∀ 'for all' and ∃ 'there exists'. However, by the time he published his definitive paper the next year, he restricted the model to first-order predicate logic, which Peirce and Frege independently introduced, recent research acknowledges.[328] In this form of conventional logic, predicates are not themselves predicates or functions defined on the domain. In IRL, on the other hand, domains of values could be relations. So, we can regard

Integral Relational Logic as an *n*th-order, nonlinear predicate calculus, explicitly describing how humans actually reason, rather than in the convoluted linear way mathematicians describe in arcane notation.

Of course, as I am a human being and not a computer, I do not need to make the system of coordinates for all knowledge explicit in my external world. Indeed, such a semantic model is so complex, I would be quickly overwhelmed if I attempted to make it all explicit. Even in business, these can be extremely complex models, difficult to print on a single sheet of A0 size paper, as I managed when working as a computer consultant for a company at Stockholm World Trade Center in the late 1990s after taking early retirement from IBM at the age of fifty-five.

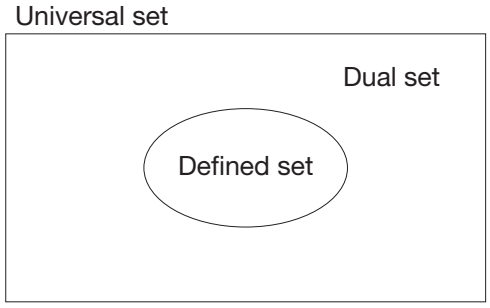
While there, I learned from the class models that the information systems architects were working on that in the Unified Modeling Language it is possible to depict the relationships of all classes in business information systems in a metamodel of the utmost abstraction, like this, reproduced from page 6. As described on page 45, I have generalized the superclass of **Object** in business systems to that of **Being**, the superclass of all classes of knowledge. In this simple way, I show that all beings in the Universe are related to each other in a multitude of different ways, with no divisions anywhere, the key to World Peace.





Unifying opposites

I now come to a relationship between sets that I did not mention in the previous section. Whenever I form a set of elements with a certain attribute value within the context of a particular domain of discourse or universal set, I inevitably form a dual set that does not have this value, illustrated in this diagram. Obvious examples are black and white, true and false, and 0 and 1 in binary arithmetic, used as symbols for impossibility and certainty in probability theory, false and true in logic, and the empty and universal sets in set theory. However, if I have a set of entities with the colour red, the dual set consists of those elements that don't have this colour.



So how can I handle this situation in Integral Relational Logic? If I am to integrate all knowledge in all disciplines and cultures from all times into a coherent whole, I must include all theories, as conceptual models formed from sets, and their opposites. Most notably, physicists have discovered that light behaves both as a wave and a particle, which Niels Bohr called 'complementarity', [329] which is reassuring. Complementary pairs of opposites feel much more comfortable than contradictory ones.

As another example from physics, Aristotle and Ptolemy believed that the Earth is the centre of the solar system, while Copernicus and Galileo believed that all the planets, including the Earth, circle the Sun. Eventually, Kepler and Newton developed heliocentric models of the heavens mathematically. But this does not mean that I should reject the geocentric model from consciousness. For if I did, I would not be able to heal my fragmented, split mind.

To do so, for the first month of this experiment in learning, I played about with some fundamental opposites in mathematics and logic, in particular the truth tables used in Boolean algebra, the proposition calculus, and the algebra of sets.[330] On the next page are three examples for the basic operators of negation, disjunction, and conjunction in computer circuitry, given on page 2.

p	$\neg p$
T	F
F	T

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Mathematics has many such duals, such as the exponential and logarithmic functions and differentiation and integration, as anti-differentiation, the fundamental theorem of the infinitesimal calculus, even though it is not immediately obvious that the process of developing a function for the tangent to a curve is opposite to the process of developing a function for the area under a curve. Then there are a number of instances of what is called the principle of duality in mathematics.

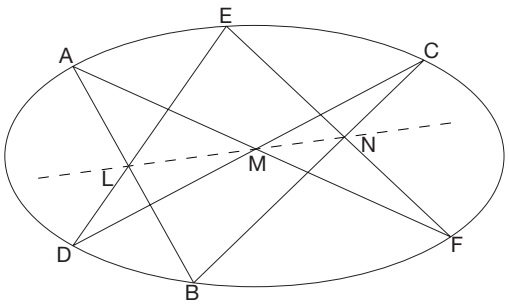
For instance, Richard Courant (1888–1972) and Herbert Robbins (1915–2001) point out in their classic mathematics textbook *What Is Mathematics?* that in the algebra of sets, the twenty-six theorems they state consist of thirteen pairs of opposites, where the symbols for subset and superset ($\subset$ and $\supset$), empty and universal set (O and I), and union and intersection (+ and ·) are interchanged. They call this the principle of duality.

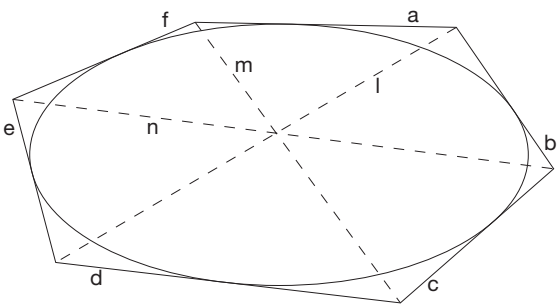
They then go on to point out that the Laws of Contradiction and Excluded Middle originating in Aristotle's *Metaphysics* at the heart of axiomatic set theory are duals of each other. The former states, "An entity cannot possess both attributes A and not-A," not-A variously written as A', $\sim A$, and $\neg A$. Its complement states, "An entity must either possess a given attribute or not possess it." Not only this, these laws apply equally to mathematical logic, as this table reproduced from page 7 indicates.

	Logic	Sets
Law of Contradiction	$A \wedge A' = 0$	$A \cap A' = 0$
Law of Excluded Middle	$A \vee A' = 1$	$A \cup A' = 1$

Another example of the coexistence of polar opposites in mathematics fascinated me as an undergraduate in the early 1960s. Florimond de Beaune (1601–1652), a student of the work of Descartes showed that a curve may be regarded both as the path of a moving point and as the envelope of a moving line, called the principle of duality in projective geometry.[331]

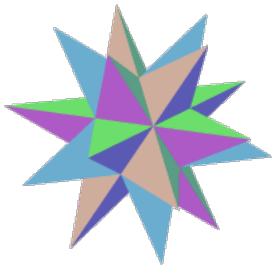
For instance, Blaise Pascal (1623–1662) discovered in 1639, when he was just sixteen years old, that if six points are placed on a conic section and joined as in the left-hand-side diagram below, then their points of intersection, *LMN*, are collinear. Because straight lines remain straight lines in conical projections, this property applies not only to the ellipse, as in the diagram, but also to the parabola and even hyperbola, consisting of two disconnected open curves. As such a property is not intuitively obvious, it is not surprising that Pascal called the six points *ABCDEF* his Mystic Hexagram.[332]



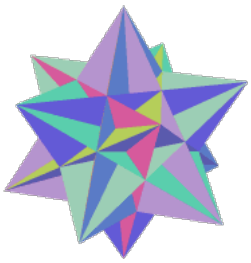


Nearly two hundred years later, in 1810, Charles Julien Brianchon (1783–1864) proved a related theorem, illustrated on the right. If six lines are drawn tangentially to a conic section to form a hexagon, as *abcedf*, then the lines joining opposite vertices, *lmn*, intersect at a single point.[333] In general, whatever theorem can

be proved about points and lines has a dual or reciprocal theorem about lines and points, where lines and points are interchanged.



Of course, the principle of duality applies not only in two dimensions. For instance, polyhedra come in dual pairs, with faces and vertices being duals of each other. For instance, the tetrahedron is self-dual as the stella octangula[334] and the great stellated dodecahedron, discovered by Kepler in 1619 in *The Harmony of the Universe*, and the great icosahedron, discovered by Louis Poinsot (1777–1859) in 1809, are duals of each other.[335]



Now while exploring some of these pairs of opposites in the spring of 1980, there were still two fundamental problems to overcome if my conceptual model of the Cosmos and hence of the psychodynamics of society was to be complete. First, if everything is symmetrical in this balanced view of the Cosmos, how could I create asymmetry from symmetry? As I was seeking to unify all opposites, including physical and psychospiritual energies, it was essential to unify symmetry and asymmetry.

Secondly, how could I include self-contradictions like "This sentence is false" in the model. Such self-referential sentences exist in the world, asserting that if they are true they are false and if they are false they are true. So, if I do not include them in the territory that I am mapping, the coherent cognitive map that I need to heal my fragmented, split mind would be incomplete. I would not have included the process of creating the conceptual model in the model, which I was pondering in the winter of 1980, and would have failed to awaken to my life's purpose, the subtitle of Eckhart Tolle's best-selling *A New Earth*.

At the time I was pondering these critical issues, I knew nothing about the theoretical foundations of mathematics and computer science. All I knew was that Ted Codd's paper on the relational model of data was the most important in the history of the data-processing industry because it described data, as the basic resource in the industry, in sound mathematical terms.

Indeed, there really is no need to spend time here on exploring what happened in mathematics and logic after paradoxes were discovered in set theory, like Russell's paradox on page 50, for Integral Relational Logic is nonaxiomatic, emerging directly from the Datum, as the Divine Origin of the Universe. Maybe one day I'll write an appendix to this book on the misguided history of Western reason and scientific method, expanding on what I wrote on this subject in *The Theory of Everything* in 2014.

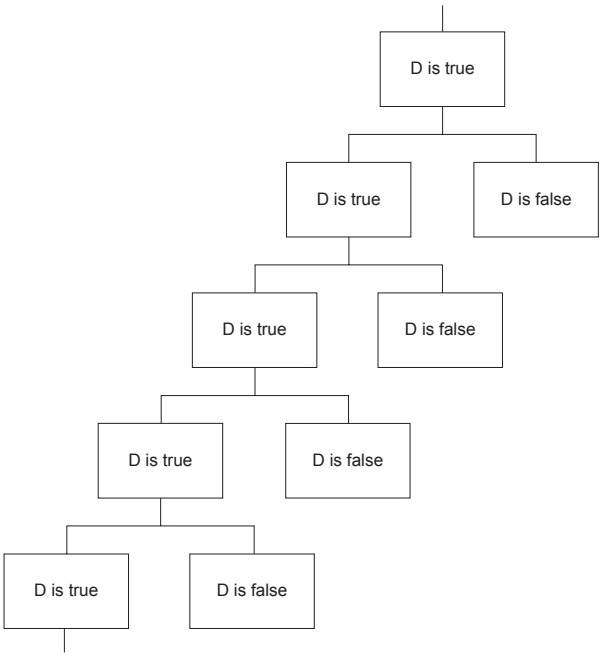
In the meantime, back to the beginnings of the thought experiment I am describing in this book. In the state of innocence I was in then, unharmed as much as possible by the deluded education system and divisive economic system, I was shown a way of unifying all opposites, including asymmetries and self-contradictions, around midsummer in 1980. Metaphorically, I passed through the mirror in Alice's living room and entered a totally different world outside, beautifully illustrated by John Tenniel for Lewis Carroll's second book on Alice's adventures in wonderland: *Through the Looking Glass*. I had entered a wondrous world where words take on quite new meanings, as Humpty Dumpty said. For the first time in my life I was free to learn what I wanted to learn,

without any cultural constraints. This looking-glass was like a two-way mirror. I could see into the room but people still living in the room could not see me. Of course, for practical purposes, I still had to function within a culture that I had not fully joined as a seven-year-old, some thirty years before, which has led to quite a few difficulties in daily life ever since, the key to my spiritual awakening.

Essentially, what happened is that I wrote down this proposition *D*, which I call the **Principle of Duality** in IRL, a fundamental primary, bootstrap concept, lying in the ontological level of the foundations of all knowledge, like the statement describing the underlying structure of the manifest Universe on page 60. This states:

A complete conceptual model of the manifest Universe consists entirely of dual sets.

But is *D* true? Well, sometimes yes and sometimes not. For instance, a collection of entities without a common attribute do not form a set, which we usually call miscellaneous. The set formed by the axiom of choice in mathematics is another set that is not really a set. This ninth axiom in the ZFC assumptions of mathematical set theory states: "Every family of nonempty sets has a choice function."[336] As Morris Kline tells us in *Mathematics: The Loss of Certainty*, around the turn of the twentieth century, mathematicians discovered that they were unconsciously forming a set from other sets to prove some fundamental theorems in mathematics. The sets that they were forming contained members arbitrarily selected from other sets. For instance, such a set could contain one citizen from every country in Europe. But, as Bertrand Russell pointed out, such a set does not have a clearly defined property and so cannot really be considered a set, by the intuitive definition of set.[337]



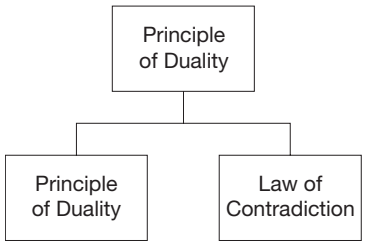
Under these circumstances, *D* is false, even though sets without a defining attribute are the opposite of sets with a defining attribute. But now something quite incredible happens! Those occasions when *D* is false are the opposite of those occasions when *D* is true, confirming that *D* is true. There is thus a **primary-secondary relationship** between the truth and falsity of the Principle of Duality, illustrated in this diagram.

I draw it as an infinite sequence of dual relationships because even though the Principle of Duality is an irrefutable truth, there is no limit to the number of times those with a dualistic outlook on life refute this irrefutable truth, for reasons we look at later. Yet every time this happens, it merely confirms the veracity of the Principle of Duality. *D* is a self-verifying proposition, impossible to deny, true in all possible worlds, an instance of a class in IRL with general attributes *A* and ¬*A*, called a paradox or self-contradiction.

When I first drew this diagram around midsummer 1980, I sat for two or three days in awe and amazement. As I realized, I had been given a universal truth that provides the key to unlocking all the innermost secrets of the Universe. Yet the Principle of Duality is not really a new idea. For instance, in the terms of Hegel's dialectical logic, if '*D* is true' is the thesis and '*D* is false' is the antithesis, then '*D* is true' is the synthesis.

Similarly, in the first of a series of five essays in 1892, titled 'The Architecture of Theories', Peirce explained what he meant by First, Second, Third in his triadic logic: "First is the conception of being or existing independent of anything else. Second is the conception of being relative to, the conception of reaction with, something else. Third is the conception of mediation, whereby a first and second are brought into relation."[338] Peirce thus came as close to discovering the fundamental law of the Universe as anyone else in the history of ideas.

Another way to interpret the Principle of Duality is that the upper diagonal represents healthy both-and reasoning, while the lower represents a divisive either-or approach to life, which I call duality and dualism on page 30. This primary-secondary relationship between the Principle of Duality and the Law of Contradiction is depicted in this simple diagram.



What this means is that while the Principle of Duality is true in all circumstances, the Law of Contradiction is only true in some situations. For instance, Euclid needed Aristotle's law to prove that a largest prime number does not exist; there are an infinity of primes.[339] The prime-counting function, $\pi(x)$, giving the number of primes less than a real number x, is given by this formula, known as the prime number theorem (PNT),[340] where ~ means 'asymptotically equivalent to':

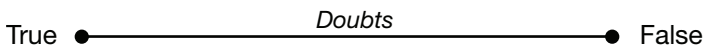
$$\pi(x) \sim \frac{x}{\ln x}$$

In other words, the proportion of primes less than n is $1/\ln(n)$, which is what you might intuitively expect, although better approximations have been discovered. As n gets larger, the primes thin out, with the average gap between them being $\ln(n)$.[341] The probability that a random number less than n is prime thus diminishes indefinitely.

But before we look at some of the psychospiritual implications of the primacy of the Principle of Duality, there are three other relationships between opposites that we need to explore a little. These are the **circle of duality**, **triangle of duality**, and **cross of duality**, three more bootstrap concepts lying in the ontological level of IRL, beneath the epistemological level, prior to interpretation within particular contexts.

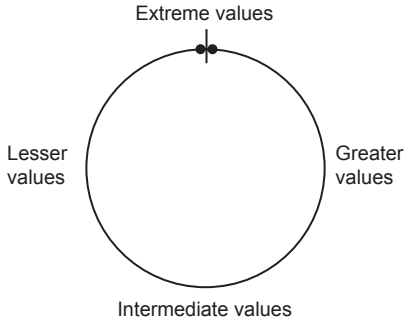
Most obviously, the world we live in is not black and white; there are many shades of grey. So, Aristotle's Law of Excluded Middle, stating, "There cannot be any intermediate between contrary statements,"[342] is not universally true. To include what Aristotle excluded, let us first consider a domain of discourse, 'all meaningful statements'. So, if these are all either true or false, we can represent the Law of Excluded Middle by two points with nothing in between:

Let us then draw a line between the extreme points of the range to include the excluded middle, which represents statements whose truth or falsity we are doubtful about, Descartes' starting point:



We now have a continuous domain of values for all meaningful statements, which is bounded by those statements that are certainly true or false. So we can say that those statements that are either true or false are in a set of certainties, which is the dual of the set containing those statements that are uncertain. In other words, the ends of the true-false spectrum of values can be considered to be the dual of the intermediate values. Any domain of values that consists of a range from one extreme to the other can be put into the set of all entities with this property. For instance, in the theory of probability, certainties and

impossibilities are denoted by 1 and 0, with uncertain probabilities being real numbers lying between these two extremes.



Now as the limits of such a domain of values have the common property that they are extreme values, we can bend the line that represents the spectrum of values to form a circle so that the ends join. I call this circle the **Circle of Duality**, illustrated here, which I could also call the Law of Included Middle. When I formed this construct in the early 1980s, I was reminded of a model of political systems taught to me in the late 1950s in a general studies lesson in high school, with memories of the dominant political systems prevalent at that time, particularly during the Second World War.

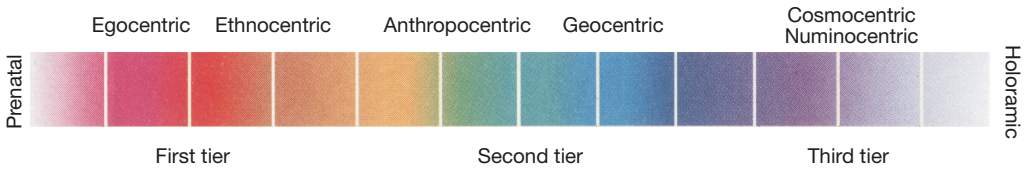
In this model, totalitarian regimes are at the extremes, with the left and right being communism and fascism, respectively. Opposite to these poles, which join at the top, is liberalism, from the Latin *liber* 'free'. In between, we have socialism and conservatism, on the left and right, respectively. Perhaps not surprisingly, my teacher was an active member of the British Liberal party, as it was then, who also happened to be a cleric.

Today, as fourteen billion years of evolution passed through their Accumulation Point in chaos theory around 2004, the polarizing political scene is much more complex, with demagogy, populism, libertarianism, and individualism on the rise in both Europe and the USA. If human society is to become governable, two conditions need to be met to sort out this mess.

First, as there is a primary-secondary relationship between the semantic modelling methods used by information systems architects and the monetary modelling methods used by financiers, such as economists, bankers, and accountants, we need to cocreate a global information system based on the former, rather than the latter.

Secondly, such a global economic system, in harmony with the fundamental laws of the Universe, would only be viable through a massive awakening of intelligence and consciousness within the population at large. In *The Song of Light: Meditations on Lumenarchy*, the ecophilosopher Henryk Skolimowski aptly calls such a system of governance *lumenarchy* 'rule through Divine Light', transcending all the *archies* and *ocracies* that humans have struggled with during the five thousand years of the patriarchal epoch.

Most significantly, democratic systems of government are not viable when the vast majority of people live in the first tier of the spectrum of consciousness, with predominantly ego- and ethnocentric perspectives on life, reproduced from page xv.



Plato was well aware of this problem, lamenting about the way that the Athenian democracy had put his beloved Socrates to death for corrupting the youth of the city state. So, to Plato, philosophers, as lovers of wisdom, should be kings, a ruling elite in a totalitarian state, a guiding principle much criticized by Karl Popper in *The Open Society and Its Enemies*.

Alexis de Tocqueville pointed out other problems with the power of the people in *Democracy in America* in the middle of the nineteenth century, saying that democracies are the tyranny of the majority or masses,[343]

as tyrannous as the despotic forms of governance that they are intended to replace. This is a critical situation that John Stuart Mill further explored in *On Liberty*. As he said:

> In general, opinions contrary to those commonly received can only obtain a hearing by studied moderation of language and the most cautious avoidance of unnecessary offence, from which they can hardly ever deviate even in a slight degree without losing ground, while unmeasured vituperation employed on the side of the prevailing opinion really does deter people from professing contrary opinions and from listening to those who profess them.[344]

Yet what are the alternatives? Social and political structures are expressions of dual and dualistic structures in our minds and psyches, generally driven by the collective and cultural unconscious. Such a society is ungovernable, as Ronald Reagan pointed out in his first inaugural address as President of the USA on 20th January 1981:

> In this present crisis, government is not the solution to our problem; government is the problem. From time to time we've been tempted to believe that society has become too complex to be managed by self-rule, that government by an elite group is superior to government for, by, and of the people. Well, if no one among us is capable of governing himself, then who among us has the capacity to govern someone else? All of us together, in and out of government, must bear the burden. The solutions we seek must be equitable, with no one group singled out to pay a higher price.[345]

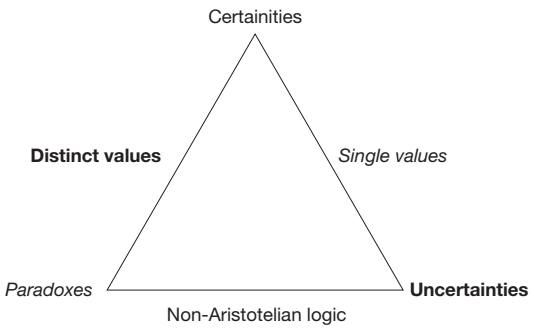
Yet, today, we see ideologues at both ends of the political spectrum, creating ever-greater divisions between peoples, deriding bipartisanship, where every individual takes responsibility not only for their own lives, but also for those of others, as a whole. I have written much more about this in other books and thought a lot more. But until there is a much greater movement from the second to the third tier in the spectrum of consciousness and from the first to the second tier, developing a transcultural system of governance, grounded in Love, the Divine Essence we all share, looks like an impossible dream.

Nevertheless, let us move on with this exposition of the science of thought, reason, and consciousness that we need to manage our affairs with full Awareness of what we are doing. Another idea I had at the very beginning of the thought experiment I am describing in this book is three ways of looking at the relationships between opposites. I was particularly concerned with showing how Aristotle's Law of Contradiction, the Law of Included Middle, and the Principle of Duality could be represented in Integral Relational Logic.

In *Metaphysics*, Aristotle said, "It is impossible for the same attribute at once to belong and not to belong to the same thing and in the same relation … as some imagine Heraclitus says,"[346] a statement known today as the Law of Contradiction, the implicit axiom for linear deductive logic and mathematical proof. In contrast, Heraclitus, known as the 'Obscure' to his contemporaries, said in the few fragments of his writings that have survived, "The Hidden Harmony is better than the obvious," "Opposition brings concord; out of discord comes the fairest harmony," and "People do not understand how that which is at variance with itself agrees with itself."[347]

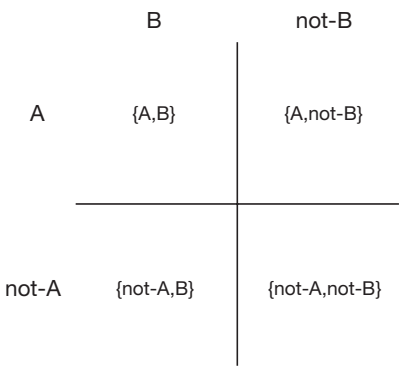
To resolve this dichotomy, in conformity with the aim of the model to be complete and whole, I classify all meaningful statements into three all-inclusive categories:

1. **Certainties**, which are either true or false.
2. **Uncertainties**, which are neither true nor false.
3. **Paradoxes**, which are both true and false.



These three classes form a complete set that can be represented by the vertices of a triangle, in which each vertex is the dual of the other two in some sense. I call this triangle the **Triangle of Duality**, encapsulating the three different ways

that opposites can relate to each other: certainties (either-or), uncertainties (neither-nor), and paradoxes (both-and), the last of these being the most fundamental, encapsulated in the Principle of Duality.



I did not add the third construct to IRL until the mid 1990s, after I read Ken Wilber's *Sex, Ecology, Spirituality: The Spirit of Evolution*. This book led me to notice that pairs of opposites can occur in two or more dimensions, which I call the **Cross of Duality**, illustrated here. This figure arises when we consider two pairs of opposites, *A* and not-*A* and *B* and not-*B*.

Carl Jung's theory of psychological types is a three-dimensional example, the three dimensions being rational (thinking and feeling), irrational (intuition and sensation), and relating (extrovert and introvert).

Extraversion (E)	Introversion (I)
Sensing (S)	Intuition (N)
Thinking (T)	Feeling (F)
Judging (J)	Perception (P)

Katharine Cook Briggs and her daughter Isabel Briggs Myers have extended this psychological typography into four dimensions with their Myers-Briggs Type Indicator (MBTI), with 'dichotomies' listed in the adjacent table.

Ken Wilber's four-quadrants model—known as AQAL, short for "all quadrants, all levels", which is short for "all quadrants, all levels, all lines, all states, all types"—is a two-dimensional example, the dimensions being interior and exterior and individual and social. The exterior quadrants are labelled 'It' and 'Its', while the individual and social interior quadrants are called 'I' and 'We', respectively. [348]

Ken calls AQAL an 'Integral Operating System', or IOS, "a neutral framework" that "can be used to bring more clarity, care, and comprehensiveness to virtually any situation".[349] Since April 2014, AQAL has been called a Superhuman Operating System, which Ken has been teaching in a ten-module Internet course, intended to "Install a Revolutionary New Operating System for Your Mind to Illuminate the Full Spectrum of Your Human Potential, and Become the Greatest Possible Version of Yourself". I did this course in the winter of 2018, learning of some differences between his books and the course, focused mainly on what he calls the second tier of the spectrum of consciousness, involving around 5% of the population, in contrast to 95% in the egocentric and ethnocentric first tier.

Yet, this Superhuman Operating System is but a two-dimensional example of the multidimensional Cross of Duality, and therefore not all encompassing. In contrast, IRL is more like a virtual machine operating system, such as IBM's Virtual Machine (VM), which can run many different operating systems including itself, than Microsoft's Windows or Apple's MacOS. It is thus far beyond the capabilities of so-called superhuman machines that can beat humans at games, mentioned on page 14.

Finally, in this section, I need to consider the existential question, who is this being who is conducting the thought experiment described in these pages? Methods in object-oriented programming languages generally refer to objects in their own classes by the keywords `this` or `self`, possibly from PIE base **s(w)e-* 'pronoun of the third person and reflexive (referring back to the subject of a sentence)', also root of Sanskrit *svāmi*, man who knows or understands himself and thus acts as a master, like a learned Brahman or Pandit.[350]

For myself, I need to refer to myself in order to solve the problem that I was wrestling with during the winter of 1980: how can I include the process of developing a comprehensive model of all business processes within the territory being mapped? As I'm explaining in this book, the Divine Logos that awakens Self-

reflective Intelligence is within, enabling me to understand the essential distinction between my own intelligence and so-called artificial general intelligence in machines.

However, such self-referencing approaches to reasoning lead to paradoxes in conventional linear logic, which greatly troubled Bertrand Russell at the beginning of the twentieth century. As he wrote in 'Reflections on my Eightieth Birthday' in 1952,

> I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. But I discovered that many mathematical demonstrations, which my teachers wanted me to accept, were full of fallacies, and that, if certainty were indeed to be found in mathematics, it would be a new kind of mathematics, with more solid foundations than those that had hitherto been thought secure.
>
> But as the work proceeded, I was continually reminded of the fable about the elephant and the tortoise. Having constructed an elephant upon which the mathematical world could rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling. But the tortoise was no more secure than the elephant, and after some twenty years of arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable.[351]

Russell is referring here to *Principles of Mathematics*, published in 1903, and *Principia Mathematica*, which Russell co-wrote with A. N. Whitehead in three volumes between 1910 and 1913, with second editions published in 1925 and 1927. They famously took 360 pages to prove the proposition (*54·43) that would eventually lead to the statement '1 + 1 = 2', arithmetical addition not yet having been defined.[352]

In an appendix to the first book, titled 'The Doctrine of Types', Russell proposed a tentative solution to paradoxes, which he called the theory of types, suggesting that this would be "a first step towards the truth". In this, he distinguished terms and individuals from their ranges of significance, determined, for instance, when grouped in classes.[353]

To avoid what he and A. N. Whitehead called a 'vicious circle', he thereby defined a hierarchy of types in which "Whatever involves all of a collection must not be one of the collection." As Morris Kline concisely explains, "Expressed in terms of sets, the theory of types states that individual objects are of type 0; a set of individuals is of type 1; and set of sets of individuals is of type 2; and so forth."[354] Whitehead and Russell therefore said that the proposition "all propositions are either true or false" is meaningless and an illegitimate totality because new propositions cannot be created by statements about 'all propositions'.[355]

The basic reason why Whitehead and Russell took so much trouble to write *Principia Mathematica* is that if the axioms of a linear system of thought, such as mathematical proof or deductive logic, contain contradictions, then any formula is derivable from them using the rules of transformation in the system. Using four axiomatic formulae of the tautological propositional calculus, Ernest Nagel and James R. Newman provide a simple proof of this characteristic of mechanistic systems of thought in *Gödel's Proof*.[356] The importance of what mathematicians call consistency in linear systems of thought is also well illustrated by this little anecdote:

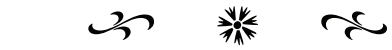
> The analyst G. H. Hardy once made this remark at dinner, and was asked by a sceptic to justify it: 'Given that 2 + 2 = 5, prove that McTaggart is the Pope'. Hardy thought briefly, and replied, 'We know that 2 + 2 = 4, so that 5 = 4. Subtracting 3 we get 2 = 1. McTaggart and the Pope are two, hence McTaggart and the Pope are one.[357]

Yet, all this effort to avoid paradoxes in axiomatic linear reasoning was to no avail. In 1931, Kurt Gödel proved in a paper titled 'On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems I' that it is not possible to prove the axioms of mathematics to be consistent. He did so with an ingenious metamathematical numbering system, proving a theorem that asserted its own unprovability. Truth in mathematics, observed by human intelligence, is more powerful than mechanistic proof.

Then, in 1936, Alan Turing developed another proof that leads to a paradox in his design for a universal computing machine that could supposedly execute any algorithm. In a paper titled 'On Computable

Numbers, with an Application to the *Entscheidungsproblem*' (decision problem), he proved that it is not possible to decide algorithmically whether a universal computing machine will ever halt.

All attempts to eliminate self-contradictions from human reason lead to paradoxes, the most fundamental feature of the dual and dualistic world we live in.



It is a pity that Russell and Whitehead ruled out the possibility of embracing and viewing Totality with Self-reflective Intelligence by standing outside ourselves, for this is just what we need to deal equanimously with the most fundamental implication of the Principle of Duality: all beings in the Universe are born to die, in conformity with the Cosmogonic Cycle, described on page 25. This naturally includes our bodies, Western civilization and the patriarchal epoch, and *Homo sapiens*, as is becoming increasingly clear today, as we accelerate blindly towards the sixth mass extinction of the species on Earth.

The central issue here is the precarious sense of identity and security that governs most people's lives, *identity* deriving from Latin *idem* 'same'. So, even though Consciousness is the Cosmic Context for all our lives and Love is the Divine Essence we all share, there is a tendency in society to favour one of a pair of opposites, rejecting the other. Such a dual or dualistic approach to life does not enable us to realize Nonduality in the utmost depth of being, leading to much fear and ignorance, called *avidyā* in Sanskrit, which we have seen in the world for many millennia.

So, faced today with a multitude of existential risks, not the least the potential for abrupt climate change, in my experience, the only way forward for humanity is to transcend the categories in the Eternal Now, free of the sense of a separate self, living in union with the Immortal Ground of Being we all share, as the Divine.

Transcending the categories

I first became acutely aware that one day a generation of children will be born who will not grow old enough to have children of their own in April 1982, when I was working in Kuwait as a computer consultant in the middle of the Falklands War. It was, at once, the most exciting and terrifying moment in my life, my joy arising from the insight that in just two years, I had been carried in a massive surge of creative energy from the Alpha to the Omega Point of evolution, much as Teilhard had prophesied a couple of years before I was born. I could see the holographic Cosmos as a coherent whole, albeit still very hazily.

On my return to England, I set out to resolve these conflicting emotions by seeking to include the concept of Wholeness in IRL. For so far, IRL had just been mapping structures in the relativistic world of form. But if it were to be complete, it must also include its opposite: the Formless Absolute. Now, while the Absolute is inaccessible to our five physical senses of sight, hearing, taste, smell, and touch, we can nevertheless feel its Presence, literally 'before being' or 'prior to existence', from Latin *præsentia*, participle of *præesse* 'to be before', from *præ* 'before' and *esse* 'to be'.

We can also call the Absolute prior to interpretation the Datum of the Universe, the most fundamental primal concept named on page 44, enabling me to complete my conceptual model of the Universe at the beginning, thereby unifying the Alpha and Omega Points of evolution. To see how this is possible, in conformity with the egalitarianism of IRL, I form the concept of the Absolute in exactly the same way as I form concepts in the relativistic world of form; by carefully observing the similarities and differences in the data patterns of my experience, the simple way of bringing our thoughts into universal order, as described on page 51.

To do this, in conformity with the Principle of Duality, I look at the Absolute in terms of two pairs of opposites: conceptually and experientially and as both a unity and an aggregate, a two-dimensional example

of the Cross of Duality, which cannot be avoided, even when we look at Ultimate Reality. Viewing the Absolute conceptually as a unity, I see that it differs from all its parts, for all these parts are limited in some way. In contrast, the Datum cannot be defined, for to do so would be to give it boundaries, to say what it is and what it is not. This is obvious from the word *define*, which comes from the Latin *dēfīnīre* 'to limit' or 'to end'. The Absolute is thus indefinable and unanalysable, qualities that are **transcendent** with respect to a knowing being.

On the other hand, when I view the Absolute as the Totality of Existence, I see that the structure of all its parts is exactly the same as the structure of any of its parts, for the Universe has an underlying, unified structure, independent of and prior to interpretation by a knowing being, as we see on page 60. The relationships that form this web of life lie within everything there is; they are the glue that holds the entire Universe together. From this perspective, we can say that the Absolute possesses the property of **immanence** with respect to all beings in the relativistic world of form, with meaningful relationships being the motive power of the Universe.

If I now feel into the Absolute experientially, through meditation and self-inquiry, I discover that the Essence of the Universe is Stillness or Emptiness, resulting in the exquisite sense of Nondual Love and Peace, which has no opposite. I am now in union with the Divine, in Oneness, in a state of Unity Consciousness. From this perspective, the Divine is **immanent**.

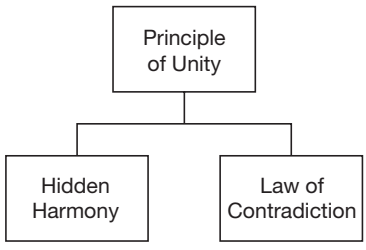
Conversely, if I feel into the Cosmos as an aggregate of all its parts, I experience the Universe simply as a web of relationships, like a mathematical graph, whose nodes consist of meaningful relationships between forms. Then as I sink ever deeper into myself, passing through infinite levels of structure, I approach the Transfinite, as all these nodes become singularities between relationships. Then, as I dissolve even further in an involutionary process, even these relationships disappear, and I am left with the magnificent feeling of translucent Wholeness, Fullness, or Cosmic Consciousness, which is **transcendent** with respect to any knowing being.

In summary, there are two pairs of dual ways in which I understand and experience the Absolute, given in this matrix, thus systemically establishing God as a rational and hence scientific concept:

	Oneness	**Wholeness**
Conceptual	Transcendent	Immanent
Experiential	Immanent	Transcendent

By including the Absolute Whole in IRL, the Principle of Duality becomes the **Principle of Unity**, which can be elegantly expressed in just seven words:

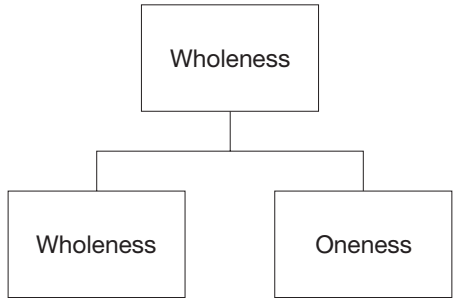
Wholeness is the union of all opposites.

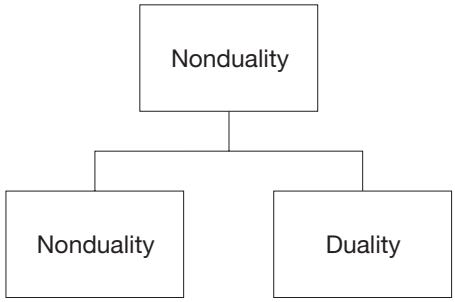


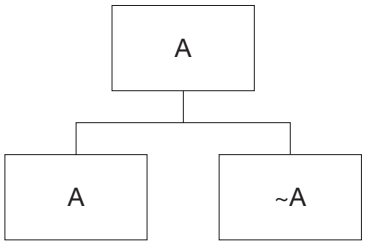
Through a rational system of thought, I have thus revealed the fundamental law of the Universe, which I intuited at the beginning to get this thought experiment off the ground. This is what Heraclitus called the Hidden Harmony. So, I redraw the diagram on page 67 to show the relationship between three fundamental terms for looking at opposites. The Principle of Unity thus lies in the mezzanine level of the foundations of all knowledge, depicted on page 28, between the ontological and Gnostic levels.

Like the Principle of Duality, the Principle of Unity shows that there is a primary-secondary relationship between some of the most fundamental opposites, not the least between Wholeness and Oneness, as two

aspects of the Absolute, and between the Formless, Nondual Absolute and the relativistic, dual world of form, depicted here.







At the utmost level of abstraction, here is a diagram of this irrefutable, universal truth. This is the closest we can get to expressing the Ineffable, Nondual, Absolute Truth in symbolic form, although it would perhaps be better to say *signate*, to distinguish what Carl Jung called signs and symbols, symbols having a profounder meaning than signs. A hypothetical superintelligent extraterrestrial being would instantly recognize this pattern, the paradigm that underlies all others, the key that unlocks all the innermost secrets of the Universe.

Then, around 2013, following the broadcast of a BBC Horizon drama documentary titled 'Einstein's Unfinished Symphony' on Swedish television, I realized that I could express the Principle of Unity and the Hidden Harmony in the notation of mathematical logic, revealing the equation that Einstein sought at the heart of his unified field theory.[358] I call this the *Cosmic Equation*, expressed here as the unification of Western mathematics and Eastern mysticism:

$$W = A = A \cup \neg A = 陰陽 = \text{ॐ}$$

Here *W* is any whole, including Wholeness, *A* is any being, including the Supreme Being and all human beings, ∪ is union, and ¬ is not. The Chinese characters denote *yin* and *yang*, as inseparable dark and light, moon and sun, female and male, etc., unified in the symbol for OM or AUM, the union of *Brahman* and *Atman* in the *Mandukya Upanishad*.

Another who has been seeking this simple equation at the heart of all knowledge is Stephen W. Hawking. In *The Theory of Everything*, a biopic of his life, Stephen told Jane, his future wife, when he first met her, that he was a cosmologist, worshipping "one, single, unifying equation that explains everything in the universe". A few years later, when being awarded a Ph.D. for his extraordinary theory about a space-time singularity as a black hole at the origin of the universe, he told his professors that he was seeking, "One, simple, elegant equation that can explain everything." But "What is the equation?" Jane had asked Stephen when she met him. "That is the question. And a very good question. I'm not quite sure yet. But I intend to find out," was his reply.

However, it is vitally important to remember that the transcultural and transdisciplinary Theory of Everything—consisting of all knowledge as a coherent whole—is a form of insight, as we see on page 41, not a collection of signs and symbols laid out on the page. So, I cannot write out what this means for me. All I can say is that as the result of the experiment in learning that I have been conducting for nearly forty years, my individual consciousness has deepened and expanded to such an extent that it has become coterminous with Consciousness itself, as Ultimate Reality.

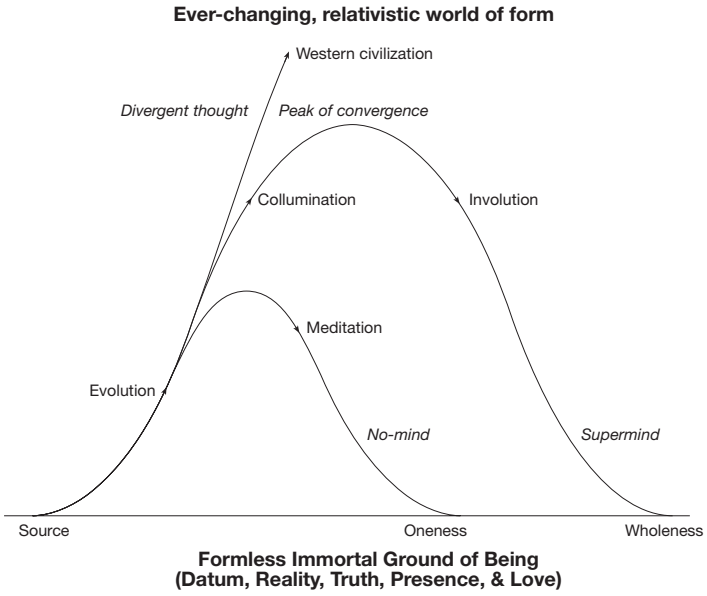
I thus experience Consciousness both as a seamless, borderless continuum with no divisions anywhere and also as the container for the entire world of form, a worldview that is virtually unknown in the West.

However, it is quite familiar to mystics in the East, as the beautiful Sanskrit word *Satchidananda* illustrates, from *Sat* 'Absolute, Eternal, Unchanging Being, Truth', *Chit* 'Absolute Consciousness', and *Ananda* 'Bliss, Absolute Joy'. We can see the relationship between *being* and *truth* from the PIE base **es-* 'to be', which is also the root of many other words in Indo-European languages, such as English *Presence*, *Essence*, *is*, *interesting*, *ontogeny*, and *soothe*, from *sooth*, an archaic word for 'truth', Sanskrit *satyagraha* 'Truth-force' and *satsang* 'gathering of Truth seekers', and Swedish *sann* 'true'.

I call this coherent cosmology of cosmologies *Weltanschauung* rather than the English *worldview*, for this German word derived from *Welt* 'world' and *Anschauung* 'view', from Middle High German *anschouwunge* 'observation, mystical contemplation'. So *Weltanschauung* has a deeper meaning than *worldview*, indicating both scientific observation and spiritual meditation.

Yet, while this Gestalt was emerging in Consciousness, I struggled for many years to understand the relationship of what was happening to me to those around me. On the one hand, I was using the concepts and language of mathematics, computer science, and information systems architects to develop a self-inclusive, integrated model of the world we live. On the other, since the early noughties, which I now see as the end of the beginning of my spiritual awakening, I had had a sense of Gnosis, no different from that of mystics and my spiritual teachers.

The turning point came in the summer of 2008, when my friend Nukunu, a teacher of Nonduality, invited me to attend a retreat in the Altai Mountains in southern Siberia, the original home of the shamans. While listening to Nukunu speak on the first morning, I resolved this difficulty by drawing this diagram.



The path marked 'Western civilization' represents the predominant way of life in today's secular society, accelerating away from Reality with every day that passes. And the small bell curve represents the traditional path of the mystics, taking a short cut to God, towards Oneness and union with the Divine, with No-mind.

The middle path that unifies these extremes is one that I have been following during my lifetime, turning evolutionary divergence into the peak of convergence, moving from the Alpha Point of evolution to its Omega Point and back again, resting in Wholeness with what Aurobindo called 'Supermind: "The Supermind is the Vast; it starts from unity, not division, it is primarily comprehensive, differentiation is only its secondary act."[359]

As I mention on page 42, Collumination is both a form of meditation, as I watch thoughts emerging directly from their Divine Source, and a term for the Coherent Light of Consciousness that reveals the holographic Universe we live in, observed with Self-reflective Intelligence, the eyesight of Consciousness.

On my return to Sweden, I then undertook the mammoth task of rewriting all my writings on my own Gnostic understanding, rather than attempting to present this transcultural, transdisciplinary Weltanschauung in terms of Western civilization, remote, as it is, from Reality.

When writing the definitive description of *Integral Relational Logic* in the early years of this decade, as Part I of a trilogy on *Wholeness*, I discovered a kindred spirit in Charles Sanders Peirce, borrowing all his published works from Gothenburg University library. These told me that Peirce passed through a profound transformation during the four years either side of his fiftieth birthday in 1889.

We can see the beginnings of Peirce's endeavours to integrate all knowledge into a coherent whole from an unpublished piece he wrote in 1885, when he felt that he may have "found the key to the secret of the universe",[360] writing to his lifelong friend William James, "I have something very vast now. I shall write it for Mind. They will say that it is too vast for them. It is … an attempt to explain the laws of nature, to show their general characteristics and to trace them to their origin & predict new laws by the law of the laws of nature."[361] Nevertheless, reflecting on this endeavour nearly twenty years later, he wrote that he was applying a method that any intelligent person could master.[362]

Following an unfinished attempt to write an ambitious book titled *A Guess at the Riddle* on his triadic architectonic, we can see the direction of Peirce's thoughts from five metaphysical essays he wrote from 1891 to 1893 for the *Monist*, edited by Paul Carus, who was the compiler of *The Gospel of Buddha: Compiled from Ancient Records*, the classic text on Buddhism that first introduced many Westerners to Buddha and his teachings, first published in 1894.[363]

Shortly after these essays were published, Peirce had what he called a mystical experience, submitting a much shorter essay to the magazine titled 'Immortality in the Light of Synechism', by far the most profound expression of Peirce's architectonic. However, this article was not published due to a misunderstanding with Paul Carus and was not published until 1958, when Arthur W. Burks edited the seventh volume of Peirce's *Collected Works*.

Peirce called this basic principle of existence synechism, because this word derives from Greek *synekhēs* 'holding together, continuous, contiguous', from *sun* 'together, with' and *ekhein* 'to have, hold', not unlike *synergy* 'working together'. He regarded continuity to be an idea of prime importance in philosophy.[364] But not only in philosophy. The synechistic principle, similar to *holistic* and *integral* tendencies today, is fundamental to all aspects of human endeavour, challenging virtually everything that we have learnt about ourselves and of our relationship to God and the Universe since the dawn of history.

In Peirce's case, he first illustrated synechism with Georg Cantor's infinitesimal continuum, questioning the opinion of many mathematicians at the time "that an infinitesimal quantity is an absurdity."[365] For a few years earlier, Cantor had shown that that there is not just one infinite cardinal; there are at least two. First, there is an infinite set of rationals between any two rationals, which can be mapped to the integers in a one-to-one correspondence, which Cantor called countable. However, Cantor also showed that the real numbers cannot be mapped to the integers and so an innumerable infinite cardinal must also exist,[366] an idea that fascinated Peirce, as it has done many mathematicians ever since, as we see in the next chapter.

In this highly significant three-page essay, Peirce showed that the principle of synechism requires us to look deeply into what it truly means to be a human being. As he wrote, likening synechism to a Brahmanical hymn on the Bliss of the pure and infinite Self:

> Nor must any synechist say, "I am altogether myself, and not at all you." If you embrace synechism, you must abjure this metaphysics of wickedness. … Your neighbours are, in a measure, yourself, and in far greater measure than, without deep studies in psychology, you would believe. Really, the selfhood you would like to attribute to yourself is, for the most part, the vulgarest delusion of vanity.[367]

What this means is "Synechism refuses to believe that when death comes, even the carnal consciousness ceases quickly,"[368] a phenomenon widely accepted today, as Anne Baring describes in *The Dream of the Cosmos*.[369] For "A man is capable of a spiritual consciousness, which constitutes him one of the eternal

verities, which is embodied in the universe as a whole."[370] So when we realize that our Authentic Self is nothing but the Absolute Whole, we become Immortal Beings, free of the fear of death.

Peirce concluded his essay by saying, "though synechism is not religion, but, on the contrary, is a purely scientific philosophy, yet should it become generally accepted, as I confidently anticipate, it may play a part in the "onement of religion and science".[371]

The key point here is that the continuity of consciousness is not sufficient, encapsulated in the new-age mantra "We are all one," often sung while still holding on to an egoic notion of self. If we are to realize that our True Nature is Immortal, we need to realize the seamless continuousness of Cosmic Consciousness, corresponding to the continuum in mathematics. In this respect, as Joseph Brent, Peirce's biographer, points out, the continuous doctrine of synechism is very similar to Bohm's concept of "unbroken wholeness in flowing movement",[372] inspired by the process philosophy of Heraclitus and A. N. Whitehead, which Bohm called the holomovement, which he likened to a flowing stream, whose substance is never the same.[373]

We look Bohm's work in the next section. But first, to complete this brief overview of IRL, we can use this universal system of reason to answer the question "Who are we?" to discover our True Identity. For *identity* derives from Latin *idem* 'same'. And what is the same for all beings in the Universe is the Absolute Whole. So, if we could all realize this in the depths of our beings, all the Holy wars—wars about the Whole that human beings have been fighting for millennia—would come to an end.

To bring about World Peace, we also need to remember that the ontological level of the foundations of all knowledge, including the Principle of Unity, is common to us all. For, if this were not the case, the transcultural, transdisciplinary Internet could not exist. It only does so because the underlying structure of the Universe is an infinitely dimensional network of hierarchical relationships, as we see on page 60, most obvious in the Internet's domain name structure.

So how we can we reconcile the True Identity that we all share with our uniqueness as individuals? For most people, including many spiritual seekers, still seem to consider that our bodies, minds, and souls determine our identity, denoted by names and social security numbers. This is a far remove from the root meaning of *individual*, which derives from Medieval Latin *indīviduālis*, from Latin *indīviduus* 'indivisible', from *in-* 'not' and *dīvidere* 'to divide'. Well, as with all things, we can use the Principle of Unity to resolve this apparent dichotomy. As holographic individuals, we are all both the entire Ocean of Consciousness, which is Immortal, and individual waves and currents on and beneath the surface, which are constantly rising and falling, with a primary-secondary relationship between them.

So knowing that Consciousness is our True Nature, Authentic Self, and Genuine Identity, we can become free of the fear of death, recognizing that the entire world of form is what is called *māyā* 'deception, illusion, appearance' and *līlā* 'play of the Divine' in Sanskrit. Such a realization is essential if we are to complete the Cosmogonic Cycle, returning to the Unmanifest in Total Freedom, denoted in Sanskrit by *jīvanmukti* 'liberated while still alive in the body'.

This emphasis on Wholeness in the East is best seen in a Chinese Buddhist school that developed a synthesis of all Buddhist teachings, embracing all the others, not one among many. This is variously called *Hwa Yen*, *Hua-yen*, and *Huayan* in transliterations from Chinese, meaning 'Flower Ornament', 'Flower-Decoration', or 'Garland' from the *Avatamsaka Sūtra*, originally written in India in Sanskrit during the first and second centuries. In 1971, Garma C. C. Chang introduced Hwa Yen Buddhism to the West in *The Buddhist Teaching of Totality: The Philosophy of Hwa Yen Buddhism*, further developed by Francis H. Cook

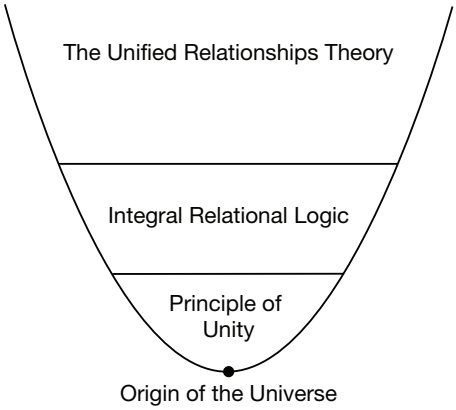
in 1977 in *Hua-yen Buddhism: The Jewel Net of Indra*, beginning his book with this visionary sentence: "Western man may be on the brink of an entirely new understanding of existence." As Cook also tells us, Hua-yen thus "came to serve as the philosophical basis for the other schools of Buddhism more concerned with practice and realization. … As D. T. Suzuki remarked, Hua-yen is the philosophy of Zen and Zen is the practice of Hua-yen."

In terms of understanding our True Identity, a central notion of the *Avatamsaka Sūtra* is Indra's Net, named after Indra, the king of the gods in the *Rig Veda*. Alan Watts likened Indra's Net to a dewy spider's web, saying, "Imagine a multidimensional spider's web in the early morning covered with dewdrops. And every dewdrop contains the reflection of all the other dewdrops. And, in each reflected dewdrop, the reflections of all the other dewdrops in that reflection. And so *ad infinitum*. That is the Buddhist conception of the universe in an image."

Indra's net is also represented as a net of jewels, each one representing us as an individual, reflecting the brilliant light shining through all the other jewels, thereby showing that we are all one whole, with no separation between any of us. So just as all structures in the Universe are holographic, they also have the property of self-similarity, like fractals, as we showed cognitively on page 60.

The Unified Relationships Theory

The *Unified Relationships Theory* (URT) is all knowledge in all cultures and disciplines at all times, past, present, and future, integrated into a coherent whole. I am able to visualize such an immense body of knowledge because Integral Relational Logic acts as a skeleton, framework, or system of coordinates for the entire world of learning. The URT is thus open-ended, as this diagram reproduced from page 28 illustrates.



As the URT includes an integral science of causality and holistic theory of evolution, it is far beyond what Albert Einstein visualized as the unified field theory and Stephen W. Hawking thought of as the theory of everything (TOE), also called the grand unified theory (GUT). As IRL provides the Cosmic Context, coordinating framework, and Gnostic Foundation for all knowledge, the URT is the Holy Grail, Philosophers' Stone, and Apotheosis of human learning.

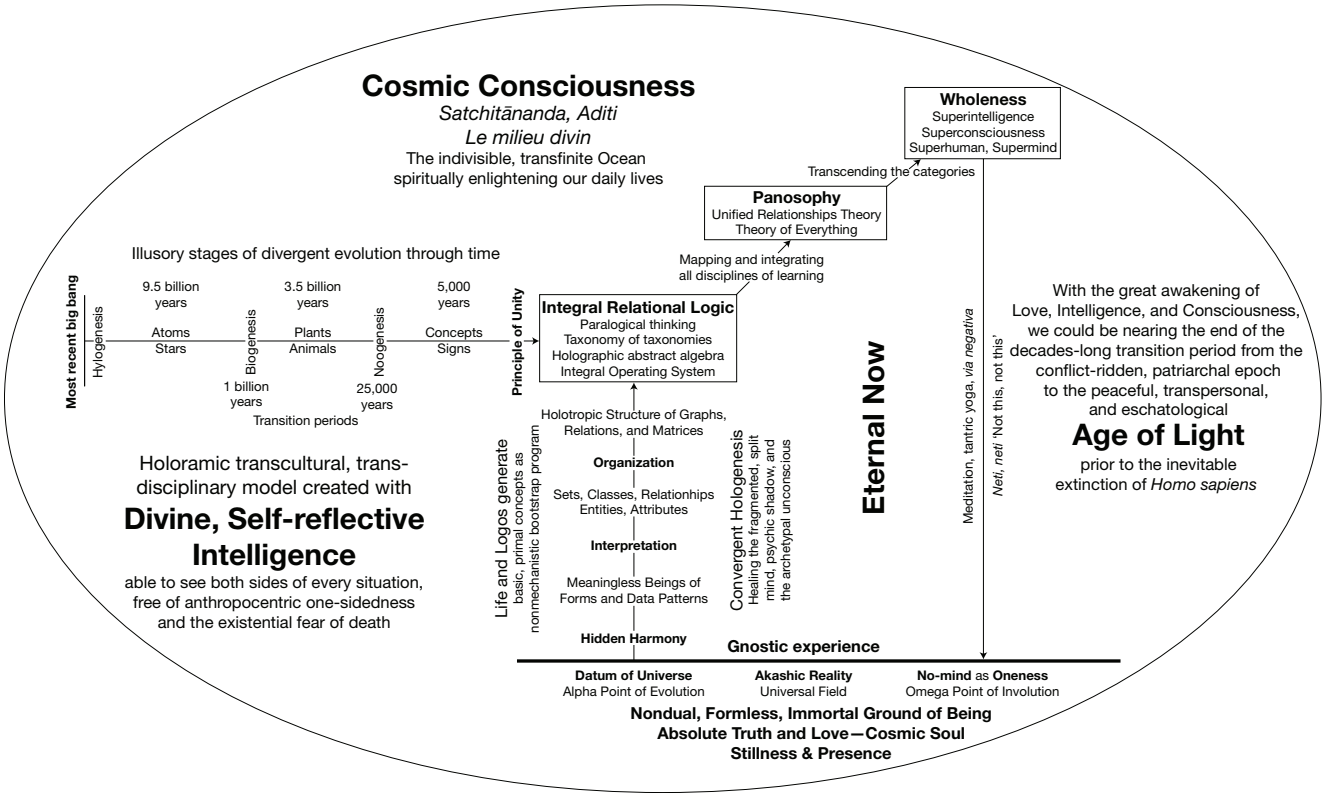
The URT thereby answers this advertisement, posted on the front cover of the *New Scientist* magazine on 30th April 2005. The accompanying cover story stated the purpose of such a theory of everything: "Physicists believe that there was only one force just after the big bang, and as the universe cooled it split into the four forces we now observe: gravity, electromagnetism, and the strong and weak [nucleic] forces. The physicists' dream is to find a theory describing this unified force."[374]

Most significantly, as the Universe, viewed as Consciousness, consists of nothing but meaningful structure-forming relationships, interpreted from the underlying data patterns, the only possible cause of change in the Universe is these relationships, whether these be psychospiritual or physical. Relationships are a

generalization of physicists' notion of fields, which the biologist Rupert Sheldrake extended into morphogenetic fields in *A New Science of Life* in 1981,[375] the year after I embarked on my own studies of such relationships. For *morphogenetic* derives from Greek *morphē* 'form, shape', the driving force of the Universe viewed as an information system once we also admit the Divine power of Life into science. Sadly, however, this obvious idea did not please the scientific establishment, John Maddox, the editor of *Nature*, famously saying of *A New Science of Life*, "This infuriating tract … is the best candidate for burning there has been for many years."[376]

Nevertheless, the URT completes what the BBC called Einstein's unfinished symphony in 2005. As Michio Kaku said in the broadcast, if Einstein had been successful in his aim of developing the unified field theory, "The theory of everything would have been the holy grail of science; it would have been the philosophers' stone. It would have been the crowning achievement of all scientific endeavours ever since humans walked the face of the Earth."[377] Similarly, in *The Elegant Universe*, Brian Greene defined the much sought-for but derided theory of everything as "a theory capable of describing nature's forces within a single, all-encompassing, coherent framework".[378]

Despite the vastness and profundity of the Unified Relationships Theory, which exists in Consciousness, in May 2017, I managed to express a vision of the Grand Design of the Universe on a single A4 page, which I have slightly revised since:



Tim Freke, a spiritual philosopher in England, coined the word *paralogical* in *The Mystery Experience: A Revolutionary Approach to Spiritual Awakening*. Tim points out that we live in a profoundly paradoxical world, so mechanistic, linear logic cannot help us to live in harmony with the basic law of the Universe. Paralogical thinking thus denotes our explorations of the utmost depths of existence, not obvious when we live superficial lives. As he says, "We see the paradoxity of something when we understand it from two opposite perspectives at once." Tim aptly uses the simple word *WOW* to denote such an awakened state of

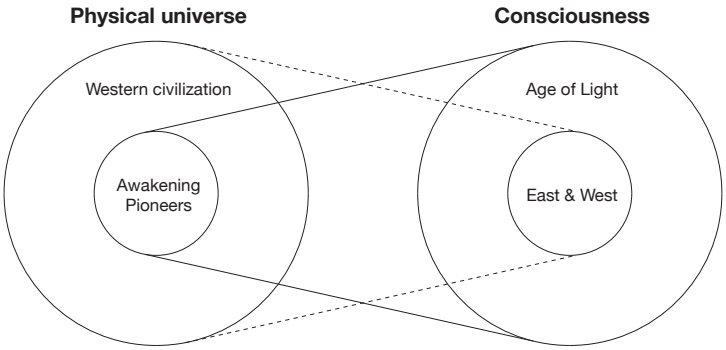
being, for there is nothing more wonderful in human existence. Not surprising, this is something "everyone is searching for", as he says.[379]

However, *Panosophy*, with this spelling, is my own coinage, denoting the transdiscipline that is the Unified Relationships Theory, integrating all specialist disciplines of learning into a coherent whole. To make it easier to refer to the URT, *panosophy* is modelled on *philosophy*, from Greek *pan* 'all' and *sophia* 'wisdom'. The ancient Greeks used the word *pansophos* to mean 'very wise', literally 'all-wise'. Then, in the mid 1630s, Jan Ámos Komenský (Comenius), who has been called the 'father of modern education', wrote books titled *Pansophiæ Prodromus* 'Forerunner of Pansophy', as 'universal wisdom', *Pampædia* 'universal education', and *Didactica Magna* 'The Great Didactic', in which he proposed that "all men are taught all subjects in all thoroughness."[380]

Comenius's friend the intelligencer Samuel Hartlib translated some of these works into English in 1642 as *A Reformation of Schooles*, referring to pansophy as specifically Christian.[381] Nevertheless, *pansophy*, occasionally spelled *pantosophy*, came to mean 'universal or cyclopædic knowledge; a scheme or cyclopædic work embracing the whole body of human knowledge'.[382] Pansophy formed the basis of Pansophia, 'a dream of science', the vision of a Utopian society, to this day still not realized, as Frank E. and Fritzie P. Manuel point out in their scholarly tome *Utopian Thought in the Western World*.[383]

To realize this dream, in 1642, Comenius attempted to set up a Pansophic College in London, to no avail. For as Matthew Spinka, his biographer, comments in 1943, "Were the grandiose project accomplished in our day, what a boon it would be! But alas! the world is still waiting for its realization, and we seem to be further away from it than ever."[384] I look further at this possibility in the Epilogue, to be written in the spring of 2019, when the prospects for humanity into the 2020s should be much clearer than they are today.

In the meantime, in this chapter, I feel the need to explain in a little more detail where Integral Relational Logic and the Unified Relationships Theory fit into the history of human learning. Essentially, establishing Consciousness as Ultimate Reality, as the Cosmic Context for all our lives, goes much further than the paradigm shift or change much talked about today. What we are engaged in is a total contextual inversion, as this diagram illustrates.



The first indication of a scientific revolution taking place today that I came across were two New Paradigm Symposia held in November/December 1985[385] and April 1986, the latter titled 'Charting Paradigm Shifts: The Growth of a New, Holistic Worldview', sponsored by the Elmwood Institute, the Melia Foundation, and the Institute of Noetic Sciences (IONS). At the second Symposium, Willis Harman, then the President of IONS, described this vision in these words:

> Most educated people in this country [the USA] would think it pretty preposterous to suggest that the change that is taking place is at as deep a level as the change that took place during the Scientific Revolution, because that would imply,

> of course, that the near future—the early part of the next century—would be as different from present times as present times are from the Middle Ages."[386]

The central issue here is the recognition that Consciousness is the primary reality, not the physical universe. However, in *Global Mind Change*, Willis Harman hedged his bets, defining three metaphysical perspectives: M-1, in which matter gives rise to mind (materialistic monism), M-2, in which matter and mind coexist as two fundamentally different kinds of stuff, à la Descartes (dualism), and M-3, in which the ultimate stuff of the Universe is recognized as consciousness, mind thus giving rise to matter (transcendental monism).[387]

We saw a similar situation during the first scientific revolution. In between the geocentric view of Aristotle and Ptolemy and the heliocentric view of Aristarchus and Copernicus, Tycho Brahe developed a compromise in which the inner planets revolve around the Sun, while the Sun, Moon, and outer planets revolve around the Earth.[388] In *New Astronomy*, Kepler wrote in Part I that these three models are equally valid mathematically. But then realizing that the Sun plays a central causal role in the orbits of planets, he discovered the first two laws of planetary motion hidden among the measurements that Tycho had made.

Marilyn Schlitz, IONS President Emeritus, is following in Willis Harman's footsteps, saying, in a One-Minute-Shift video on the Web:

> When Copernicus proved that the Earth revolves around the Sun, he literally changed the world as we knew it. Darwin and Einstein did the same in their day. What if we are now going through the next scientific revolution, one every bit as profound? For centuries, science and religion have been at odds. Science has focused on the physical, denying the reality of what most religions believe. However, today's science is showing that some spiritual insights are actually scientific truths; that psychic abilities may be real; that we are all fundamentally interconnected; and that we all have innate abilities to heal and transform ourselves. Science and technology without wisdom can endanger life as we know it. But when we marry the best of science with the best of our wisdom traditions, humanity will have the capacity to create a more just, compassionate, and sustainable future.[389]

Then on 20th July 2013, Stephen Dinan, founder of the Shift Network and formerly IONS Director of Membership and Marketing, convened a teleseminar titled 'The Next Scientific (R)evolution The Emergence of the Akashic Paradigm with Consciousness at the Core', with Ervin Laszlo, Ken Wilber, Barbara Marx Hubbard, Riane Eisler, and Duane Elgin.

As the systems philosopher Ervin Laszlo said, we need to give up the idea that the world is a giant mechanism. Rather the Universe is "most like an Internet, a kind of Cosmic Internet. What you know about this information system, which we call the Internet, all things are somehow connected. You can reach any and all items on the Internet from any and all points. And they all hang together somehow." Indeed. When we look at the Universe and hence society as an information system, as this book outlines, we can complete today's revolution in science.

Ervin Laszlo calls this great revolution in science the 'Akashic paradigm', using the word *Akasha* to refer to the Universal Quantum Field. He took the word from Vivekananda's *Raja Yoga*: "Everything that has form, everything that is the result of combination, is evolved out of this *Akasha*. … Just as *Akasha* is the infinite, omnipresent material of this universe, so is this *Prana* the infinite, omnipresent manifesting power of this universe."[390]

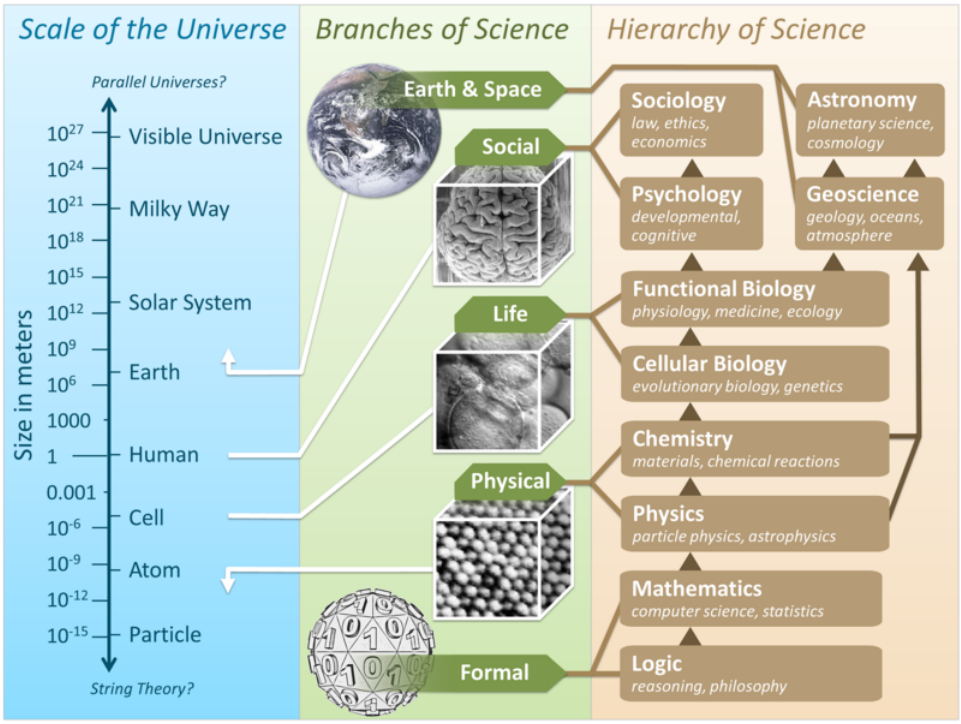
The word *Akasha* derives from Sanskrit *Ākāsha*, corresponding to Greek *aither* 'pure, fresh air', in Latin *æther*, "the pure essence where the gods lived and which they breathed", which is *quintessence*, the fifth element, the others being fire, air, earth, and water, of course. But what is this quintessential æther and how can we know of its existence, never mind that it is Ultimate Reality? Well, in 1887, Albert Michelson and Edward Morley showed in a famous experiment that an 'æther wind' could not be physically detected as the Earth passed through the supposed æther.[391] Although Albert Einstein did not specifically mention

the Michelson–Morley experiment in his 1905 paper on the special theory of relativity,[392] he did say that the notion of 'aether-drift'[393] is 'superfluous' in his theory.[394]

Now while the Æther cannot be detected with the physical senses, humans through the ages have been well aware of it existence. For instance, the principal religions of the world have called the Absolute *Jehovah*, *God*, *Allah*, *Brahman*, *Dao*, *Shūnyatā* 'emptiness', and *Tathatā* 'Suchness', which is ever present, as the etymology of *Presence* indicates. So, as none of us is ever separate from the Divine for an instant, none us is ever separate from any other being.

This principle of interconnectedness is the central theme of *What is Reality?: The New Map of Cosmos, Consciousness, and Existence*, published in 2016, written by Ervin Laszlo and others. In the Preface, Ervin writes, "The new reality perceives embracing interconnection among all things in the universe. … Connection and coevolution are the core of the new map of cosmos and consciousness."[395]

What this epoch-making revolution in science means is that we need to re-evaluate what scientists regard as the most fundamental science on which all the sciences can be built. In general, most people think that physics is the primary science, although some biologists are today attempting to usurp physics' crown. However, Martin Rees, the Astronomer Royal of the United Kingdom, and a former President of the Royal Society and Master of Trinity College, Cambridge, says that the sciences are sometimes likened to the different levels in a building, with logic in the basement, mathematics on the first floor, the various materialistic sciences forming the next floors, with the upper floors representing the human sciences, depicted in this diagram, posted on Wikipedia in 2013.



But just look at how far apart logic—as the science of mind and reason—and psychology—as the science of mind and consciousness—have moved. This was not George Boole's intention when he wrote *Laws of Thought*, as we see on page 2. Perhaps it is not surprising that a split opened up between mathematical logic and psychology because while the former led to the invention of the stored-program computer in the late 1940s, its linearity can tell us little about how humans actually think and reason.

We can see one reason why this split happened from the later development of mathematical logic. For

instance, in 1865, in the first of a series of lectures 'On the Logic of Science', Charles Sanders Peirce, a great admirer of Boole's pioneering work, said that all the definitions of logic that had evolved during the previous two millennia could be divided into two classes: "those which do not and those which do give to logic a psychological or human character".[396]

In examining the relative merits of these two views of logic, Peirce said, "we ought to adopt a thoroughly unpsychological view of logic", for three reasons. First, "I say that the logical form is already realized in the symbol itself; the psychologists say that it is only realized when the symbol is understood." So "logic needs no distinction between the symbol and the thought; for every thought is a symbol and the laws of logic are true of all symbols." Secondly, Peirce said, "The second advantage of the unpsychological view is that it affords a most convenient means for exploding false notions of the subject," going on to say, "The third advantage of the unpsychological view is that it points to a direct and secure manner of investigating the subject."[397]

Peirce reiterated his determination to keep logic separate from psychology in 1898, when he gave a series of lectures on *Reasoning and the Logic of Things* in Cambridge, Massachusetts. In the exordium for the third lecture titled 'The Logic of Relatives', he said, "My proposition is that logic, in the strict sense of the term, has nothing to do with how you think."[398]

In 1903, Bertrand Russell, co-author of *Principia Mathematica*, and Gottlob Frege, generally regarded as the founder of first-order predicate logic with a very strange notation,[399] agreed with Peirce. For when being troubled by the paradoxes that had been found in the foundations of mathematics, Russell wrote a famous letter to Frege, notably agreeing with his rejection of any psychological element in logic.[400]

On the other side of the coin, I've seen no reference to Boole's Laws of Thought in any book on psychology that I have ever read. For instance, in *Psychology*, William James does not mention his friend Charles Sanders Peirce. And there is no mention of Boole in the eighteen volumes of the *Collected Works* of Carl Gustav Jung, for instance.[401]

Even so, William James summarized the challenges and opportunities in 1892 in the final paragraph of *Psychology: Briefer Course*, an abridgement of the two-volume *Principles of Psychology*, written two years earlier. He saw psychology, which George Trumbull Ladd defined "as the *description and explanation of states of consciousness as such*",[402] as:

> A string of raw facts, a little gossip and wrangle about opinions, a little classification and generalization on the mere descriptive level; a strong prejudice that we have states of mind, and that our brain conditions them: but not a single law in the sense in which physics shows us laws, not a single proposition from which any consequence can causally be deduced. We don't even know the terms between which the elementary laws would obtain if we had them. This is no science, it is only the hope of science. … But at present psychology is in the condition of physics before Galileo and the laws of motion, of chemistry before Lavoisier and the notion that mass is preserved in all reactions. The Galileo and the Lavoisier of psychology will be famous men indeed when they come, as come they some day surely will. … Meanwhile the best way in which we can facilitate their advent is to understand how great is the darkness in which we grope, and never to forget that the natural-science assumptions with which we started are provisional and revisable things.[403]

At the beginning of the twentieth century, Eugen Bleuler, who coined the words *schizophrenia* and *ambivalence*, held a similar view as the director of the prestigious Burghölzli Mental Hospital in Zürich. As Sonu Shamdasani tells us in his introduction to Jung's monumental *The Red Book*: "It was held that by turning psychology into a science through introducing scientific methods, all prior forms of human understanding would be revolutionized. The new psychology was heralded as promising nothing less than the completion of the scientific revolution."[404]

However, progress was slow. In 1935, Jung was bold enough to call psychology the 'science of consciousness' in the first of a series of five lectures he gave on the theory and practice of analytical

psychology to the Institute of Medical Psychology (Tavistock Clinic). He added, "[Psychology] is the science of what we call the unconscious psyche," a science he said had not yet left the cradle.[405]

Then in 1976, after three decades exploring the symptoms of our grievously sick society, Erich Fromm wrote in his greatest masterpiece *To Have or To Be?* that if we are to avoid economic and psychological catastrophe, "We need a Humanistic Science of Man as the basis for the Applied Science and Art of Social Reconstruction."[406] However, he was uncertain of success, saying,

> Whether such a change from the supremacy of natural science to a new social science will take place, nobody can tell. If it does, we might still have a chance for survival, but whether it will depends on one factor: how many brilliant, learned, disciplined, and caring men and women are attracted by the new challenge to the human mind. [407]

For myself, I am still uncertain of success within even a comparatively small community of souls, even though Integral Relational Logic re-unifies logic and psychology, in the manner that George Boole intended, becoming the primary science on which both the sciences and humanities can be built, as illustrated in the diagram on page 28.

Rebuilding the entire world of learning with what is essentially an Eastern worldview enables us to answer many of the scientific and social questions facing humanity today. For instance, Martin Rees has said, "Einstein's theory and the quantum theory cannot be meshed together: both are superb within limits, but at the deepest level they are contradictory. Until there has been a synthesis, we certainly will not be able to tackle the overwhelming question of what happened right at the very beginning." As he goes on to say, "Interpretations of quantum theory today may be on a 'primitive level', analogous to the Babylonian knowledge of eclipses: useful predictions, but no deep understanding."[408]

Similarly, even though the human genome has been sequenced, it seems that the more the secrets of the DNA molecule are revealed, the more questions that remain unanswered. As Steve Jones, Professor of Genetics at University College London has said, "We don't understand genetics at all."[409]

And from a social perspective, we could theoretically use Integral Relational Logic and the Unified Relationships Theory to cocreate a society living in harmony with the fundamental laws of the Universe, governed by what Henryk Skolimowski aptly calls lumenarchy 'rule by Divine Light'.

To set this art and science of thought and consciousness into the history of ideas, to counteract the constant bifurcation of evolutionary processes, during the past four hundred years, scientists have made a short series of discoveries, each of which has served to unify pairs of opposites, in conformity with the Hidden Harmony. Johannes Kepler set the ball rolling in 1609 by unifying the split between causal physics and mathematical astronomy,[410] which Aristotle had opened up in *Physics*.[411] Isaac Newton (1642/3–1727) produced the second term in this series in 1687 by unifying Kepler's celestial physics with Galileo Galilei's terrestrial dynamics in *Principia*. [412]

Albert Einstein introduced the next two terms in this series with the special and general theories of relativity. First, in 1905, he developed the special theory of relativity by reconciling the incompatibilities between the principle of relativity, which states that physical phenomena run their course relative to different coordinate systems according to the same general laws, and the observed constancy of the speed of light.[413] Einstein did this by replacing Newton's absolute framework of space with a relativistic space-time continuum, in which the notion of simultaneity is relativistic. In the general theory of relativity, published in 1915, Einstein went on to show the equivalence of gravitational and inertial mass during acceleration,[414] and in so doing abandoned the Euclidean–Cartesian rectilinear model of space, replacing it with the view that space-time is curved.

In 1980, David Bohm continued this unifying process by showing how we can unify the incompatibilities between quantum physics and relativity theory in *Wholeness and the Implicate Order*. For the theories of relativity and quantum mechanics, which Bohm said should really be called 'quantum *non*-mechanics', display opposite characteristics, the former having the properties of continuity, causality, and locality, with the latter being characterized by noncontinuity, noncausality, and nonlocality.

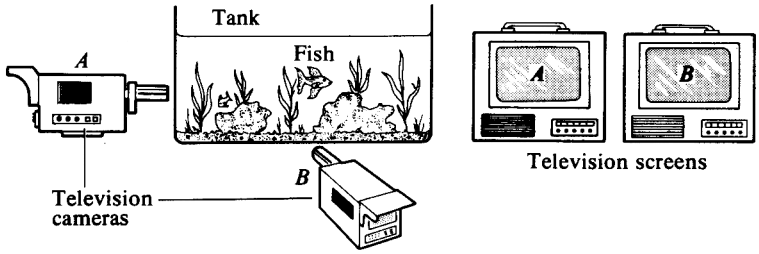
Unifying Mysticism and Mathematics introduces the sixth and final term in this series, describing how *all* opposites can be unified in Wholeness. In so doing, it changes the meaning of *cosmos*, as the physical universe, to *Cosmos*, as Consciousness, embracing and lying within the entire Universe, as we can discover by mapping inner space, as the Cosmic Psyche, rather than outer space.

David Bohm used the hologram as a metaphor for the undivided wholeness of both relativity and quantum theories, illustrating a quite new type of order—the implicate order—underlying the explicate, where we see phenomena as being separate from each other, including each of us as human beings. For *hologram* derives from Greek *ὅlos* 'whole' and *gramma* 'letter of the alphabet', from *graphein* 'to write'. So a hologram or holograph is something that 'writes the whole', like collumination.

Viewing Consciousness as an ocean enables us to fully complete David Bohm's unification of quantum and relativity theories. Inspired by the process thinking of Heraclitus and A. N. Whitehead, Bohm could see that underlying the appearance of separation, which science has traditionally focused its attention on, is an undivided flowing stream, which he called the holomovement, whose substance is never the same. As he said, "On this stream, one may see an ever-changing pattern of vortices, ripples, waves, splashes, etc., which evidently have no independent existence as such. Rather, they are abstracted from the flowing movement, arising and vanishing in the total process of the flow."[415]

In IRL, this river of life in the horizontal dimension of time becomes the vast Ocean of Consciousness, which we can visualize as a ball of water with infinite radius, which psychologists like Sigmund Freud and Stanislav Grof have talked about in their writings. To give this ocean some structure, we need to visualize it with a finite diameter, with the surface then representing the physical universe, the waves and ripples accessible to our physical senses. But beneath the surface lies the Cosmic Psyche, the 99% of the Universe where all knowledge, wisdom, and joy dwell, as described in Kabbalah. And at the very centre of the Ocean of Consciousness is the Origin of the Universe, the Divine Source of Life, giving rise to all forms in the manifest universe.

As well as using a river as a metaphor for what underlies the material universe, Bohm used the metaphor of a fish swimming in a tank with two television cameras filming it to show how relativity and quantum theories could be unified. The television screens would then display opposite characteristics of this single, underlying reality, illustrated here:



But what is the fish to make of all this? Well, the Sufi poet Kabir wrote in the fifteenth century, "I laugh when I hear that the fish in the water is thirsty,"[416] using water as a metaphor for Consciousness, as the

Numinosphere. But that is not how astrophysicists understand our Environment, or the Arena in which we live, leaving much to be understood. For instance, Martin Rees has said, "In the twenty-first [century], the challenge will be to understand the arena itself, to probe the deepest nature of space and time," going on to say, "A fish may be barely aware of the medium in which it swims."[417] For as Kabir the weaver says in the fish poem, "You do not see that the Real is in your home, and you wander from forest to forest listlessly."

In the words of the popular Sufi poet Rumi, "Love is the sea of not-being and there intellect drowns."[418] For me, this sea is the Ocean of Consciousness, a multidimensional generalization of Bohm's one-dimensional holomovement, which we first experience in the womb. As Stanislav Grof says in *The Holotropic Mind*, our early experiences in the womb "have strong mystical overtones; they feel sacred or holy. … In this state of cosmic unity, we feel that we have direct, immediate, and unlimited access to knowledge and wisdom of universal significance." This rapturous period in our lives, a reminder of "Gardens of Paradise in the mythologies of a variety of the world's cultures", can be referred to as 'oceanic ecstasy'.[419]

However, almost no physicists have, as yet, fully understood Bohm's synthesis of quantum and relativity theories, for cognitive, experiential, and psychosocial reasons. Concerning the former, in a review of *Wholeness and the Implicate Order* in a Sunday newspaper in the summer of 1980, Dana Zohar said that Bohm was seeking to develop an algebra of algebras that would give his cosmology a sound mathematical foundation. I've mislaid the original newspaper cutting and was unable to find the reference during a brief visit to the British Library in September 2018. We also did not talk about this possibility during our talks during the 1980s, even though I introduced Integral Relational Logic to him in embryonic form.

Nevertheless, it has occurred to me in recent years that as IRL takes the abstractions of modern algebra to the utmost level of generality, it can be considered the algebra of algebras that he was seeking. To explain this, I plan to write three further chapters in this book during this coming winter and spring, showing how mathematics can be viewed as a emergent, generating science of patterns and relationships, leading to what Alfred North Whitehead called *Universal Algebra* in 1898.[420]

3. From Zero to Transfinity

I now come to the heart of this book: to show how I build a coherent conceptual model of mathematics using Integral Relational Logic as a taxonomy of taxonomies, building on a few basic concepts. As I have said earlier, rather than viewing mathematics as an axiomatic, deductive proof system, I experience mathematics as a generative science of patterns and relationships emerging directly from the Divine Origin of the Universe. In this regard, mathematics is no different from any other specialist branch of knowledge, all of which emerge from the multidimensional Ocean of Consciousness, whose centre and transfinite 'surface' are the Immanent and Transcendent Datum of the Universe, as the Continuum.

So, as I begin this chapter, I sink into emptiness and fullness, as Wholeness, rather as I do when explaining how Integral Relational Logic emerges in consciousness. But now I have a system of coordinates for all knowledge. So I can apply this framework to develop a coherent view of mathematics, as a whole, free of the axioms or assumptions with which mathematical reasoning has begun its linear proof reasoning for millennia. The validity of the logic is 'proof by coherence', confirmed by the exquisite experience of inner well-being in the Presence of the Divine, viewing mathematics as a multidimensional structure.

As mathematics has traditionally been viewed as the science of number and space, I begin with the first of these, painting geometric and other figures when these are appropriate to illustrate the discourse. The next step in building a map of the whole of mathematics is to look at the way that operations on numbers grow inductively, just like numbers themselves. Then before I look at the way that the infinity of infinite ordinals and cardinals leads to Transfinity, I look at their opposite, the infinitesimal in differential and integral calculus.

Growth of number types

Number is a subclass of the superclass **Being** in Integral Relational Logic, just like that of **Human being**, of which you and I are instances. The word *number* derives from Latin *numerus* 'a measure, quantity', cognate with *numerare* 'to count'. The concept of number has evolved over the centuries, as additional properties have been found useful in various domains, reconciling anomalies as they have arisen. A character in blackboard bold typeface style (ℤ) denotes each class, although simple bold characters (**H**) are used in some instances.

As well as numbers themselves, mathematicians have defined a number of binary operations on them, which are all based on addition (+), with multiplication (×) being repeated addition, and subtraction (-) and division (÷) being inverses of these operations, which we look at in the next section.

As I am not a professional mathematician, I use IRL to provide an overall picture of the **Number** class. First, I showed on page 51 that I cannot define the concept of number until I define the primal concept of **set**, the basic interpretative concept of semantics, giving meaning to the meaningless data patterns arising from the **Datum** of the Universe.

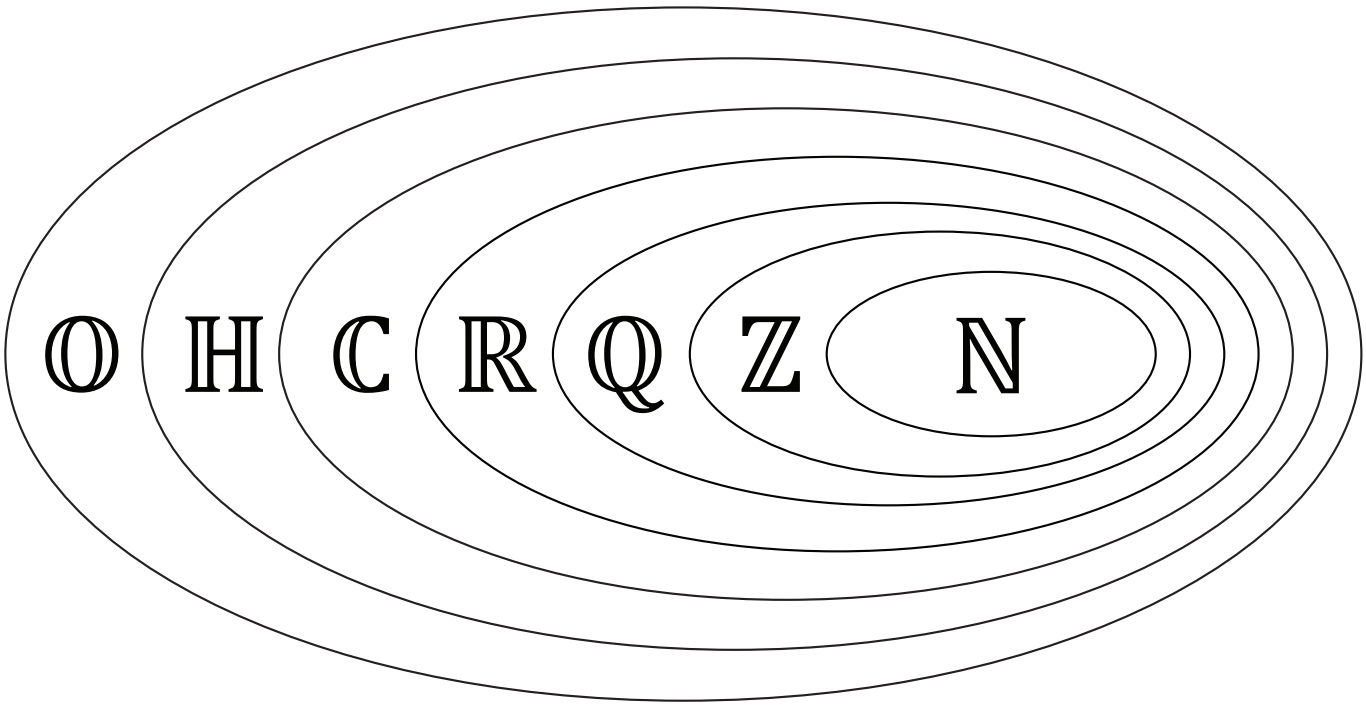
The basis of doing so is to carefully examine the **similarities** and **differences** in the **attributes** of the

entities in the set, which is how David Bohm brought order to quantum physics. It is just simple common-sense. I then showed how these entities could be organized in **relations** within the context of a **class**, the attribute values being drawn from **domains of values**, as a way of measuring the domain. Numbers provide **quantitative** domains, while all other forms of measurement provide **qualitative** domains. In information systems modelling in business, both of these are treated in exactly the same way.

In this chapter, I now need to regard numbers as entities in various classes, distinguished by their various attribute values. I could in principle organize all these in relations, as a form of set. But I don't feel the need to do this here, as it would be rather cumbersome. Nevertheless, it is important to remember that they exist behind the scenes, so to speak. Also, mathematically relations, like sets, are normally unordered. However, for practical purposes relations are generally ordered, based on a name or internal ID. Yet, by their very nature, numbers are often the basis for ordering, something that I need to remember. Here then is a list of the basic classes of number, as I understand them.

- $\mathbb{N}$ Class **Natural number**, the set of numbers, used for counting and sequencing, as integers 1, 2, 3 …, adding zero to overcome some difficulties in arithmetic and in denoting numbers as numerals.
- $\mathbb{Z}$ Class **Integer**, set of all integers, positive, negative, and zero, from German *Zahlen* 'numbers'.
- $\mathbb{Q}$ Class **Rational number**, set of numbers in the form a/b, where a and b are integers ($a, b \in \mathbb{Z}$) and $b \neq 0$, from *quotient*.
- $\mathbb{R}$ Class **Real number**, set of numbers, both rational and irrational, as the measure of a line.
- $\mathbb{C}$ Class **Complex number**, set of numbers, in the form $a + bi$, where a and b are reals ($a, b \in \mathbb{R}$) and $i = \sqrt{-1}$, as a measure of the complex plane.
- $\mathbb{H}$ Class **Quaternion**, set of quaternions, as complex numbers extended into three/four dimensions, in the form $a + bi + cj + dk$, where a, b, c, and d are reals ($a, b, c, d \in \mathbb{R}$) and $i = j = k = \sqrt{-1}$, from William Rowan *Hamilton*, who discovered them in 1843.
- $\mathbb{O}$ Class **Octonian**, set of *octonions*, a set of octets (or 8-tuples) of real numbers $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, obeying complex multiplicative rules.

This diagram shows the relationship of the major classes of number, an extension of one in Wikipedia.



However, the different types of number, and the algebras for operating on them, do not stop with

octonions. Arthur Cayley (1821–1895) and Leonard Eugene Dickson (1874–1954), in their Cayley–Dickson construction, showed that there is an infinite sequence of algebras from the real numbers onwards, in which each higher-dimensional algebra is defined as the product of two pairs of numbers from the next lower dimension. For instance, a complex number is the product of two pairs of reals in two dimensions, a quaternion is the product of two pairs of reals in four dimensions, and an octonion is the product of two pairs of quaternions in eight dimensions. The next class of number is **Sedenion** in sixteen dimensions, although this and classes in higher dimensions don't seem to be particularly useful, basically because the patterns found in the lower dimensions break down. Nevertheless, I find it interesting that the concept of number can be extended indefinitely, like so many other constructs in mathematics.

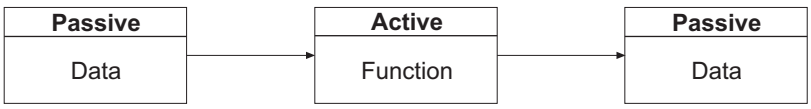
Natural numbers

To view mathematics as a generative science in the vertical dimension of time, I begin with the basic primal concepts in Integral Relational Logic. As I explain on page 51, I interpret the meaningless **beings** underlying the Totality of Existence by carefully observing the **similarities** and **differences** in the **attributes** of their **data patterns**, putting them into **sets** as appropriate—lots and lots of them.

These sets have many different sizes, the minimum set consisting of a single entity, denoted by |. In Book VII of *The Elements*—the first of three books exploring the foundations of number theory rather than geometry[421]—Euclid defined | like this: "An (*sic*) **unit** is that by virtue of which each of the things that exist is called one." The second definition is then, "A **number** is a multitude composed of units."[422] So I can use the basic method of interpretation to put all sets of a particular size into another set, assigning a corresponding natural number to *quantify* or *count* its elements, denoted by strings of |'s, such as ||||||| or ||||||||||,[423] like the way the ancients used tally sticks to count in the Upper Palaeolithic period, between 50,000 and 10,000 years ago.[424] So the notions of number and numeral, corresponding to concept and signifier in the meaning triangle, defined on page 4, are not bootstrap concepts in IRL.

In turn, these entities have the common attribute that they are all used for counting. So they can all be put into a single set that I denote by ℕ, as mathematicians do. Within consciousness, I can thus see with my inner eye numerous sets with countable attributes, some of which are larger or smaller than others. This gives rise to the notion of *sequence* or *order* in a linear sense. So there are two basic types of natural number, which mathematicians call *cardinal* and *ordinal*. These are the same for finite sets, but are different for infinite sets, which we look at in the last section of this chapter.

To ensure that the set of natural numbers ℕ includes all possible ones, I borrow an idea from the axioms or postulates of the natural numbers, which Giuseppe Peano (1858–1932) introduced in 1889, drawing on earlier work by Charles Sanders Peirce[425] and Richard Dedekind (1831–1916).[426] These axioms are formed in a mechanistic manner, following the basic data-processing structure in business modelling, repeated here from page 4:



So let us begin with a meaningless data element denoted by the Greek letter zeta (ζ) and a function S, producing an output S(ζ). This, in turn, can be input to the function S, producing S(S(ζ)) and so on without limit. But what are ζ and S? Well, as I form sets by looking at the similarities and differences in the data patterns of my experience, all sets formed in this way must have at least one element. So it would appear that ζ should represent |, the count of the set with just one element.

However, this situation can lead to difficulties when it comes to basic arithmetical operations on numbers. In particular, addition and its inverse subtraction are dyadic functions in the basic data-processing structure, taking two inputs and producing an output. For instance, the addition of ||||| and ||| produces ||||||||. Reversing this function, subtracting ||||| from ||||||||| results in |||. However, if I repeat this operation, subtracting ||| from |||, I am left with nothing, nought, or zero, denoted by the empty set in mathematics: {}.

This is zero as a number, as a concept. Moreover, there is also the need for *zero* as a numeral, as a sign, which took many hundreds of years to be recognized and resolved. Early on, Sumerians, Greeks, and Romans, for instance, recognized that endless strings of |'s to denote numbers was unmanageable. So they assigned symbols for groups of signs in a hierarchical fashion. For instance, the Romans used letters of the alphabet for this purpose, like this:

Symbol	I	V	X	L	C	D	M
Value	One	Five	Ten	Fifty	Hundred	Five hundred	Thousand
Denotes	|	IIIII	VV or IIIIIIIII	XXXXX	LL or XXXXXXXXXX	CCCCC	DD or CCCCCCCCCC

So two hundred and fifty six is represented in Roman numerals as CCLVI and three thousand, six hundred and two by MMMDCII. There is no need to include the symbols X or L in the second example for there are no groups of ten in this number. The highest denominations were placed first, with a few exceptions, such as IV for four and XCIX for ninety-nine, not IC.

One obvious limitation of this numeral system is that large numbers over ten million cannot conveniently be denoted. Another is that the omission of an order of magnitude in decimal notation is implicit not explicit. The breakthrough leading to the Indo-Arabic numeral system that we use today arose from the conjunction of three fundamental ideas that had been gradually evolving since humans began counting and accounting over five thousand years ago:

1. A glyph could take on different meanings dependent on its *position* in the string.
2. Glyphs should be *abstract*, with no ideographic significance.
3. A glyph is needed for *zero*, for the omission of a symbol in the string.[427]

But what should the glyph for the absence of a symbol be called? Well, the Indians called this symbol *shūnya* 'empty', cognate with *shūnyatā* 'emptiness, void',[428] denoting the mystical experience of not-being in union with the Divine, free of the sense of a separate self. The Arabs then made a loan translation of *shūnya* as *çifr* 'empty', related to *çafara* 'to be empty', from which we get *cipher*, originally meaning 'zero' before it came to mean in the sixteenth century any numeric figure used as a code in encrypted communications.

When Fibonacci (Leonard of Pisa) introduced the Indo-Arabic numerals in *Liber Abaci* (*Book of the Abacus*) in 1202 he wrote, "It first described the 'nine Indian figures' [9, 8, 7, 6, 5, 4, 3, 2, 1], together with the sign 0, 'which is called zephirum in Arabic'."[429] *Zephirum* is a near homophone of Latin *zepherus* 'warm west wind', root of *zephyr* 'gentle breeze' or 'something that is airy or insubstantial'. This is how the word *zero* entered the English language, as a linguistic doublet of *cipher*, with a shared mystical root meaning 'empty, void', denoted in mathematics as {}, the empty set. In some sports, like tennis, *zero* is denoted by *love*, meaning a game played for the love of it, not for money. In the words of the popular Sufi poet Rumi, "Love is the sea of not-being and there intellect drowns."[430]

So the function S(*n*), where *n* is a natural number, denotes a natural number that is next in order, or one more than the previous natural number. In this way, I can generate an infinite sequence of natural numbers, as all members of the set ℕ, like this:

1. $\zeta = 0$.
2. $S(\zeta) = | = 1$.
3. $S(S(\zeta)) = || = 2$.
4. $S(S(S(\zeta))) = ||| = 3$.

⋮

It is therefore most convenient to interpret ζ (zeta) as zero, the smallest natural number. However, zero sometimes needs to be omitted in arithmetic operations on the natural numbers. For instance, if q is a factor or divisor of p, p/q is a natural number provided q is not zero. In programming languages, indexing arrays can start with either 0 or 1. In general languages, like C and Swift, 0 is the base index to denote the offset from the first address, while in mathematical languages, like Fortran and Mathematica, 1 is the base index in conformity with the way that matrices are addressed.[431] APL leaves this choice to the programmer, with the default being 1.[432]

In this simple way, I have thus shown that the number system in mathematics emerges from Nothing, as the Divine Source and Origin of the Universe, in essentially the same way as all concepts are formed in Integral Relational Logic. In terms of counting, 1, 2, 3, and so on are the most fundamental integer sequence, number A000027 in the On-Line Encyclopedia of Integer Sequences (OEIS).[433]

There was thus no need for Bertrand Russell and Alfred North Whitehead to spend many exhausting years writing *Principia Mathematica*, taking 360 pages to prove the proposition '1 + 1 = 2.'[434] As I am using the natural, intuitive way of forming sets, I also don't need the complex way mathematicians use to determine the properties of the countable set of natural numbers ℕ. These properties are known as Peano's axioms, as already mentioned. Nevertheless, for the record, here is a brief overview of these postulates:[435]

1. $\zeta \in \mathbb{N}$.
2. If $n \in \mathbb{N}$, then $S(n) \in \mathbb{N}$.
3. $S(n) \neq \zeta$.
4. If $n, m \in \mathbb{N}$ and $S(n) = S(m)$, then $n = m$.

However, these axioms are not quite rigorous enough. They do not prevent loops, such as $m = S(n)$ and $n = S(m)$. To exclude such a situation, there is a fifth axiom of induction:

5. Suppose there is a subset T of ℕ such that:
   - $\zeta \in T$ and
   - $n \in T \Rightarrow S(n) \in T$

   then T is ℕ itself.[436]

Now like my own intuitive method for the creation of the set of natural numbers, the symbols are so abstract, they exist prior to interpretation by a cognitive being. To interpret this set as class **Natural number** ℕ, I consider ζ to be zero and the function $S(n)$ to be successor or next. So mathematicians and I are in agreement with similar provisos.

Excluding 0 and 1 from the natural numbers, I then define two subclasses **Prime**[437] and **Composite**, as Euclid did in Book VII of *The Elements*: "A **prime number** is that which is measured by an (*sic*) unit alone" and "A **composite number** is that which is measured by some number."[438] In modern terms, a prime number has the defining attribute that its only divisors are itself and 1. These categories lead immediately to two fundamental laws of arithmetic: the infinitude of the primes and the fundamental theorem of arithmetic.

Euclid proved these in Propositions 20 and 14 in Book IX of *The Elements*,[439] respectively, although it was not until 1801 that Carl Friedrich Gauss (1777–1855) expressly stated the latter in *Disquistiones Arithmeticæ*.[440]

First, there are an infinite number of prime numbers, which can be proven by assuming that Aristotle's Law of Contradiction is true in this context and that there are finite number of primes n, whose maximum is p_n. Let us then create a subclass **Primorial**, defined as the product of the first n primes, denoted by:

$$p_n\# = \prod_{i=1}^{n} p_i$$

where p_i is the ith prime. Euclid proved that there are on infinite number of primes by supposing that $p_n\# + 1$ is the largest prime. Such numbers are called Euclid numbers, which I can denote by P, the supposedly largest prime. Yet, the number P is not divisible by any of the primes p_i, for if it is divided by any of these, this would leave a remainder of 1. So, either P is a prime larger than p_n, or it is a composite number whose factors are larger than p_n. For instance, if the supposed largest prime is 5, then:

$$P = (2 \cdot 3 \cdot 5) + 1 = 31$$

I once assumed that all Euclid numbers are prime. But, as I now see, it is easy to prove that this is not the case with P = 13:

$$P = (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) + 1 = 30{,}031 = 59 \cdot 509$$

In either case, I have found a prime larger than the supposed largest. I can extend this proof indefinitely by assuming that the new prime I have found is the largest that exists, thus proving by contradiction that there are an infinite number of primes.[441]

Viewing mathematics as a generative science, there are a number of sequences that arise from these constructions. First, there are the Euclid numbers themselves, growing very rapidly, even faster than factorials, for the composites are omitted. These are OEIS A006862, one more than the primorials themselves, sequence OEIS A002110:

3, 7, 31, 211, 2 311, 30 031, 510 511, 9 699 691, 223 092 871, 6 469 693 231, 200 560 490 131, …

Some of these are primes, known as primorial primes, listed in OEIS A005234 with p_n in the primorial being:

2, 3, 5, 7, 11, 31, 379, 1 019, 1 021, 2 657, 3 229, 4 547, 4 787, 11 549, 13 649, 18 523, 23 801, …

The indices n for these in the OEIS A014545 are:

1, 2, 3, 4, 5, 11, 75, 171, 172, 384, 457, 616, 643, 1 391, 1 613, 2 122, 2 647, …

We can see from this last list that such primorial primes are rather sparse. As of September 2001, the largest known one has index 33,237 (392113# + 1) with 169,966 digits. It is not known if there are an infinite number of such primorial primes.[442]

However, in writing this section, I have discovered another form of primorial primes, which may be infinite in number. These have the form $p_n\# - 1$, which Ernst Kummer (1810–1893) used to provide a more elegant proof that there are an infinite number of primes. If p_i in $p_n\#$ are all the primes, then $p_n\# - 1$ must be composite, divisible by at least one p_i. So both $p_n\#$ and $p_n\# - 1$ would be divisible by p_i, which is impossible.[443] So, either $p_n\# - 1$ is a larger prime than p_n or it is a composite with larger prime factors than p_n.

This class of numbers, which I suppose we could call Kummer numbers, not all of which are prime, one less than the primorials and two less than the Euclid numbers, begins with this sequence, not defined in the OEIS:

1, 5, 29, 209, 30 029, 510 509, 9 699 689, 223 092 869, 6 469 693 229, 200 560 490 129, …

Some, but not all of these are prime, as we can see in these two examples. For $p_n = 5$:

$$(2 \cdot 3 \cdot 5) - 1 = 29$$

However, the next in the list is composite, both the factors being larger than 7:

$$(2 \cdot 3 \cdot 5 \cdot 7) - 1 = 209 = 11 \cdot 19$$

The primorial primes in this sequence begin with these values for p_n, OEIS A006794:

3, 5, 11, 13, 41, 89, 317, 337, 991, 1 873, 2 053, 2 377, 4 093, 4 297, 4 583, 6 569, 13 033, ...

The indices *n* for these in OEIS A057704 are:

2, 3, 5, 6, 13, 24, 66, 68, 167, 287, 310, 352, 564, 590, 620, 849, 1 552, 1 849, 67 132, 85 586, …

As of March 2018, the largest known primorial prime in this form is 1098133# − 1 (*n* = 85586) with 476,311 digits, found by the PrimeGrid project.[444]

Secondly, the fundamental theorem of arithmetic, also called the unique factorization theorem or the unique-prime-factorization theorem, states, *Every natural number greater than 1 is either a prime or the product of primes; this representation is unique except for the order of the factors*. So prime numbers (2, 3, 5, 7, 11, 13, …) are the basic building blocks of arithmetic and thus hold a position of central importance, as mathematicians have long sought patterns among the primes, some of which are still elusive. For example, 1200 is uniquely defined in this way:

$$1200 = 2^4 \times 3^1 \times 5^2 = 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 5.$$

1 is not a prime, for it can be added any number of times to the factorization of a natural number, invalidating the fundamental principle of arithmetic. So 1 is simply a **Unit** and 0 is **Zero**. The class **Natural number** thus contains four subclasses **Zero**, **Unit**, **Prime**, and **Composite**.

From the time of the ancient Greeks, mathematicians have sought patterns within the natural numbers, an exploration that is continuing today. For instance, Srinivasa Ramanujan (1887–1920) defined the subclass **Highly composite number** as one that has more divisors than any number smaller than it.

Number	1	2	4	6	12	24	36	48	60	120	180	240	360	720	840	1260	1680	2520	5040	7560
# factors	1	2	3	4	6	8	9	10	12	16	18	20	24	30	32	36	40	48	60	64

Numbers that have many factors are most useful for dividing them equally into groups. For instance, we have 24 hours in the day (divisors 1, 2, 3, 6, 8, 12, 18, and 24) and 60 seconds in a minute and minutes in an hour (divisors 1, 2, 3, 5, 6, 8, 10, 12, 15, 18, 30, and 60), 60 being the base of Sumerian mathematics. Plato was particularly interested in 5040, mentioning "in his *Laws* that 5040 is a convenient number to use for dividing many things (including both the citizens and the land of a city-state or polis) into lesser parts, making it an ideal number for the number of citizens (heads of families) making up a polis."[445] For the unique factorization of 5040 is:

$$5040 = 2^4 \times 3^2 \times 5 \times 7$$

Coincidently, 5040 is an instance of **Factorial**, another subclass of class **Natural number**, known to the ancients. A factorial, denoted by *n*!, is the product of all natural numbers greater than zero and less than or equal to *n*. Here is a table of the first dozen, showing that this sequence grows very fast:

n	1	2	3	4	5	6	7	8	9	10	11	12
n!	1	2	6	24	120	720	5,040	40,320	362,880	3,628,800	39,916,800	479,001,600

Then we have class **Perfect number**,[446] defined as one that is equal to the sum of its proper divisors, that is excluding the number itself. The sum of *all* the divisors of number is the basic sigma function, so the sum of just the proper divisors ($s(n)$), known as the aliquot sum, is:

$$s(n) = \sigma(n) - n$$

The first four perfect numbers, known to the ancient Greeks,[447] are given in this table:

n	Perfect number	Product of divisors	Sum of divisors
1	6	2×3	$1 + 2 + 3$
2	28	$2^2 \times 7$	$1 + 2 + 4 + 7 + 14$
3	496	$2^4 \times 31$	$1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$
4	8,128	$2^6 \times 127$	$1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1016 + 2032 + 4064$

Now there is a pattern here, which Euclid proved in the last of his propositions on number theory.[448] The first divisors are powers of 2, followed by the product of one less than the next power of 2 and successive powers of 2. In mathematical terms, if $2^k - 1$ is prime, then all the divisors of $N_k = 2^{k-1}(2^k - 1)$ are:

$$1, 2, 2^2, \ldots 2^{k-1}, (2^k - 1), 2(2^k - 1), 2^2(2^k - 1), \ldots 2^{k-1}(2^k - 1)$$

These are two geometric sequences, whose sum is equal to $2N_k$, using the formula on page 179. So, leaving out N_k itself, the sum of the proper divisors of N_k is N_k, as a perfect number. Then in the eighteenth century, Euler proved that an even number N_k is perfect if and only it satisfies the formulation:

$$N_k = 2^{k-1}(2^k - 1)$$

where $2^k - 1$ is prime, denoted as M_k. That is, while no odd perfect numbers have been found or proven to not exist, all even perfect numbers have this form.[449] Many early writers erroneously believed that M_k is prime for all prime k, until Hudalrichus Regius, probably working with Roman numerals, showed in 1536 that M_{11} is composite:

$$2^{11} - 1 = 2{,}047 = 23 \times 89$$

Regius went on to prove that M_{13} is prime, finding the first perfect number since the Ancient Greeks:[450]

$$P_5 = 2^{12}(2^{13} - 1) = 4{,}096 \times 8{,}191 = 33{,}550{,}336$$

Then in 1603, Pietro Cataldi (1548–1626) showed that the next two primes, k = 17 and 19, generate perfect numbers P_6 and P_7. He did so by first publishing a list of primes up to 750 and then by laboriously dividing $2^k - 1$ by all primes not exceeding the square root of this number.[451] Cataldi also showed that if k is composite, equal to ab, then M_k is composite, for both $2^a - 1$ and $2^b - 1$ are factors.[452]

2	**3**	**5**	**7**	11	**13**	**17**	**19**
23	29	**31**	37	41	43	47	53
59	61	**67**	71	73	79	83	89
97	101	103	107	109	113	**127**	131
137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223
227	229	233	239	241	251	**257**	263
269	271	277	281	283	293	307	311
313	317	331	337	347	349	353	359
367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457
461	463	467	479	487	491	499	503
509	521	523	541	547	557	563	569

So, by the beginning of the seventeenth century, the first seven perfect numbers had been found, shaded in cyan in the first row of this table of the first 104 prime exponents p. However, as you can see from the other cells shaded in cyan, perfect numbers become rarer and rarer. As it turns out, while eight of the first eleven primes p give rise to a perfect number, only 43 of the first two million prime exponents (up to 32,452,843) do so.[453]

Unbeknownst of this fact, in 1644, Marin Mersenne (1588–1648), a French friar, asserted that M_p—known today as a Mersenne number—is a Mersenne prime for p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, and 257, marked in red and bold in this table, and composite for all other primes $p \leq 257$. He did so even though it was obvious to other mathematicians that Mersenne could not have tested for the primality of all these numbers—and neither could they. Although Mersenne supposedly found only two perfect numbers with exponents greater than 19, making five mistakes, numbers of the form $2^p - 1$ are named after him today. Perhaps this is because he was intuitively aware of the sparseness of the perfect numbers, especially the huge

gap after exponent 127, by far the largest proportional gap to the next perfect number among those that have been discovered to date.

In the event, it took nearly three hundred years for mathematicians to discover the mistakes in Mersenne's list. Euler and Édouard Lucas (1842–1891) confirmed that M_{31} and M_{127} are prime in 1732 and 1876, respectively, generating what we now know as P_8 and P_{12}. However, Lucas and Frank Nelson Cole (1861–1926) showed in 1876 and 1903 that M_{67} is composite[454] and Maurice Kraitchik (1882–1957) showed that M_{257} is composite in 1922,[455] the last error to be found because proving Mersenne numbers to be composite is more difficult than proving them to be prime. So exponents p = 67 and 257 do not generate perfect numbers, shaded in yellow in the table. Mersenne omitted P_9, P_{10}, and P_{11} with p = 61, 89, and 107, not boldened in the table, which Ivan Pervusin (1827–1900) discovered in 1883 and Ralph Ernest Powers (1875–1952) found in 1911 and 1914.[456] Even Euler made mistakes in his calculations, at first asserting that p = 41 and 47 generate perfect numbers before detecting his error in 1753.[457]

Then in 1952, Raphael M. Robinson (1911–1995) tested the primality of M_p for all primes p < 2304 and found five more Mersenne primes generating perfect numbers (P_{13} to P_{17}) for p = 521, 607, 1279, 2203, and 2281. He did so by writing a program in machine language to code a primality test for Mersenne numbers that Lucas had devised and which Derrick Henry Lehmer (1905–1991) had simplified.[458] Amazingly, the Standards Western Automatic Computer (SWAC) he used had only seven instructions and 256 37-bit words of memory.[459]

Since then, as of January 2018, a total of just fifty perfect numbers has been found, sixteen by Great Internet Mersenne Prime Search (GIMPS),[460] a collaborative project of volunteers who use freely available software to search for Mersenne prime numbers. The seven largest are generated by the seven largest primes yet discovered.[461] Here is a complete list of Mersenne prime exponents known today, sequence OEIS A000043:[462]

2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609, 57885161, 74207281, and 77232917.

As you can see, the largest Mersenne prime, confirmed on 26th December 2017, has p = 77,232,917 with 23,249,425 digits, available as a 23.7 MB file.[463] If this were printed with a font size of 7 with ten thousand digits per page, then this would require 2,325 pages, not unmanageable. However, such perfect numbers and largest primes are minuscule compared with the unimaginable way that even finite numbers can grow, never mind infinite ones, which we look at later in this chapter on page 130.

Another subclass of natural number, closely related to perfect numbers, is **Amicable pair**, also known to the Pythagoreans, who credited them with many mystical properties.[464] The smallest pair are 220 and 284, for $s(220) = 284$ and $s(284) = 220$ (OEIS A259180). That is, the sum of the proper divisors of 220 {1, 2, 4, 5, 10, 20, 22, 44, 55, 110} equals 284, whose proper divisors {1, 2, 4, 71, 142} sum to 220. Mathematicians, from Thâbit ibn Kurrah (826–901) to Euler developed formulae for generating amicable pairs. But they missed the second smallest (1184, 1210), which 16-year old B. Nicolò I. Paganini found in 1866, this pair having eluded his more illustrious predecessors.

In general, amicable pairs (m, n) satisfy these two equations, a modification of the self-referencing function for the aliquot sum of perfect numbers on page 94:

$$\sigma(m) - m = n \quad \text{and} \quad \sigma(n) - n = m$$

These give:

$$\sigma(m) = \sigma(n) = m + n$$

As mathematics is an ever-generalizing science, mathematicians have naturally extended these amicable pairs into amicable tuples (n_1, n_2, n_3, … n_k), following this formula:

$$\sigma(n_1) = \sigma(n_2) = \sigma(n_3) = \cdots = \sigma(n_k) = n_1 + n_2 + n_3 + \cdots + n_k$$

For instance, the smallest amicable triple (OEIS A125490) is (1980, 2016, 2556), for

$$\sigma(1980) = \sigma(2016) = \sigma(2556) = 1980 + 2016 + 2556 = 6552$$

And the smallest amicable quadruple (OEIS A036471) is (3,270,960, 3,361,680, 3,461,040, 3,834,000), for the sigma function for each of these equals their sum, which is 13,927,680.

Actually, the sigma function that I have been using is a special case of the general divisor function, given by this formula, where $d|n$ denotes a divisor of n:

$$\sigma_k(n) = \sum_{d|n} d^k$$

In other words, $\sigma_0(n)$ is the count of the number of factors of n, including n, itself. That is $\sigma_0(24)$ is 8, with divisors {1, 2, 3, 4, 6, 8, 12, 24}, $\sigma_1(24)$ is 60, $\sigma_2(24)$ is 850, and so on. So, what I have been calling σ is actually σ_1. Wikipedia tells us that the divisor function appears in a number of remarkable identities, including relationships on the Riemann zeta function, of much interest to mathematicians for the last century and a half, which we look at in the next chapter.

Finally, on the natural numbers, as they are used for counting, just how many of them are there? Fairly obviously, there are an infinite number of them generally denoted by the sign ∞. Now if we removed all the odd numbers, it might appear that the count of the remaining even numbers is smaller than the count of all of them. However, we can place the even numbers in a one-to-one correspondence with each natural number in turn, like this, for n = 1 upwards:

$$2n \to n$$

So the count of even numbers is exactly the same as the count of all the natural numbers. We thus obtain the remarkable result that ∞ + ∞ = ∞. Or 2 × ∞ = ∞. But this result applies to any prime number p. For instance, all the numbers that are just multiples of a prime number, no matter how large, map to all the integers. So $p \times \infty = \infty$. We see later how this relationship leads to the Transfinite, and hence to the Divine.

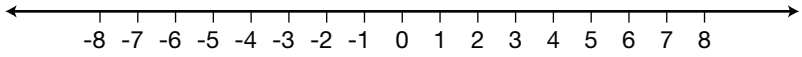
Integers

One major difficulty with the natural numbers is that they are unbalanced; they do not form a whole. While they are closed with respect to addition and multiplication, they are not closed with respect to the inverse operations of subtraction and division, in contradiction of the Principle of Unity, the fundamental law of the Universe. To address the first of these, we need to add negative numbers, forming the class **Integer**, from Latin *integer* 'complete, whole, intact, untouched', from *in-* 'not' and *tangere* 'to touch', cognate with *entire*, *integrity*, and *tangible*.

We can thus define **Positive** and **Negative** as subclasses of **Integer** (ℤ), excluding **Zero**, denoted as $\mathbb{Z}^+$ and $\mathbb{Z}^-$, respectively. To create $\mathbb{Z}^-$, we apply the unary operator negative (-)—distinct from the binary operator minus (-)—to each member of $\mathbb{Z}^+$. So the infinite set of integers is zero plus all the positive and negative natural numbers. This set is also an instance of an infinite group in abstract algebra, which we look at in Chapter 5.

Another pair of dual subclasses is **Even** and **Odd**, where an even number is an integer that is a multiple of two or is divisible by two. But is zero even? Well, zero satisfies these two criteria, as well as all other attributes of evenness. For instance, when two even numbers are added, the result is an even number. Also, an even number is one that lies between odd numbers. In the case of zero, it lies between -1 and +1. So zero is even. Indeed, if it weren't even, it would break the fundamental rules of arithmetic.[465]

However, there are some peculiar asymmetries with the number line, as Alberto A. Martinez points out in *Negative Math: How Mathematical Rules Can Be Positively Bent*.[466]



As he says, "This figure seems to suggest a perfect *symmetry* among negatives and positives as merely numbers to the left or right of zero." Yet, while squaring a positive number results in a positive number, squaring a negative number results in a positive one, not a negative number. Also, taking the cube root of a -8, for instance, leads to -2, while taking the square root of -4 is $2i$, not even on the number line. Furthermore, we usually think of numbers on the left to be smaller than numbers on the right. But what are we to make of this equation, where in the first term the denominator is apparently smaller than the numerator, whereas in the second it is the other way round?

$$\frac{-4}{2} = \frac{4}{-2} = -2$$

Such formal asymmetries "seem to privilege positive numbers over negatives", as Martinez points out. So how can we intuitively make sense of all this in a thoroughly consistent manner? This is the subject of a few YouTube videos I discovered when researching this topic on the Internet. For instance, Burkard Polster, a maths professor at Monash University in Melbourne, Australia, with the moniker Mathologer, asks why does a negative times a negative lead to a positive? He answers this question by regarding (-3)(-4) as unknown quantity, x say. He begins with this equation just for (-3):

$$(-3) + 3 = 0$$

Then multiplying both sides by (-4), we have:

$$(-3)(-4) + (-12) = 0$$

From this we can see that the unknown x = (-3)(-4) is +12, making the sign of the result explicit. So this result is entirely consistent with the basic laws of arithmetic, as he shows with another example involving a more complex multiplication operation.[467] Another way of intuitively understanding this is to regard the two terms in $a \times b$ in different ways, as Heather Brooks describes. The sign of the first factor determines whether someone walking along the number line is walking to the right or the left, while the second factor denotes whether the person is walking forwards or backwards. So someone walking backwards to the left, where both a and b are negative, is actually moving to the right.[468]

This result is entirely consistent with the black-and-white world of Aristotle's Law of Contradiction. Something that is not-not A is A. As some politicians say, "If you are not with us, you're against us." While this law is necessary in linear mathematics, it is hopelessly inadequate in human relationships, where the both-and Hidden Harmony, which Aristotle rejected, governs the Universe, and hence our lives.

The integers can also be divided into congruence or residue classes with respect to an integer $n > 1$, studied by Euler about 1750. Then in 1801 in *Disquistiones Arithmeticæ*, Gauss produced a comprehensive study of what is now called modular arithmetic or clock arithmetic. In its most elementary form, modular arithmetic is "arithmetic done with a count that resets itself to zero every time a certain whole number n

greater than one, known as the modulus (mod), has been reached. Examples are a digital clock in the 24-hour system, which resets itself to 0 at midnight ($n = 24$), and a circular protractor marked in 360 degrees ($n = 360$)."[469]

In general, a congruence class is the set $\{\ldots, a - 2n, a - n, a, a + n, a + 2n, \ldots\}$, consisting of integers congruent to a modulo n, known as $\mathbb{Z}_n$. For instance, if $a = 5$ and $n = 7$, then $\{\ldots, -9, -2, 5, 12, 19, \ldots\}$ is a congruence class. So there are n exclusive congruence classes for each modulus n, most conveniently denoted by a when $a < n$, the remainder or residue when any member of the set is divided by n. For instance, for $n = 7$, the residues form the set $\{0, 1, 2, 3, 4, 5, 6\}$.

For a positive integer n, two numbers a and b are said to be congruent modulo n, if their difference $a - b$ is an integer multiple of n (that is, if there is an integer k such that $a - b = kn$). This congruence relationship is denoted:

$$a \equiv b \pmod{n}$$

For instance:

$$-8 \equiv 16 \equiv 4 \pmod{12}$$
$$-4 \equiv 10 \equiv 3 \pmod{7}$$

Modular arithmetic has many applications, such as cryptography, and plays a central role in number theory, especially in the theory of groups, which we look at a little more in Chapter 5.

Extending the notion that the multiples of prime numbers map to all the natural numbers, we can similarly map all the integers to the natural numbers, with zero mapping to itself, like this:

$n \to 2n - 1$ if n is positive

$n \to -2n$ if n is negative

In other words, $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{Z}^-$ all have the same countable cardinality, which has an even more astonishing consequence with the rational and real numbers, which we look at in the next two subsections.

Rational numbers

We now address the second weakness of the natural numbers, and hence integers: the lack of a division operation or function—as inverse of multiplication—within the class. In general, when an integer p is divided by a positive integer q to form p/q, the result is a rational number. Henry Billingsley (c. 1532–1606), an English merchant and Lord Mayor of London, introduced this term in 1570 in the first English translation of Euclid's *Elements*, giving Definition 3 in Book V as "Such magnitudes or quantities, which may be expressed by numbre, are called rationall."[470]

Rational derives from Latin *ratiōnālis* 'reasonable, rational', from *ratiō*, 'reckoning, account, computation, calculation', from *rērī* 'to reckon, calculate; to think, consider, judge', from PIE base **ar-* 'to fit together', cognate with *reason*, *order*, *artist*, *harmony*, and *arithmetic*. Rationals, as instances of the conceptual class **Rational number**, are most commonly represented in signs as common or vulgar fractions, from Latin *vulgaris* 'ordinary, everyday', from *vulgus* 'the common people', and Latin *frangere* 'to break in pieces, shatter'.

The roots of these words illustrate a central dilemma of Western thought over the millennia, experientially and cognitively detached from Nondual Reality. To reason logically, we break everything into pieces so that they can be fitted together again in order and harmony, not knowing that the entire exercise is an illusory activity. But that is the nature of the relativistic and dualistic world we live in for practical

purposes. This situation is even more critical when we come to consider the irrational numbers in the next subsection, leading to continuity, that which cannot be divided into parts.

For myself, I give meaning to rationals, as fractions, by using them to denote the way we could equitably share a finite number of entities among individuals, including the finite money supply. For *common* derives from Latin *commūnis* 'shared, common, general, universal, public', originally in sense 'sharing burdens', from *cum* 'together with' and *mūnus* 'office, duty; gift, present', from *mūnare* 'to give, present', from PIE base **mei-* 'to change, go, move; with derivatives referring to the exchange of goods and services within a society as regulated by custom or law', also root of *municipal* 'service performed for the community'. This PIE base is also the root of Sanskrit *maitreya* 'friendly, benevolent', from *maitrī* 'friendship, benevolence, good will', from *mitra* 'friend, companion, associate', and Pāli *mettā* 'loving-kindness', akin to Buddhist compassion (*karunā*) and love or charity (*agapē*) in Christianity.

Indeed, the very word *number* derives from a PIE base **nem-* 'to assign, allot', the root of Greek *nemein* 'to manage, arrange', the root of *economy* 'the management of the household', *astronomy* 'an arrangement of the stars', and *taxonomy* 'an arrangement of an arrangement', from Greek *taxis* 'arrangement, order'.

The result of dividing one integer by another is a quotient, from Latin *quotiens* 'how many times', from *quot* 'how many'. Rationals are thus signified by ℚ, following Peano, who denoted them in this way in 1895 from Italian *quoziente*.[471] The two numbers in the quotient are numerator, from Late Latin *numerātor* 'counter, numberer', denoting the number of equal parts to be shared, and denominator, from Latin *dēnōminātor* 'thing that names or designates'.

Now while rationals, as ratios or proportions, have had many applications over the years, it would take us away from the central theme of this book to study them in any detail here. Regarding sharing, I would just like to mention a concept I discovered on Wolfram Alpha when I entered a fraction into its text box on the Web. We learn from the Rhind papyrus from around 1650 BCE that Egyptians defined a number of fractions as a sum of positive (usually) distinct unit fractions, those that are the reciprocals, which Fibonacci studied in *Liber Abaci* in 1202.[472]

The theory of Egyptian fractions is not all that clear. Nevertheless, Wikipedia gives a couple of simple examples to illustrate the general principle. For instance, to share five pizzas among eight people, we could give each a half of four of them plus an eighth of the remaining, for

$$\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$$

Similarly, 13 pizzas could be shared among 12 diners by giving each diner one pizza and splitting the remaining pizza into 12 parts, rather messy. However,

$$\frac{13}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

So, more conveniently, we could split 6 pizzas into halves, 4 into thirds and the remaining 3 into quarters, and then give each diner one half, one third and one quarter.

One of the earliest uses of rational numbers is that of Pythagoras, who is credited with discovering that concordant intervals in music are related by small integer ratios. "Strings with lengths in the ratio 2:1 produced the interval of an octave known to the ancient Greeks as *diapason*. Those in the proportion 3:2 produced the interval of the fifth, known to the Greeks as *diapente*. Strings of equal tension with length in the proportion 4:3 produced the interval of a fourth known to the Greeks as *diatessaron*. … All of these intervals are present between strings with relative lengths 2, 3, and 4. Thus the most harmonious of intervals

are contained in the number progression 1:2:3:4."[473] This gave rise to what is called Pythagorean tuning, in which all intervals in the 12-tone scale are of the form $3^m/2^n$, where *m* and *n* are both either positive or negative.[474]

However, the insatiably curious Kepler, seeking a unifying harmony in music, geometry, and the heavens, considered the ratios 6:5, 5:4, 8:5, and 5:3 to be consonant,[475] *consonance* being defined as "Acoustically, the sympathetic vibration of sound waves of different frequencies related as the ratios of small whole numbers; psychologically, the harmonious sounding together of two or more notes".[476] Strings tuned in these ratios are called 'just intonation'.[477]

But harmonies are a little more complex than these simple ratios. If polyphonic music is to be composed in all twelve major and minor keys and sound harmonious in each of them, instruments really need to be capable of equal temperament, a tuning system based on the division of the octave into twelve equal semitones. This means that the ratios between successive semitones should be the same, leading to a geometric progression, not an arithmetical one. Specifically, as an octave is double the frequency of the tonic, then each semitone ratio is $\sqrt[12]{2}$ in equal temperament, a ratio of 1.059463094 to 1, very far from the ratio of two small integers.

To resolve this difficulty, music theorists today further divide a semitone into a hundred parts, each cent having a ratio of 1.00057779 to 1, five cents being the smallest interval perceptible to the most discerning human ear. However, to simplify calculations and understanding, a logarithmic scale is used, converting nature's geometric series into a more familiar arithmetical one. The number of cents (*n*) between two notes of frequency *x* and *y* is thus given by this formula:[478]

$$n = 1200\log_2\frac{x}{y}$$

Things get even more complicated when we consider the fact that instruments do not generally sound a pure sine wave, but a complex one consisting of harmonics, integer multiples of the fundamental frequency. These harmonics alter the timbre of the sound, greatly enhancing the pleasure of listening to music, even though some of these ratios might appear more dissonant than consonant. For as this table shows, harmonics can differ quite considerably from equal temperament, presenting this knowledge in the tabular formulation of IRL. The difference columns denote the difference of the ratios to equal temperament in cents. Curiously, Pythagorean tuning is often closer to equal temperament that just intonation. What did Pythagoras know that we don't know today?

Interval	Note	Eq. Temp.	Pythagorean tuning		Just intonation		Harmonic series		
		Cents	Ratio	Diff.	Ratio	Diff.	No.	Ratio	Diff.
Tonic	C	0	1/1	0	1/1	0	0	1/1	0
Minor second	C♯, D♭	100	256/243	-10	16/15	12	17	17/16	5
Major second	D	200	9/8	4	9/8 10/9	4 -18	9, 18	9/8	4
Minor third	D♯, E♭	300	32/27	-6	6/5	16	19	19/16	-2
Major third	E	400	81/64	8	5/4	-14	5, 10, 20	5/4	-14
Perfect fourth	F	500	4/3	-2	4/3	-2	21	21/16	-29
Tritone	F♯, G♭	600	1024/729	-12	7/5	-17	11, 22 23	11/8 23/16	-49 28
Perfect fifth	G	700	3/2	2	3/2	2	3, 6, 12, 24	3/2	2
Minor sixth	G♯, A♭	800	128/81	-8	8/5	14	25 13, 26	25/16 13/8	-27 41
Major sixth	A	900	27/16	6	5/3	-16	27	27/16	6
Minor seventh	A♯, B♭	1000	16/9	-4	7/4	-31	7, 14, 21, 28 29	7/4 29/16	-31 30

Interval	Note	Eq. Temp.	Pythagorean tuning		Just intonation		Harmonic series		
		Cents	Ratio	Diff.	Ratio	Diff.	No.	Ratio	Diff.
Major seventh	B	1100	243/128	10	15/8	-12	15, 30 31	15/8 31/16	-12 45
Octave	C	1200	2/1	0	2/1	0	2, 4, 8, 16, 32	2/1	0

Wolfram Alpha also gives some other properties of rationals that are relevant to the central theme of this book, showing how mathematical structures, like structures in general, evolve from simplicity to complexity. First, if the denominator q is in the form 2^a5^b, then the decimal expansion is finite. For instance, 43/80 = 0.5375. Otherwise, the decimal expansion of the fraction has a repeating or recurring decimal, with a related period, which can also apply to 1/10, as 0.09999999…, where 9 repeats indefinitely. For instance, if q = 3, then there are two periods with a single digit, consisting of values 3 and 6. If q = 11, then there are five periods of two digits, whose values are 09, 18, 27, 36, 45, which are reversed when p > 5. If q = 7, then there is just one period of six digits, which cycle for each value of p: 142857. And if q = 13, there are two periods of six digits, marked in red and blue in this table:

1/13	2/13	3/13	4/13	5/13	6/13	7/13	8/13	9/13	10/13	11/13	12/13
0.076923	0.153846	0.230769	0.307692	0.384615	0.461538	0.538461	0.615384	0.692307	0.769230	0.846153	0.923076

So what is the underlying pattern here? Well, first of all, when we multiply a repeating decimal by increasing powers of 10, then all we do is move the decimal point. So in terms of sevenths, we have:

1/7	10/7	100/7	1,000/7	10,000/7	100,000/7	1,000,000/7
$\frac{1}{7}$	$1\frac{3}{7}$	$14\frac{2}{7}$	$142\frac{6}{7}$	$1428\frac{4}{7}$	$14285\frac{5}{7}$	$142857\frac{1}{7}$
0.142857142857	1.42857142857	14.2857142857	142.857142857	1428.57142857	14285.7142857	142857.142857

So by the time we multiply by 10^6, the repeating decimal starts over again, written like this: $.\overline{142857}$. In general, the length of the cycle for prime p is the smallest value l for which:

$$10^l \equiv 1 \pmod{p}$$

The number of cycles n is thus $(p - 1)/l$ or

$$n \times l = p - 1$$

For instance, we have:

$10^1 \equiv 1 \pmod 3$, with 2 cycles 3 and 6.

$10^2 \equiv 1 \pmod{11}$, with 5 cycles 09, 18, 27, 36, and 45.

$10^6 \equiv 1 \pmod 7$, with 1 cycle 142857.

$10^6 \equiv 1 \pmod{13}$, with 2 cycles 076923 and 153846.

We can see why these patterns arise from the prime factors of $10^n - 1$, the first thirty given in the following list. The ones marked in red are sequence A007138 in the OEIS database, defined as: "Smallest primitive factor of 10^n - 1. Also smallest prime p such that $1/p$ has repeating decimal expansion of period n." In *The Book of Numbers*, John H. Conway and Richard Guy call the primes that have only one cycle with length p - 1, long primes.[479] Those primes that are not the smallest primitive factors and are not long primes are marked in green, also the first time they appear in the list.[480]

$10^1-1 = 3^2$

$10^2-1 = 3^2\times11$

$10^3-1 = 3^3\times37$

$10^4-1 = 3^2\times11\times101$

$10^5-1 = 3^2\times41\times271$

$10^6-1 = 3^3\times7\times11\times13\times37$

$10^{7}-1 = 3^{2}\times239\times4{,}649$
$10^{8}-1 = 3^{2}\times11\times73\times101\times137$
$10^{9}-1 = 3^{4}\times37\times333{,}667$
$10^{10}-1 = 3^{2}\times11\times41\times271\times9{,}091$
$10^{11}-1 = 3^{2}\times21{,}649\times513{,}239$
$10^{12}-1 = 3^{3}\times7\times11\times13\times37\times101\times9{,}901$
$10^{13}-1 = 3^{2}\times53\times79\times265{,}371{,}653$
$10^{14}-1 = 3^{2}\times11\times239\times4{,}649\times909{,}091$
$10^{15}-1 = 3^{3}\times31\times37\times41\times271\times2{,}906{,}161$
$10^{16}-1 = 3^{2}\times11\times17\times73\times101\times137\times5{,}882{,}353$
$10^{17}-1 = 3^{2}\times2{,}071{,}723\times53{,}63{,}222{,}357$
$10^{18}-1 = 3^{4}\times7\times11\times13\times19\times37\times52{,}579\times333{,}667$
$10^{19}-1 = 3^{2}\times1{,}111{,}111{,}111{,}111{,}111{,}111$
$10^{20}-1 = 3^{2}\times11\times41\times101\times271\times3{,}541\times9{,}091\times27{,}961$
$10^{21}-1 = 3^{3}\times37\times43\times239\times1{,}933\times4{,}649\times10{,}838{,}689$
$10^{22}-1 = 3^{2}\times11^{2}\times23\times4{,}093\times8{,}779\times21{,}649\times513{,}239$
$10^{23}-1 = 3^{2}\cdot11{,}111{,}111{,}111{,}111{,}111{,}111{,}111$
$10^{24}-1 = 3^{3}\times7\times11\times13\times37\times73\times101\times137\times9{,}901\times99{,}990{,}001$
$10^{25}-1 = 3^{2}\times41\times271\times21{,}401\times25{,}601\times182{,}521{,}213{,}001$
$10^{26}-1 = 3^{2}\times11\times53\times79\times859\times265{,}371{,}653\times1{,}058{,}313{,}049$
$10^{27}-1 = 3^{5}\times37\times757\times333{,}667\times440{,}334{,}654{,}777{,}631$
$10^{28}-1 = 3^{2}\times11\times29\times101\times239\times281\times4{,}649\times909{,}091\times121{,}499{,}449$
$10^{29}-1 = 3^{2}\times3{,}191\times16{,}763\times43{,}037\times62{,}003\times77{,}843{,}839{,}397$
$10^{30}-1 = 3^{3}\times7\times11\times13\times31\times37\times41\times211\times241\times271\times2{,}161\times9{,}091\times2{,}906{,}161$

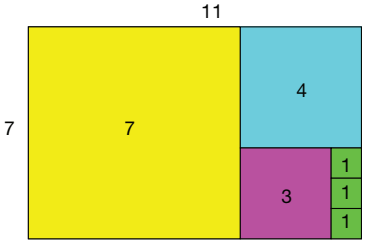
Multiplication of any factor by $2^{a}5^{b}$ gives the same period as the factor alone. A denominator obtained by a multiplication of two factors has a period equal to the first power of 10 in which both factors appear. For instance, the period of 49 is 42 for 7^{2} first appears in the factorization of 10^{42} - 1 and the period of 77 is 6, for both 7 and 11 are factors of 10^{6}-1.

In general, these patterns don't just apply to the decimal representation of rational numbers. They apply to any base b by the 'little' theorem of Pierre de Fermat (1607–1665), for when b does not divide p:

$$b^{p-1} \equiv 1 \pmod{p}$$

Taking a holistic perspective of the rationals, another way to view them is as hierarchical sequences of finite simple continued fractions, whose initial terms are the reciprocals of the positive integers. This is a subject that I did not study during my formal mathematics education in the 1950s and 60s, even though it has ramifications in many branches of mathematics.

A continued fraction is formed by repeated application of Euclid's algorithm for finding the greatest common divisor (GCD) of two positive integers. For Euclid showed that if two numbers contain a common factor, their differences also contain this GCD.[481] So by repeatedly subtracting the smaller from the larger as many times as is necessary to leave a remainder smaller than the smaller, the GCD can be found.[482] If this is 1, then the numbers are coprime, with no common factor.[483]



For instance, for $p/q = 7/11$, this diagram[484] shows how repeatedly subtracting the smaller from the larger leaves a GCD of 1. The continued fraction is formed from this process like this:

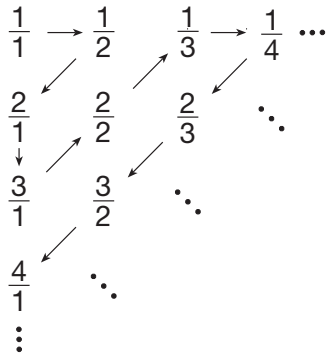
$$\frac{7}{11} = \frac{1}{\frac{11}{7}} = \frac{1}{1+\frac{4}{7}} = \frac{1}{1+\frac{1}{\frac{7}{4}}} = \frac{1}{1+\frac{1}{1+\frac{3}{4}}} = \frac{1}{1+\frac{1}{1+\frac{1}{\frac{4}{3}}}} = \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{3}}}}$$

In general, a finite simple continued fraction takes this form, where $a_0 = 0$ if $p < q$. As continued fractions take much space on the page, they are most conveniently expressed as $[a_0; a_1, a_2, \ldots a_n]$. This table shows the first few rows and columns of the continued fractions, illustrating the way that 5/11, 7/11, and 9/11 are related to earlier terms in their sequences.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

col/ row	2	3	4	5	6	7	8	9	10	11
1	[0; 2]	[0; 3]	[0; 4]	[0; 5]	[0; 6]	[0; 7]	[0; 8]	[0; 9]	[0; 10]	[0; 11]
2		[0; 1, 2]	[0; 2]	[0; 2, 2]	[0; 3]	[0; 3, 2]	[0; 4]	[0; 4, 2]	[0; 5]	[0; 5, 2]
3	[1; 2]		[0; 1, 3]	[0; 1, 1, 2]	[0; 2]	[0; 2, 3]	[0; 2, 1, 2]	[0; 3]	[0; 3, 3]	[0; 3, 1, 2]
4		[1; 3]		[0; 1, 4]	[0; 1, 2]	[0; 1, 1, 3]	[0; 2]	[0; 2, 4]	[0; 2, 2]	[0; 2, 1, 3]
5	[2; 2]	[1; 1, 2]	[1; 4]		[0; 1, 5]	[0; 1, 2, 2]	[0; 1, 1, 1, 2]	[0; 1, 1, 4]	[0; 2]	[0; 2, 5]
6			[1; 2]	[1; 5]		[0; 1, 6]	[0; 1, 3]	[0; 1, 2]	[0; 1, 1, 2]	[0; 1, 1, 5]
7	[3; 2]	[2; 3]	[1; 1, 3]	[1; 2, 2]	[1; 6]		[0; 1, 7]	[0; 1, 3, 2]	[0; 1, 2, 3]	[0; 1, 1, 1, 3]
8		[2; 1, 2]		[1; 1, 1, 2]	[1; 3]	[1; 7]		[0; 1, 8]	[0; 1, 4]	[0; 1, 2, 1, 2]
9	[4; 2]		[2; 4]	[1; 1, 4]	[1; 2]	[1; 3, 2]	[1; 8]		[0; 1, 9]	[0; 1, 4, 2]
10		[3; 3]	[2; 2]		[1; 1, 2]	[1; 2, 3]	[1; 4]	[1; 9]		[0; 1, 10]
11	[5; 2]	[3; 1, 2]	[2; 1, 3]	[2; 5]	[1; 1, 5]	[1; 1, 1, 3]	[1; 2, 1, 2]	[1; 4, 2]	[1; 10]	

Now while the positive integers are used to count distinct entities, the rationals are used to count parts of entities. So how many rationals are there? Can we count them like we counted the integers? Indeed, we can, with an even more surprising result. As Georg Cantor (1845–1918) showed,[485] the rational numbers, of the form p/q, map to the counting numbers, even just those that lie between 0 and 1, or between any two rational numbers, no matter how close they are to each other.



This diagram illustrates the way that all the rationals can be arranged in a sequence that maps to the natural numbers: 1/1→1, 1/2→2, 2/1→3, 3/1→4, 2/2→5, 1/3→6, and so on. Yes, rationals in which the numerators and denominators are not coprime are duplicated, but this doesn't matter; it merely serves to emphasize the point. This mapping shows that the count of rationals is the same as that of the positive integers, even though there are an infinite number of rationals between any two rationals! The infinite set of all rationals is enumerable, sometimes called *de*numerable.

Real numbers

However, this is not the case with what mathematicians call the set of 'real numbers', to no end of consternation. To resolve the confusion that thus arises in the foundations of mathematics, we need to take a giant leap into what for many is an unknown world. From a mystical perspective, real numbers are an attempt to measure continuity, which, as an undivided whole, cannot be divided into parts and is therefore uncountable and unmeasurable. Bohm showed in *Wholeness and the Implicate Order* that we need to view the Totality of Existence as an unbroken flowing movement, as we see on page 85, very similar to Peirce's notion of synechism on page 76. However, as I say there, over the years, I have learnt to extend this notion into the multidimensional Ocean of Consciousness, which I experience Gnostically.

But how can we express this notion of continuity mathematically? Well, to begin with the number line on page 97, this depicts a mathematical object of infinite length and zero width, divided into parts by the integers. Then in the subsection on the rational numbers, we saw how the rationals between 0 and 1, for instance, divide the number line into ever-greater fragments, which nevertheless have the same cardinality as the natural numbers.

Now, in this section, we look at the way that the so-called real numbers divide the number line into minuscule pieces, so small that they cannot be mapped to the countable numbers, as we look at later in this chapter. They eventually merge into a continuous line, at the core of the infinitesimal calculus, which we look at on page 133.

In the meantime, just what is this class **Real number**, shortened to **Real**? Well, *real* derives from late Middle English (as a legal term meaning 'relating to things, especially real property'), from Anglo-Norman French, from late Latin *reālis*, from Latin *rēs* 'thing, object, matter, affair, circumstance', putatively from PIE base **rē* 'to bestow, endow', contracted from **reə-*, suffixed form **reə-i-* 'goods, wealth, property', also root of *republic*.

In accordance with the etymology, the OED gives this primary definition of *real*: "Having an objective existence; actually existing as a thing", citing Shakespeare from 1601 as the first person to use the word in English with this sense: "Is there no exorcist Beguiles the truer office of mine eyes? Is't real that I see?"[486] However, other etymologies indicate that *real* was used in English from the early 1300s to indicate that what has physical existence is true and from 1440 or 1550 to denote something that is genuine or authentic.[487]

So, historically, what people regard as reality relates to our everyday external affairs. What is actual is that which pertains to acts, from Late Latin *āctuālis*, from Latin *āctus*, past participle of *agere* 'to drive, set in motion', from PIE base **ag-* 'to drive, draw, move', also root of *exact*, *agitate*, *agony*, and *axiom*.

We see this mechanistic and materialistic perspective most clearly in three dictionary definitions of *real-world*: "The realm of practical or actual experience, as opposed to the abstract, theoretical, or idealized sphere of the classroom, laboratory, etc.",[488] "The set of situations most humans have to deal with in their lives, rather than what happens in stories, films, etc.",[489] and "The existing state of things, as opposed to one that is imaginary, simulated, or theoretical."[490]

However, the OED gives this citation for the meaning of *real-world* from 1966: "The Vice Chancellor of Lancaster University strongly believes 'that the university must keep contact with the real world outside'. May I take this opportunity to ask..: (a) what is real about the real world? (b) why it is always outside?"[491] Yes, indeed. Aren't our inner experiences real, genuine, and authentic?

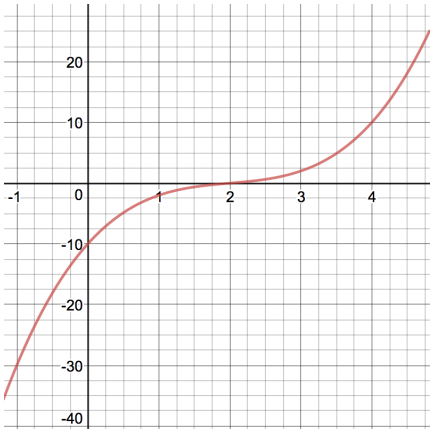
Yet this raises the tricky question of what is genuine knowledge. David Hume showed for both logical and psychological reasons, Francis Bacon's inductive way of reasoning could not lead to certain knowledge, which led Bertrand Russell to write in his inimitable manner:

> It is therefore important to discover whether there is any answer to Hume within the framework of a philosophy that is wholly or mainly empirical. If not, there is no intellectual difference between sanity and insanity. The lunatic who believes that he is a poached egg is to be condemned solely on the grounds that he is a minority, or rather—since we must not assume democracy—on the grounds that the government does not agree with him. This is a desperate point of view, and it must be hoped that there is some way of escaping it.[492]

For myself, I have discovered what it truly means to be a human being, as compared to computers, by looking inwards in a thought experiment, which I outline in the previous chapter. I am thus my own authority, not needing endorsement from any recognized external authority. However, as Integral Relational Logic is not based on any previously existing system of thought, I'm still not sure to what extent I can share my wonderful experiences with others. Nevertheless, mathematics has many similarities with the mystical worldview, which I'm endeavouring to illustrate in this book.

Although the term *real number* is not recorded in English before 1909,[493] one of the first uses of the adjective *real* to number was that of Descartes, who used the term in 1637 in *Geometry*, originally written as an application of his method of properly conducting one's reason and of seeking the truth in the sciences.

He was studying cubic polynomials, observing, "For the rest [note that] the true roots, as well as the negative ones, are not always real, but sometimes only imaginary."[494] So, as so often happens in human learning, we form concepts and words to denote them by comparing them with their opposites, in this case that of imaginary numbers, a notion that began appearing during the 1500s.



As an example, he wrote, "Thus, although we can conceive three roots in the equation

$$x^3 - 6x^2 + 13x - 10 = 0$$

there is nevertheless only one real root, 2, no matter how we may augment, diminish, or multiply the other two, in the way just explained, they will still be imaginary [$2 \pm i$]."[495] As you can see, this cubic only crosses the x-axis once, coincidentally at its point of inflection, and thus has only one 'real' root.

The key point here is that Descartes' cubic equation is an example of a continuous function $f(x)$, which maps the values of x on the continuous x-axis, including all numbers, the rationals and their opposites the irrationals. The notion of continuity is a central concept in mathematics, corresponding to Ultimate Reality, which is a seamless Continuum with no borders or divisions anywhere. We can see this from the word *Tantra*, which figuratively has the sense of weaving opposites together in Wholeness, with other original meanings indicating 'groundwork, principle, system' and 'Context, Continuum'. For *tantra* derives from Sanskrit *tantram* 'loom', unifying 'warp' and 'weft', from *tan* 'to stretch', and *-tra-m* 'instrument'. So *tantra* literally means 'an instrument for stretching', at the heart of Integral Relational Logic, as the ultimate Integral Tantric Yoga.

So, unifying the rationals and irrationals in a continuous real number line—mapped to functions of any type—is a vital step in moving towards the Formless Absolute in direct experience. The notion of continuity in mathematics has a long history, going back to the Pythagoreans, who thought that space and time could be thought of as both a 'continuity' and as points and instants.[496] The ancient Greeks were well aware of the problem of incommensurability, which Eudoxus of Cnidus (c. 390–c. 337 BCE) resolved with his 'axiom of continuity', very similar to the notion of limit in modern mathematics, which is equivalent to the first proposition in Book X of Euclid's *Elements*:

> If from any magnitude there be subtracted a part not less than its half, and if from the remainder one again subtracts not less than its half, and if this process of subtraction is continued, ultimately there will remain a magnitude less than any preassigned magnitude of the same kind.[497]

Then, in 1604, in *Introduction to Vitello's Optics*, Kepler enunciated the 'principle of continuity'—the continuous change of a mathematical entity from one state to another—treating the parabola as the limiting case of either an ellipse or a hyperbola, in which one of the two foci moves off to infinity.[498] At the time, Kepler was also working on his *New Astronomy*, which was eventually published in 1609, showing that the planets orbit the Sun in ellipses, with the Sun at one of the foci, a word that Kepler coined from Latin *focus* 'hearth, fireplace', apparently from the 'burning point of a lens or mirror'.

As the notions of limit and continuity properly belong to the infinitesimal calculus, we look more at these in the section on this subject, beginning on page 133, and as the notion of continuum (𝔠)[499] of the real number line from -∞ to +∞, belongs to studies of the infinite, we look at this in the section beginning on page 166.

In the meantime, let us look at some of the characteristics of the class **Irrational number**, the set of numbers that cannot be expressed as the ratio of two integers. The existence of such measures has been known since the ancient Greeks. For instance, the ratios of the diameter of a square to its side ($\sqrt{2}$) and of the circumference of a circle to its diameter (π) cannot be expressed in terms of p/q, where p and q are natural numbers. So what exactly are such irrationals?

Well, we saw on page 101 that when q is not of the form 2^a5^b, a rational number is expressed as a decimal in a repeating pattern that continues indefinitely. But with irrational numbers, there is no discernible pattern in their infinite decimal representations. For instance,

$\sqrt{2}$ = 1.41421356237309504880168872420969807856967187537694807317 6…

and

π = 3.14159265358979323846264338327950288419716939937510582097 4…

The first point to note here is that these two real numbers ($\mathbb{R}$) are instances of two subclasses of **Irrational number** ($\overline{\mathbb{Q}}$ or $\mathbb{R}\backslash\mathbb{Q}$): **Algebraic number** ($\mathbb{A}$) and **Transcendental number** ($\mathbb{T}$), the latter defined as anything that is not algebraic. The numbers that are not roots of an algebraic equation are called 'transcendental', for as Euler said, "They transcend the power of algebraic methods."[500] These real algebraic numbers are defined as the roots of a polynomial equation, like Descartes' conic:

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

where $a_0 \neq 0$, n is a natural number, and each a_i is a complex number, which we haven't yet looked at. This leads to the Fundamental Theorem of Algebra: "Every polynomial equation having complex coefficients and degree >=1 has at least one complex root. This theorem was first proven by Gauss."[501] Another related theorem is the Factor Theorem, which states that if r is a root of a polynomial $f(x)$, then $(x - r)$ is a factor of $f(x)$ giving $f(r) = 0$.

This leads naturally to the result that a polynomial $f(x)$ of degree n has n roots in the complex plane, if multiplicities are counted. Not that all these roots are radicals. Niels Henrik Abel (1802–1829) and Évariste Galois (1811–1832) showed independently—when the former was nineteen and shortly before the latter was shot dead in a duel—that only polynomials of degree up to and including quartics have radical solutions, which I plan to look at in Chapter 5 if I have the ability. So, to find the roots of higher degree polynomials, it is sometimes necessary to use numerical methods, such as the one that Newton introduced, which I learned at university.

However, what I am more interested in at the moment is the count of the algebraic and transcendental numbers. In 1874, Cantor gave us an answer to the first of these questions using a more restrictive definition of algebraic number where $a_i \in \mathbb{Z}$. In a famous paper that launched Cantor's studies of infinite sets, he proved that algebraic numbers, like $\sqrt{2}$ and φ, the Golden Ratio, are countable. He did this with the concept of the 'height' of an algebraic equation, defined as

$$h = n + a_0 + |a_1| + \cdots + |a_{n-1}| + |a_n|$$

For instance, $3x^2 - 2x + 1 = 0$ has height $h = 2 + 3 + 2 + 1 = 8$. David M. Burton then explains Cantor's proof in this way:

> For any fixed height h, the integers n, a_0, a_1, …, a_{n-1}, a_n can be specified in only a finite number of ways, thereby leading to a finite number of equations; each such equation can have at most as many different roots as its degree. Thus, there are just a finite number of algebraic numbers arising from equations of a given height. By grouping the algebraic equations according to height, starting with those of height 2, then taking those of height 3, and so on, one can write down the set of algebraic numbers in a sequence.[502]

But what about the transcendental numbers? Well, Cantor proved that their number far exceeds the count of the algebraic numbers. They are not countable, not mappable to the countable natural numbers. He did this using what is known as the 'diagonal method'. The method is by indirect proof, which George Pólya (1887–1985) likened "to a politician's trick of establishing a candidate by demolishing the reputation of his opponent", related to the method of *reductio ad absurdum*, which Pólya likened to the irony of a satirist, stressing and overstressing a point until it leads to an absurdity.[503]

Let us assume that the real numbers, consisting of all algebraic and transcendental numbers, are countable. In that case, it is possible to map the reals strictly between 0 and 1 to the natural numbers, expressing nonterminating decimals as infinite decimals, such as 0.333… and 0.4999… for ⅓ and ½, respectively. Such a set, arranged in denumerable or countable order, could look like this:

$$\begin{array}{l} 1\leftarrow 0.a_{11}a_{12}a_{13}\ldots \\ 2\leftarrow 0.a_{21}a_{22}a_{23}\ldots \\ 3\leftarrow 0.a_{31}a_{32}a_{33}\ldots \\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \end{array}$$

where a_{ij} is a digit between 0 and 9 inclusive. In the words of Carl B. Boyer, who wrote a classic book on the history of mathematics, "Cantor then exhibited an infinite decimal different from all of those listed. To do this, simply form the decimal $b = 0.b_1b_2b_3\ldots$, where $b_k = 9$ if $a_{kk} = 1$ and $b_k = 1$ if $a_{kk} \neq 1$." This real number is different from all those in the list, which is supposed to include all reals. Hence the set of real numbers, which Cantor denoted with $\mathfrak{c}$ for 'continuum', is not countable.[504] As this substitution is a binary one, the proof can be expressed in binary arithmetic, as is done in Wikipedia.[505]

Cantor did not stop there. In 1878, he published a paper that showed that the points in a finite square, 'clearly two-dimensional', can be put into one-to-one correspondence with the points of a straight-line segment, 'clearly one-dimensional'.[506] Indeed, there is no need to stop at two dimensions. This mapping can be extended into the three dimensions of a cube and then into any number of dimensions. Furthermore, the points in a finite *n*-dimensional hypercube can be mapped into infinite 'space'. "Dimensionality is not the arbiter of the power of a set."[507] In other words, the cardinality of the points in all of *n*-dimensional hyperspace is just $\mathfrak{c}$.

Cantor was so shocked by this result that in 1877 he wrote to his friend Richard Dedekind, "I see it, but I do not believe it." Not surprisingly, he had considerable difficulties in getting his revolutionary discoveries published, with Leopold Kronecker (1823–1891) being particularly hostile, leading the hypersensitive Cantor to suffer a series of 'nervous breakdowns' from 1884 to the end of his life, dying in a mental institution.[508]

However, with the authority of Dedekind to support him, Cantor's paper was published, allowing him to continue with his studies of infinite sets. He went on to prove some even more surprising results. There are not just two infinite cardinals; there are an infinite number of them, which Cantor called *transfinite*, which has a quite different meaning in Integral Relational Logic, as we see at the end of this chapter.

Although the transcendental numbers far outnumber the algebraic ones, those that have been proven to be transcendental are very rare. Indeed, proofs of transcendence are so difficult, apparently not following any pattern, such proofs are more a mathematical curiosity, a friend, who is a retired mathematical teacher, tells me.

Whether irrationals are algebraic or not caused a lot of botheration in mathematical circles until Joseph Liouville (1809–1882) proved in 1844 that numbers of this form are transcendental:

$$\sum_{n=1}^{\infty} \frac{a_n}{10^{n!}}$$

where a_n is an arbitrary number between 1 and 9, inclusive. For instance, if all the a_n's are 1, this strange number, consisting of 1's separated by a number of zeroes increasing by factorial n ($n!$), is transcendental: 0.11000100000000000000000010….[509] Burkard Polster gives an elegant proof that this Liouville number is not a solution to an algebraic function on his Mathologer YouTube channel, based on a similar proof that John H. Conway and Richard Guy give in *The Book of Numbers*.[510] Indeed, Mathologer goes even further, showing that if the digits of any other real number, including the rational ones, are used as values of a_n—if necessary, adding a natural number to the Liouville number—then an infinite set of clones is produced with what mathematicians call a measure of zero, not perceptible within the totality of transcendental numbers.[511]

But what about other irrational numbers, like e and π? Are they algebraic or transcendental? Well, while they have been proved to be irrational, it was not until 1873 that Charles Hermite proved that e is transcendental. Then nine years later, C. L. F. Lindemann (1852–1939) proved that π is transcendental.[512] However, I am not aware of any proof of the transcendence of the Feigenbaum bifurcation velocity constant δ (4.669201609), which I use in proving that evolution passed through its Accumulation Point in chaos theory terms around 2004,[513] explaining why global institutions, projected from fragmented, schizoid minds, have been degenerating into chaos during the last several years.

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}$$

Now while we saw on page 103 that rational numbers can be expressed in a terminating continued fraction, the irrational numbers can only be expressed as an infinite continued fraction, in this generalized form, where a_i and b_i are real [or complex] numbers, with b_0 being a leading integer. If $a_i = 1$ for all i, then the continued fraction is a simple one. However, very few irrational numbers display any recognizable pattern. Joseph-Louis Lagrange (1736–1813) proved that continued fractions are periodic just for algebraic numbers of degree 2,[514] some of which I look at here, ever seeking to view mathematics as a generative science of patterns and relationships. If there is a repeating pattern, both a_i and b_i need to display a pattern, which should therefore be algorithmic. Before we look at this possibility, let us look first at a particular example:

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694807317\,6\ldots$$

As Steve Chow, a mathematics professor at Pierce College, Woodland Hills, California, explains on his blackpenredpen YouTube channel,[515] we can rewrite this as:

$$\sqrt{2} = 1 + \frac{(\sqrt{2}-1)(\sqrt{2}+1)}{\sqrt{2}+1} = 1 + \frac{1}{1+\sqrt{2}}$$

Here, we have a recursive definition for $\sqrt{2}$. So we can replace $\sqrt{2}$ by this expression:

$$\sqrt{2} = 1 + \frac{1}{1+1+\frac{1}{1+\sqrt{2}}} = 1 + \frac{1}{2+\frac{1}{1+\sqrt{2}}}$$

This can be continued indefinitely, giving:

$$\sqrt{2} = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\ddots}}}$$

As a more compact form, this can be written as:

$$\sqrt{2} = 1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\ldots$$

And in even more concise terms for simple continued fractions, this can be written as $\sqrt{2} = [1; \overline{2}]$, where the bar denotes a repeating continued fraction, like a recurring decimal. This example is an instance of a general algorithm that Ron Knott gives on his comprehensive page on continued fractions:[516]

$$\sqrt{n} = b + \frac{n-b^2}{b+\sqrt{n}} = b + \frac{n-b^2}{2b + \frac{n-b^2}{b+\sqrt{n}}} = \frac{n-b^2}{2b + \frac{n-b^2}{2b+\frac{n-b^2}{b+\ddots}}}$$

where $b = \lfloor \sqrt{n} \rfloor$ (OEIS A000196, the integer part of $\sqrt{n}$). So if $b_i = n - b^2$ and $a_i = 2b$, we can generate continued fractions for all n that are not square numbers. In these cases, $\sqrt{n}$ is an integer, which can nevertheless be represented as an infinite continued fraction, with $b_i = \lfloor \sqrt{n} \rfloor - 1$ and $a_i = \sqrt{n}$, as blackpenredpen described in his YouTube video on continued fractions for the case of $n = 4$, that is $\sqrt{n} = 2$. For example:

$$\sqrt{3} = 1 + \frac{2}{2+}\frac{2}{2+}\frac{2}{2+}\frac{2}{2+}\frac{2}{2+}\frac{2}{2+}\frac{2}{2+}\ldots$$

$$\sqrt{4} = 2 = 1 + \frac{2}{1+}\frac{2}{1+}\frac{2}{1+}\frac{2}{1+}\frac{2}{1+}\frac{2}{1+}\frac{2}{1+}\ldots$$

$$\sqrt{5} = 2 + \frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\ldots$$

$$\sqrt{19} = 4 + \frac{3}{8+}\frac{3}{8+}\frac{3}{8+}\frac{3}{8+}\frac{3}{8+}\frac{3}{8+}\frac{3}{8+}\ldots$$

As you can see from this last example, this algorithm for generating continued fractions for square roots rarely results in a simple continued fraction. However, Ron Knott gives another algorithm for finding a simple continued fraction for any square root.[517] This is periodic in the values of b_i, somewhat more complicated to explain. If $m = \lfloor \sqrt{n} \rfloor$, then, for some real x,

$$\sqrt{n} = m + \frac{1}{x}$$

From this, we derive:

$$x = \frac{1}{\sqrt{n} - m} = \frac{\sqrt{n} + m}{(\sqrt{n} - m)(\sqrt{n} + m)} = \frac{\sqrt{n} + m}{n - m^2}$$

For instance, if $n = 2$, then $m = 1$, we begin with:

$$\sqrt{2} = 1 + \frac{1}{x}$$

giving:

$$x = \sqrt{2} + 1$$

We stop here, because this expression is the square root plus the original first integer. We therefore have:

$$\sqrt{2} = 1 + \frac{1}{x} = 1 + \frac{1}{\sqrt{2}+1} = 1 + \frac{1}{1+1+\frac{1}{x}} = 1 + \frac{1}{2+\frac{1}{1+\ddots}}$$

So $\sqrt{2} = [1; \overline{2}]$, as before. If $n = 3$, then $m = 1$, we begin with:

$$\sqrt{3} = 1 + \frac{1}{x_1}$$

giving:

$$x_1 = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{\sqrt{3}+1}{2}$$

As this is not the square root plus the original first integer, we need to continue with the same procedure:

$$x_1 = \frac{\sqrt{3}+1}{2} = 1 + \frac{1}{x_2}$$

$$x_2 = \frac{2}{\sqrt{3}-1} = \frac{2(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \sqrt{3}+1$$

Here, we have the square root plus the original first integer. So we can set the values for x_1 and x_2 into the expression for $\sqrt{3}$:

$$\sqrt{3} = 1 + \frac{1}{x_1} = 1 + \frac{1}{1+\frac{1}{x_2}} = \frac{1}{1+\frac{1}{\sqrt{3}+1}} = \frac{1}{1+\frac{1}{1+1+\frac{1}{x_1}}} = \frac{1}{2+\frac{1}{1+\ddots}}$$

Using this recursive method, we have these continued fractions for a few small integers that are not square,

$$\sqrt{3} = 1 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\ldots$$

$$\sqrt{7} = 2 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\ldots$$

$$\sqrt{12} = 3 + \frac{1}{2+}\frac{1}{6+}\frac{1}{2+}\frac{1}{6+}\frac{1}{2+}\frac{1}{6+}\frac{1}{2+}\frac{1}{6+}\ldots$$

Here are the first twenty-five terms in the sequence for $\sqrt{n}$ using this algorithm. The last term in the sequence is $2\times\lfloor\sqrt{n}\rfloor$, and I can see a few other patterns. But I'm not sure if there are any overall, underlying rules for the periods and values for b_i, like there are for recurring decimals for the rationals, on page 101.

√1	$[1;]$	√2	$[1;\overline{2}]$	√3	$[1;\overline{1,2}]$	√4	$[2;]$	√5	$[2;\overline{4}]$
√6	$[2;\overline{2,4}]$	√7	$[2;\overline{1,1,1,4}]$	√8	$[2;\overline{1,4}]$	√9	$[3;]$	√10	$[3;\overline{6}]$
√11	$[3;\overline{3,6}]$	√12	$[3;\overline{2,6}]$	√13	$[3;\overline{1,1,1,1,4}]$	√14	$[3;\overline{1,2,1,6}]$	√15	$[3;\overline{1,6}]$
√16	$[4;]$	√17	$[4;8]$	√18	$[4;\overline{4,8}]$	√19	$[4;\overline{2,1,3,1,2,8}]$	√20	$[4;\overline{2,8}]$
√21	$[4;\overline{1,1,2,1,1,8}]$	√22	$[4;\overline{1,2,4,2,1,8}]$	√23	$[4;\overline{1,3,1,8}]$	√24	$[4;\overline{1,8}]$	√25	$[5;]$

When researching this topic, I found another algorithm for generating continued fractions for algebraic numbers on the ei pi YouTube channel,[518] which is closely related to nested radicals, which Srinivasa Ramanujan (1887–1920) much studied. If all the *a*'s and *b*'s in the generalized continued fraction are equal, then:

$$x = b + \frac{a}{b+\frac{a}{b+\ddots}}$$

As the primary numerator is, itself, *x*, we have:

$$x = b + \frac{a}{x}$$

which is the quadratic equation:

$$x^2 - bx - a = 0$$

The positive root *r* of this quadratic equation is thus:

$$r = b + \frac{\sqrt{b^2 + 4a}}{2} = b + \frac{a}{b + \frac{a}{b + \frac{a}{b + \ddots}}} = \sqrt{a + b\sqrt{a + b\sqrt{a + b \ldots}}}$$

Most famously, if $a = b = 1$, then this equation gives the golden ration φ—which Conway and Root still denote by τ,[519] as I was taught as a student—as a continued fraction:

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ddots}}} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$$

The golden ratio is known as the most irrational number, even more so than the transcendental numbers, because its successive convergents diminish more slowly than for any other continued fraction. The sequence of convergents for a continued fraction is calculated by adding each fraction to the total in turn. The convergents for φ are naturally terms in the Fibonacci sequence, which come ever closer to 1.618033988749848482..., as this table indicates.

1	1	0.61803399
2	2	-0.38196601
3/2	1.5	0.11803399
5/3	1.666666667	-0.04863268
8/5	1.6	0.01803399
13/8	1.625	-0.00696601
21/13	1.615384615	0.00264937
34/21	1.619047619	-0.00101363
55/34	1.617647059	0.00038693
89/55	1.618181818	-0.00014783
144/89	1.617977528	0.00005646
233/144	1.618055556	-0.00002157

However, there are, in general, no regular patterns in simple continued fractions for cube roots and higher radicals. For instance,

$$\sqrt[3]{3} = [1;\ 2, 3, 1, 4, 1, 5, 1, 1, 6, 2, 5, 8, 3, 3, 4, 2, 6, 4, 4, 1, 3, 2, \ldots]$$

Nevertheless, a few of the transcendental numbers do have patterns in their continued fractions. For instance, *e*, which Conway and Root call 'Napier's number', after the discoverer of logarithms, has this continued fraction:

$$e = 2 + \frac{1}{1 +}\frac{1}{2 +}\frac{1}{1 +}\frac{1}{1 +}\frac{1}{4 +}\frac{1}{1 +}\frac{1}{1 +}\frac{1}{6 +}\frac{1}{1 +}\frac{1}{1 +}\frac{1}{8 +}\frac{1}{1 +}\frac{1}{1 +}\frac{1}{10 +}$$

However, this is π as a simple continued fraction: [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, …]. Nevertheless, π can be expressed with patterns in a generalized continued fraction in several ways, such as this:[520]

$$\pi = 3 + \frac{1^2}{6 +}\frac{3^2}{6 +}\frac{5^2}{6 +}\frac{7^2}{6 +}\frac{9^2}{6 +}\frac{11^2}{6 +}\frac{13^2}{6 +} \ldots$$

Although it has been rather tedious seeking patterns among the continued fractions, this brief study has given me a feeling for the subject, which can become quite complex when all the details are explored, which is best left to those with more competence than my own.

Of course, as real numbers play a major role in science, there is much more to say about them. In this book, I am most interested in their role in the continuity of the infinitesimal calculus and their relationships to group theory, which Abel and Galois initiated when seeking general rules for finding the roots of polynomials. At a later date, I plan to return to this subject here, of particular interest being the discriminants of polynomials.

Complex numbers

Having seen how the uncountable domain of real numbers measures the entire continuum of multidimensional hyperspace, it might seem that there is nothing more to add on the subject of types of number. However, mathematicians have discovered that the real numbers do not solve all mathematical

problems. Most significantly, there are roots to quadratic, cubic, and quartic equations in one variable that are not real. So mathematicians needed to invent the class **Complex number** (ℂ), which has the form $a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, *i* standing for *imaginary*. So *bi* is in the class **Imaginary number** (𝕀).

The word *imaginary* derives from Latin *imāginārі* 'form an image of in one's mind, picture to oneself, conceive', which became English *imagine*. But our mental images or conceptual models are not just figments of our imagination, "often with [the] implication that the conception does not correspond to the [so-called] reality of things", as the OED says. For, as we are the least instinctive of all the animals, our cognitive maps determine how we view the world and hence govern our behaviour, even when these images are buried deep within the unconscious, as depth psychologists are well aware of.

The OED tells us that John Wycliffe was the first person to use *imaginary* in English in 1382 in the Prologue to Apocalypse (Revelations) in his first translation of the Bible into English. He wrote, "*For sum visioun is bodili, as whanne we seen eny thing with bodili iȝen; sum is spiritual, or ymaginarie, as whanne we seen slepinge, or ellis wakinge we biholden the ymagis of thingis, bi whiche sum other thing is signified.*" [521] Endeavouring to interpret Wycliffe's Middle English as well as I can, he seems to relate our inner vision to spirituality, and both to dreams and concepts that signify something, supposedly real, as things, from the root of *real*, something that is still, to this day, not well understood.

As we see on page 105, Descartes was the first person to use *imaginary* in mathematics in *Geometry* in 1637. However, it was not until 1706 that William Jones (1675–1749)[522] used *imaginary* in English for the root of an equation. In *Synopsis Palmariorum Matheseos*, an introduction to the principles of arithmetic and geometry for beginners, he wrote, "Since any *Quantity* composed of *Parts* may be reduc'd into *Parts*, it must inevitably follow, That the *Original Components* or *Roots* of all *Equations* may be either *Affirmative*, *Negative*, *Mix'd*, or *Imaginary*."[523]

Incidentally, in the same book, Jones, who was a friend of both Isaac Newton and Edmund Halley, also introduced π as a symbol for the ratio between the circumference and the diameter of a circle, from *periphery*. He even correctly gave π to 101 decimal places, which John Machin (c. 1686–1751) had calculated using a quickly converging series.[524]

Like radicals and negative numbers, so-called imaginary numbers were slow to gain acceptance in mathematics. The turning point came in 1545, when Gerolamo Cardano (1501–1576) , a rather notorious polymath, published *Ars Magna* 'The Great Art' (algebra), two years after Copernicus's *On the Revolutions of the Heavenly Spheres* was published. Cardano called imaginary numbers 'fictitious', while nevertheless acknowledging their existence as the solution to this quadratic equation, the roots being $5 \pm \sqrt{-15}$:[525]

$$x(10 - x) = 40$$

While solutions to particular quadratic equations had been known since the Babylonians, in *Ars Magna*, Cardano gave general solution to cubic and quartic equations, which had puzzled mathematicians for millennia, inevitably highlighting the need to include the square root of negative numbers in mathematics. This book is thus "frequently taken to mark the beginning of the modern period in mathematics".[526]

Actually, Cardano did not himself solve these equations, as he partially acknowledged. The situation is mired in controversy, due mainly to mathematicians seeking to keep their discoveries secret in case they might lose a challenge from another mathematician and thus lose their reputation and livelihood. Niccolò Fontana Tartaglia (1500–1557) is generally acknowledged as the first to solve the cubic equation, announcing this in 1535, while keeping the general solution secret. At this, Antonio Maria Fiore, believing Tartaglia to

be bluffing, challenged him to a contest, which Tartaglia won handsomely, Fiore not being able to solve cubics in the form that Tartaglia had solved.

At this, Cardano, who was much more famous than Tartaglia, known as the 'stammerer', having gained a reputation as a physician, persuaded Tartaglia to reveal his secret, promising not to publish the solution until Tartaglia had had the opportunity to do so himself. However, Cardano discovered that Scipione del Ferro (1465–1526), Fiore's teacher, had found a solution to the cubic equation before Tartaglia, and so did not feel obliged to keep Tartaglia's secret. Furthermore, at Cardano's request, Lodovico Ferrari (1522–1565), Cardano's student, found a solution to the quartic equation. Hence, Cardano felt free to publish his masterpiece, greatly upsetting Tartaglia.[527]

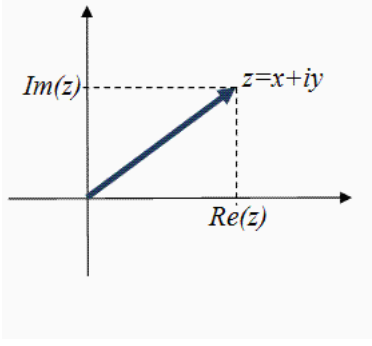
However, while non-real solutions to cubic and quartic equations were lurking within the formulae, it was not until Rafael Bombelli (1526–1572) published *Algebra*, as a successor to *Ars Magna*, shortly before his death, that complex numbers and, indeed, negative ones, became acceptable solutions to problems in mathematics. Before this, polynomials were written like this, to avoid negative coefficients:[528]

$$3x^3 + 5x = 2x^2 + 7$$

Basically, as we know today, the general solution to cubic equations involves reducing them to quadratic equations by clever substitutions and likewise the general solution to quartic equations involves reducing them to cubic equations and hence quadratic ones. The algorithms are rather involved, better amenable to computer evaluation than human calculation, which is often prone to error.[529]

Mathematicians thus naturally thought that quintic equations and polynomials of higher order could be solved by similar algebraic methods. In the event, as already mentioned, Abel and Galois proved in 1831 and 1832, respectively, that quintic polynomials and above cannot be solved algebraically, founding group theory at the heart of modern abstract algebra, which led eventually to Integral Relational Logic.

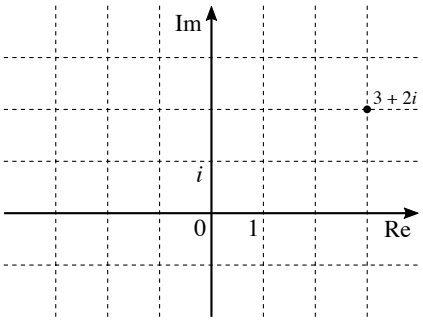
In the meantime, during the 1600s and 1700s, mathematicians generally referred to the roots of polynomials that are not real as imaginary quantities. This much troubled Euler, who remarked that "such numbers, which by their nature are impossible, are ordinarily called imaginary or fanciful numbers because they exist only in the imagination". Euler began to bring some clarity to this murky terminology in 1777, when he proposed to use the symbol i to denote just $\sqrt{-1}$. Then, in 1832, Gauss proposed that $a + bi$, as a whole, should be called *complex*, with a being *real* and bi *imaginary*.[530]



However, as James Grime points out on a Numberphile YouTube video, complex numbers are more compound than complex. For, as Gauss showed, the number $a + bi$ could be seen simply as an ordered pair or point (a, b). This could thus be represented on a 'complex' plane, with axes for real and imaginary parts, like the x- and y-axes for (x, y) in the Cartesian plane.

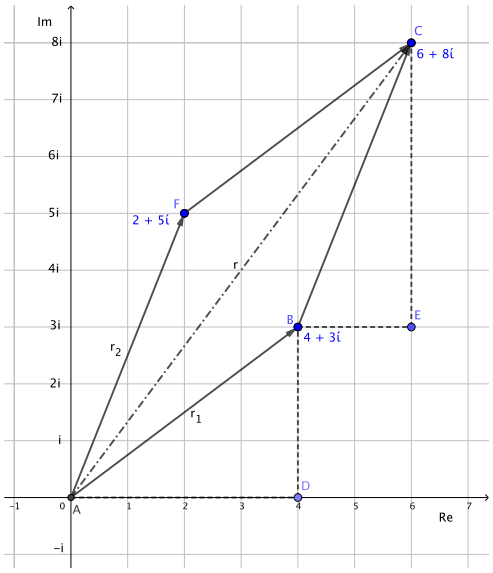
Actually, Gauss was not the first to suggest this idea. Two amateur mathematicians previously proposed a simple geometrical interpretation for representing complex numbers, making them much more comprehensible.[531] First, in 1797, a Norwegian surveyor and cartographer Casper Wessel (1745-1818) proposed that such line segments or vectors could be added and multiplied together like complex numbers, eventually leading to vector spaces and linear algebra.[532] Secondly, a Swiss bookkeeper Jean Robert Argand (1768–1822) had a similar idea in 1806, leading to the complex plane sometimes being called the Argand plane.[533]

As Wessel and Argand were little known, in the event it was the authority of Gauss that brought about the general acceptance of the geometrical interpretation of complex numbers. In the special case when (a, b)



are both integers in $a + bi$, this gives rise to **Gaussian integer**, a subclass of **Complex number**, representing a lattice on the complex plane, as in this diagram.[534]

As operations on complex numbers do not follow the simple pattern in the hierarchy of numerical, arithmetical operators, let us see how Wessel viewed them. Regarding the addition of complex numbers, these could be represented by adding one to another, as in this diagram, known as the parallelogram law for vector addition. Here $r_1 = 4 + 3i$ and $r_2 = 2 + 5i$, giving $r = 6 + 8i$. In general, the addition rules for complex numbers follow that for the real numbers:



$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

As addition is commutative for the real numbers, it is also commutative for the complex ones.

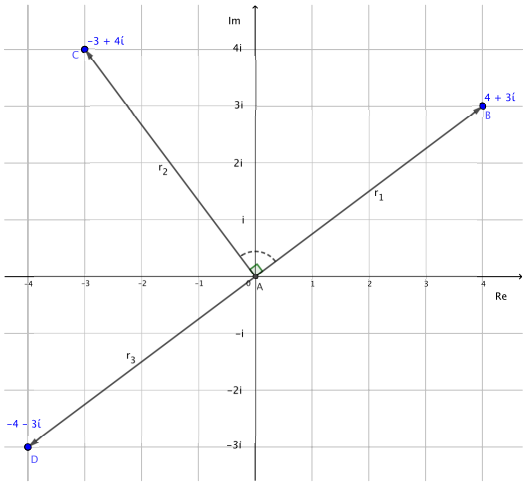
We can use the Pythagorean Theorem to calculate the values of r_1, r_2, and r as 5, 5.385164807, and 10, respectively, from:

$$r = \sqrt{a^2 + b^2}$$

In general, writing $z = x + yi$, the magnitude or modulus of z is given by:

$$|z| = \sqrt{x^2 + y^2}$$

This is also called the norm of z, written as $\|z\|$, which some writers confusing use to refer to $|z|^2$, which is more correctly known as the 'absolute square'.



Wessel also showed that multiplying a vector in the complex plane by a complex number rotates it. To take the special case of $z = i$, the vector is rotated by 90° or π/2 radians anticlockwise, as in this diagram. And when it is rotated by a further 90°, the result is the opposite vector, graphically showing that $i^2 = -1$! As the other special example, if the complex number is actually real, that is $b = 0$ in $a + bi$, then the vector is scaled by a, stretched or squished depending on whether a is greater or smaller than 1.

In general, complex multiplication is given by this formula:

$$(a + bi)\times(c + di) = (ac - bd) + (ad + bc)i$$

Another way to view a complex number is as a matrix:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

So viewing the second complex number as a vector, we have:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ ad + bc \end{pmatrix}$$

As a special case, if $a = 1$ and $b = 0$, we have what is called the identity matrix I in abstract algebra:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Like multiplying by 1 with the real numbers, multiplying a matrix by the identity matrix leaves it unchanged. The opposite special case is when $a = 0$ and $b = 1$, giving a matrix representation for i:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We can test this in matrix algebra by seeing that i^2 is:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -I$$

Matrix multiplication also tells us that the multiplication of complex numbers is commutative, not generally the case with matrix multiplication:

$$z_1 z_2 = z_2 z_1$$

In matrix form:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{pmatrix}$$

Another important concept of complex numbers is their conjugate, defined as:

$$\bar{z} = a - bi$$

In matrix form, this is the transpose of the matrix representation of z:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^T$$

This means that the product of a complex number z and its conjugate is the square of its norm or magnitude:

$$z\bar{z} = a^2 + b^2 = |z|^2$$

So the inverse of a complex number is:

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

Similarly, the determinant of a complex number gives the square of its magnitude:

$$\begin{vmatrix} a & -b \\ b & a \end{vmatrix} = a^2 - (-b^2) = a^2 + b^2 = |z|^2$$

Another powerful tool for handling complex numbers is to see them in terms of polar coordinates. For instance, a vector z terminating at a point A, with the coordinates (a, b) in the complex plane at an angle θ to the x-axis, can be written as:

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{\theta i}$$

using Euler's identity, with $r = \sqrt{a^2 + b^2}$. So multiplying two complex numbers together gives:

$$z_1 z_2 = r_1 e^{\theta_1 i} \cdot r_2 e^{\theta_2 i} = r_1 r_2 e^{(\theta_1 + \theta_2)i}$$

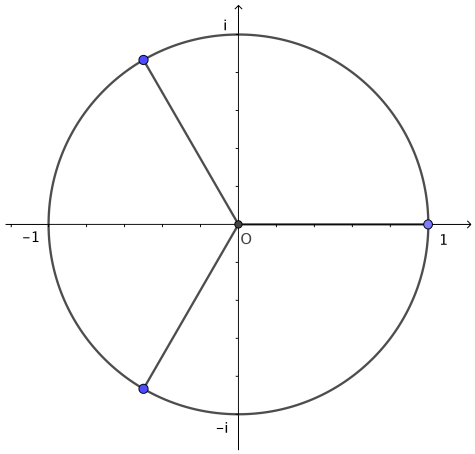
So, while the magnitudes of the complex numbers are multiplied together, the angles of rotation are added together. In the special case of $r = 1$, we have the 2D rotation matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

We can use these polar coordinates to determine the n roots of unity for this polynomial:

$$x^n = 1$$

Fairly obviously, one root is $x = 1$, while most other roots are complex numbers. The one exception is when n is even, with a root at -1. We can see this most clearly with the unit circle in the complex plane, with $n = 3$ in this example. The cubic roots of unity, often needed as the roots of cubic equations, are thus:



$$e^0 = 0 \quad e^{\frac{2\pi i}{3}} \quad e^{\frac{4\pi i}{3}} = e^{-\frac{2\pi i}{3}}$$

So the fourth roots of unity are 1, i, -1, and $-i$, as you might expect, successively multiplying by i. In general, the n roots of unity are:

$$e^{\frac{2k\pi i}{n}} \text{ for } k = 0, 1, 2, \ldots, n-1$$

So, when $k = n/2$, we have Euler's famous identity:

$$e^{\pi i} = -1$$

Quaternions

To extend complex numbers from two dimensions into three, it might seem obvious to develop an algebra based on numbers in the form $a + bi + cj$. However, this does not work, as William Rowan Hamilton (1805–1865), a polyglot and child-prodigy, discovered after thirteen years of fruitless endeavour. He could add triples, but could not multiply them, as he told his nine- and eight-year-old sons when they asked him about his progress at breakfast at the beginning of October 1843.[535]

Then, on Monday 16th October 1843, when taking a walk with his wife, in a sudden moment of inspiration, he saw that he needed four-dimensional numbers for the three-dimensional algebra that he was seeking. At this, he famously inscribed with a penknife on Brougham (now Broom) Bridge in Dublin, these relationships:

$$i^2 = j^2 = k^2 = ijk = -1$$

where i, j, k, are 'imaginaries' in numbers of this form $a + ib + jc + kd$, where a, b, c, $d \in \mathbb{R}$. Hamilton called them *quaternions*, denoted by $\mathbb{H}$ in his honour, from Latin *quaternārius*, from *quaternī* 'by fours', from *quater* 'four times'. The next month, on 20th November 1843, he publicly announced his discovery in a letter published in the *Philosophical Magazine* in 1844, enclosing a copy of a letter that he had sent the next day to his friend John T. Graves (1806–1870), who was also a jurist, who had inspired his researches. [536]

Hamilton had arrived at his discovery of the quaternions after a thirteen-year systematic study of all number systems known at his time, not unlike this chapter in this book, published in 1837 as *Theory of Conjugate Functions, or Algebraic Couples: with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time*. He was seeking to provide a sound foundation for algebra based on "order and continuous progression, or, as it might be called, PURE TIME".[537]

In this progressive way, Hamilton defined "negative numbers by a temporal opposite direction, steps backwards in time". Regarding $\sqrt{-1}$, Hamilton initially regarded this as absurd, imaginary, impossible, or contradictory, preferring to think of what we today call complex numbers as couples, or pairs of reals, not unlike Gauss at about the same time, although Gauss is not mentioned in his paper. At the end, Hamilton said that he hoped to extend his Theory of Couples into a Theory of Triplets.[538]

However, Hamilton met with difficulties when attempting to determine the modulus or norm of numbers in the form $a + bi + cj$. This left him with a term $2bcij$, which he didn't know what to do with. Eventually, he realized that he needed a third term, which he then called 'imaginary unit', in which:

×	1	i	j	k
1	1	i	j	k
i	i	−1	k	$-j$
j	j	$-k$	−1	i
k	k	j	$-i$	−1

$$ij = k, jk = i, ki = j,$$
$$ji = -k, kj = -i, ik = -j$$

This means that while multiplication in quaternion algebra is associative, it is not commutative, as the highlighted cells show in this table. If p and q are quaternions, $pq \neq qp$.

Regarding basic arithmetic operations, quaternions are added just like the real numbers. So we have:

$$(a_1 + b_1 i + c_1 j + d_1 k) + (a_2 + b_2 i + c_2 j + d_2 k) = (a_2 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k$$

Multiplication is a little more complicated:[539]

$$(a_1 + b_1 i + c_1 j + d_1 k)\,(a_2 + b_2 i + c_2 j + d_2 k) = A + Bi + Cj + Dk$$

where:

$$A = a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2$$
$$B = a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2$$
$$C = a_1 c_2 + c_1 a_2 + d_1 b_2 - b_1 d_2$$
$$D = a_1 d_2 + d_1 a_2 + b_1 c_2 - c_1 b_2$$

This means that the product of a quaternion q with its conjugate, which is the square of its norm, is given by this expression, like that for the complex numbers, because all the coefficients of the 'imaginaries' cancel themselves out:

$$q\overline{q} = (a + bj + ck + dk)(a - bj - ck - dk) = a^2 + b^2 + c^2 + d^2 = |q|^2$$

We thus have this expression for the inverse of a quaternion, like that for complex numbers:

$$q^{-1} = \frac{\overline{q}}{|q|^2}$$

Furthermore, the norm of the product of two quaternions is the product of their norms, an amazing identity that Euler had discovered in 1749, known as the Euler four-square identity.[540]

Despite the similarity with the complex numbers, the lack of commutativity among the quaternions was a great shock to mathematicians. For as Morris Kline tells us, "Here was a physically useful algebra which fails to possess a fundamental property of all real and complex numbers, namely that $ab = ba$."[541] This property of quaternions helped lead mathematicians into universal algebra, of which abstract, linear, and Boolean algebras are particular instances, where generalized objects being operated on are not necessarily numbers per se. As we explore in Chapter 5, these have led to Integral Relational Logic, which takes mathematical abstraction to the utmost level of generality.

However, while mathematicians welcomed quaternions, physicists were less able to accommodate them in their studies of space and time, as Hamilton had hoped. It is a rather complicated story, leading to a 'vector war', which Michael J. Crowe detailed in 1967 in *A History of Vector Analysis*,[542] presenting a summary in 2002.[543] Some of the major figures were three Scottish distinguished mathematical physicists, Peter Guthrie Tait (1831–1901) and his friend James Clerk Maxwell (1831–1879), together with William Thompson (Lord Kelvin from 1892) (1824–1907), and Hermann Günther Grassmann (1809–1877), Josiah Willard Gibbs (1839–1903), and Oliver Heaviside (1850–1925) in Germany, the USA, and England, respectively.

After Hamilton wrote an indigestible 762-page treatise titled *Elements of Quaternions*, posthumously published by his son William Edwin Hamilton in 1866, Tait wrote two textbooks on the subject—*Elementary Treatise on Quaternions* and *Introduction to Quaternions*, published in 1867 and 1873, respectively, with further editions in later years. Tait had begun a correspondence with Hamilton in the 1850s, when he was Professor of Mathematics at Queen's College, Belfast. He also wrote the article on 'Quaternions' in the Ninth Edition of the *Encyclopædia Britannica* in 1886. Within physics, although Maxwell did not originally utilize quaternions when formulating his famous equations on electromagnetic phenomena, in 1873 in *A Treatise on Electricity and Magnetism*, he wrote:

> As the methods of Des Cartes are still the most familiar to students of science, and as they are really the most useful for purposes of calculation, we shall express all our results in the Cartesian form. I am convinced, however, that the introduction of the ideas, as distinguished from the operations and methods of Quaternions, will be of great use to us in the study of all parts of our subject.[544]

As Maxwell went on to point out, in 1846 Hamilton had considered quaternions to consist of the sum of two parts: a *scalar*, as the real term a, and a *vector*, as $ib + jc + kd$, the first time these two words were used

in mathematics, from Latin *scālāris* 'of a ladder', from *scālæ* 'ladder, staircase', and Latin *vector* 'carrier', from *vehere* 'convey, carry'.[545]

The role of Grassmann is not unlike that of Hamilton in that his ideas grew out of an attempt to develop a mathematical system, more like non-Euclidean geometry or Boolean algebra than the Pythagorean theorem or Newton's version of the calculus, as Crowe puts it.[546] Indeed, when Alfred North Whitehead came to write *A Treatise on Universal Algebra* in 1898, he used Hamilton, Grassmann, and Boole's systems as the framework for his book.

In 1844, Grassmann published his masterpiece, *Die Lineale Ausdehnungslehre, ein neuer Zweig der Mathematik* (*The Theory of Linear Extension, a New Branch of Mathematics*), with a second edition in 1878, which Lloyd C. Kannenberg translated into English in 1995 as *A New Branch of Mathematics*. As leading mathematicians at the time, such as August Ferdinand Möbius (1790–1868) and Gauss, did not understand this book, Grassmann wrote a second version in 1862, which Kannenberg translated in 2000 as *Extension Theory*, "stripped of the philosophical drapery" of the first book.[547] This second version did not fare much better, leading Grassmann to turn to another major interest, that of language, learning Sanskrit to translate *Rig Veda*.

The 'philosophical drapery' that Kannenberg refers to is a radically new approach to mathematics that Grassmann sought to introduce, based on *the general theory of forms*, which is "a series of truths that relate to all branches of mathematics in the same way, and which thus assume only the general concepts of equality and difference, conjunction and separation". Most significantly, Grassmann wrote in the Foreword to the first edition that he was not appealing to any principle previously proven in any other branch of mathematics, showing that geometry is only a special application of general principles. He was thus not constricted to viewing space in just three dimensions, but could look at it in any number.[548]

Like Hamilton, looking at negatives in time, Grassmann began by looking at negatives on the number line. As he explained in the Foreword to the first edition, "if A, B, C are points of a straight line, then $AB + BC = AC$ is always true, whether AB and BC are directed similarly or oppositely, that is even if C lies between A and B." He then went on to see that this equation could also be true in the general case, when A, B, C do not lie on a single straight line, leading him to see that "parallelograms in general may be regarded as products of an adjacent pair of their sides, provided one again interprets the product, not as a product of their lengths, but as that of the two displacements with their directions taken into account." He was thereby led to what he described as "the most striking harmony",[549] not unlike Wessel nearly fifty years earlier, still then unknown.

As I discovered in the autumn of 2018, Grassmann's approach to developing 'a new branch of mathematics' thus bears a striking resemblance to the way that Integral Relational Logic has emerged in consciousness. In specific terms, he introduced what are today called outer and inner or cross and dot vector products, showing that Grassmann understood "the associative, commutative, and distributive laws of algebra more fully than any earlier mathematician",[550] although Hamilton might have demurred.

However, while Hamilton admired Grassmann's genius when he read his book towards the end of 1852, he wrote in letters to Augustus De Morgan the next year that Grassmann had failed to perceive that the combination of inner and outer products, which are analogous to Hamilton's scalar and vector parts, could have led to quaternions.[551]

In the event, while Grassmann's abstract ideas had little direct influence on the subsequent development of vector analysis, Hamilton's quaternions had even less influence. Mathematical physicists seem to have asked themselves, why use the four dimensions of quaternion algebra when three-dimensional vectors in

Cartesian coordinates would be simpler? Kelvin was particularly critical, even though he co-authored *Treatise on Natural Philosophy* with Tait from 1855 to 1865, when they were professors of natural philosophy at Glasgow and Edinburgh Universities, respectively.

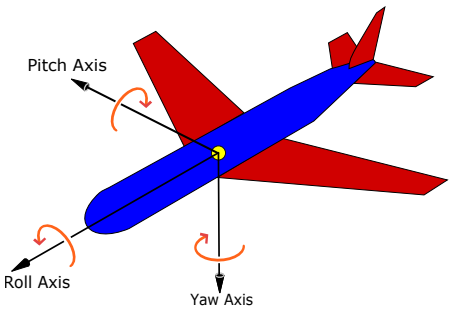
In 1892, Kelvin is reported to have said, "Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell." Then, in 1901, shortly after Tait's death, he wrote in a letter, "We [Kelvin and Tait] have had a thirty-eight years' war over quaternions," not agreeing to introduce them into Thomson and Tait's *Treatise*,[552] after which they went their separate ways, although remaining cordial from what I have read.

Heaviside was even more polemical, writing in 1893 in *Electromagnetic Theory*, after reading Tait's textbooks on *Quaternions*, "I came later to see that, as far as the vector analysis I required was concerned, the quaternion was not only not required, but was a positive evil of no inconsiderable magnitude; and that by its avoidance the establishment of vector analysis was made quite simple and its working also simplified, and that it could be conveniently harmonised with ordinary Cartesian work."[553]

Edwin Bidwell Wilson (1879–1964) was somewhat gentler in 1901, when editing a standard textbook on *Vector Analysis* based on the lectures of Gibbs, even though Tait had said in 1890 that Gibbs "must be ranked as one of the retarders of Quaternion progress", declaring that his 1884 pamphlet on *Vector Analysis* was a sort of 'hermaphrodite monster'.[554] In the General Preface to *Vector Analysis*, Wilson wrote, "Notwithstanding the efforts which have been made during more than half a century to introduce Quaternions in to physics the fact remains that they have not found wide favor."[555]

Faced with this resistance to quaternions, in the 1890s, some sixty mathematicians around the world set up a Quaternion Society, as an 'International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics'. Robert Stawell Ball and Alexander Macfarlane were the initial leaders, Tait being not well enough to serve as president. However, in the event, the Quaternion Society "became a victim of the first World War" and was dissolved.[556]

Even though quaternions are not much used in vector analysis today, in recent years, they have seen a revival in computer graphics and in flying machines like aircraft and spaceships, which can rotate around three orthogonal axes, known as yaw, pitch, and roll.



To understand why this renaissance has happened, we need to go back to Euler, who showed that the orientation of a device could be defined in terms of three angles with reference to a Cartesian coordinate system, known as Euler angles. These are normally denoted by α, β, and γ, or φ, θ, ψ for the *x*-, *y*-, and *z*-axes, giving three 3D rotation matrices around these axes, corresponding to the 2D rotation matrices for complex numbers:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \qquad R_y = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \qquad R_z = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What these show is that as two axes rotate around a third, this one stays stationary. This means that the sequence of rotations is all important. Rotations are not commutative, so different sequences can give different results. For instance, we have the following product, which is not symmetrical around α, β, and γ:

$$R_x R_y R_z = \begin{pmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{pmatrix}$$

Now, if $\beta = {}^{\pi}/_{2}$, then $\sin\beta = 1$ and $\cos\beta = 0$, giving:

$$R_x R_y R_z = \begin{pmatrix} 0 & 0 & 1 \\ \sin(\alpha+\gamma) & -\cos(\alpha+\gamma) & 0 \\ -\cos(\alpha+\gamma) & \sin(\alpha+\gamma) & 0 \end{pmatrix}$$

My understanding is that this matrix is an expression of gimbal lock, when two axes get locked together, best understood by those with a far better spatial intelligence than me, which actually occurred with Apollo 13 in 1970, as mentioned in the film of the aborted Moon landing, I am told, as I haven't seen the movie. Now while there is a solution to gimbal lock in mechanical terms, by adding a fourth gimbal, there is also a solution in mathematical terms by adding the fourth dimension in quaternions, today also used in tablets and smart phones for orientation purposes.

Grant Sanderson, known as 3Blue1Brown on YouTube, has brilliantly illustrated this with his superb graphics, even providing an interactive tool, which Ben Eater designed, available on the latter's website.[557] Basically, as Grant says, to use quaternions in three-dimensional computer graphics displayed on a two-dimensional screen, it is necessary to view them as a stereographic projection from four dimensions onto three, a visualizing skill that takes some practice.

For myself, I don't have such skills. But the mathematics is very simple. The basic formula is given by this expression:

$$f(p) = q \cdot p \cdot q^{-1}$$

where p is an object in three-dimensional space, q is a quaternion and q^{-1} is its inverse with the special property that the norm of the quaternion is 1:

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2} = 1$$

Octonians

On 26th October 1843, a couple of days after Hamilton sent Graves an expanded solution for extending complex numbers into four dimensions,[558] Graves wrote back saying, "There is still something in the system which gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties." Nevertheless, he went on to ask, "If with your alchemy you can make three pounds of gold, why should you stop there?"[559]

Indeed, this is just what Graves did during that autumn and winter. In December and the following January, he wrote to Hamilton describing a new 8-dimensional algebra,[560] which he called the 'octaves', and speculated about the possibility of extending this even further into a general theory of '2^m-ions'. He added four other imaginaries, which he called *l*, *m*, *n*, and *o*, and used them to generalize Euler's four-square identity into an eight-square identity, not knowing that Ferdinand Degen (1766–1825) had already done so around 1818.[561]

Hamilton offered to publish Graves' discovery of what would become octonions, denoted by $\mathbb{O}$, although I haven't discovered the origin of this term. However, Hamilton was so busy with developing the theory of quaternions that he omitted to do so. In the event, Arthur Cayley (1821–1895) was the first to describe the complex algebra of octonians, as a one-page postscript to an article published in the *Philosophical Magazine* in March 1845.[562] At this, Graves added a short postscript of his own to an article published in the next issue, stating that he had shown that 'The product of two sums of eight squares is a sum of eight squares,'

generalizing Euler's four-square identity, around Christmas 1843, an explanation of which "must be reserved to another time".[563]

Rather belatedly, Hamilton provided that 'other time' in an 1848 issue of the *Proceedings of the Royal Irish Academy*, giving the formulae that Graves had sent him 'On a Theorem respecting Products of Sums' in December 1843 and January 1844. However, Graves was not the only one working on extending Euler's four-square formula. John Radford Young (1799–1885), a self-taught mathematician working as Professor of Mathematics at Belfast College, presented a paper at the Royal Irish Academy on 'On an extension of a Theorem of Euler' in November 1847, appending a letter that Hamilton had sent him in March 1848, giving more details of Graves' discovery, when the paper was published.[564]

So the discovery of octonians had a rather confused genesis, which Leonard Eugene Dickson (1874–1954) documented in 1919 in 'On Quaternions and Their Generalization and the History of the Eight Square Theorem'.[565] Because, Cayley was the first to publish on the subject, octonions are sometimes called today Cayley numbers, although if they are to be attributed to anyone, they should be called Degen numbers. In this paper, Dickson showed how an infinite sequence of '2^n-ions' can be generated indefinitely, known today as the Cayley–Dickson construction, with each algebra having twice the dimension of the previous one.[566]

Without going into the intricate details of these algebras in this book, in 2001, John C. Baez presented a more elegant way in which the octonions can be visualized than the original constructs of Graves, Cayley, and Young.[567] An octonion is an 8-dimensional algebra with a basis of 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7. This can be presented in octonion form as:

$$u = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$$

where:

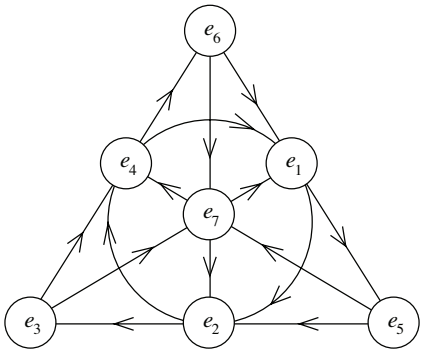
$$e_i = \sqrt{-1} \quad \text{and} \quad e_ie_j = -e_je_i \quad \text{when } i \neq j$$

Beginning with $e_1e_2 = e_4$, the 'imaginaries' are then formed into seven triplets, acting like quaternions, having these relationships, where the suffixes rotate in $\mathbb{Z}_7$, similar to modulus arithmetic:

$$e_ie_j = e_k \Rightarrow e_{i+1}e_{j+1} = e_{k+1}$$

$$e_ie_j = e_k \Rightarrow e_{2i}e_{2j} = e_{2k}$$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1



That is, the basic triplets are 124, 523, 346, 615, 574, 371, 672, given as the circle, edges, and diagonals of this Fano plane. And to the left is the multiplication table for the 'imaginaries', with the embedded quaternions marked in different colours. The most important distinguishing feature of this algebra is that it is not associative. In other words,

$$(e_ie_j)e_k \neq e_i(e_je_k)$$

if i, j, k are distinct or if $e_ie_j \neq \pm e_k$. Rather, the algebra has the property of alternative associativity, as shown here:

$$(e_ie_j)e_k = -e_i(e_je_k)$$

This makes the product of two octonions rather complex, essentially providing the eight-square identity, which Degen discovered. I don't think anything is served by going into more detail here. Apart from some other hypercomplex numbers, some of which are duals to those we have already looked at, I have fulfilled my purpose to show that the types of number can be extended indefinitely from the natural numbers, like so many constructs in mathematics.

Other hypercomplex numbers

Another feature of the hierarchy of hypercomplex numbers that I discovered when writing this book is that they bifurcate as the reals become complex numbers. The first person to notice this was James Cockle (1819–1895), who served as the first Chief Justice of Queensland from 1863 to 1879, when Australia was still a British colony.[568] When studying quaternions, Cockle noticed that he could form another type of number in the form, where $a, b, c, d \in \mathbb{R}$:

$$t = a + bi + cj + dij$$

which he called a *tessarine*, from Greek *tessara-*, combining form of *tessares* 'four'. His letter announcing such a number was published in the prestigious *Philosophical Magazine* in 1848, to be followed with further developments in the following two years.[569] In the first of these letters, Cockle gave these relationships for his 'imaginaries' i and j, where $k = ij$:

$$i^2 = -j^2 = k^2 = -1$$
$$ij = k, \qquad jk = i, \qquad ki = -1$$

In other words:

$$j^2 = \sqrt{+1}$$

Here, j is not 1, but effectively another type of mathematical object, which can be treated just like i, as $\sqrt{-1}$. In his initial letter, Cockle showed that the product of two tessarines is another tessarine, showing that they form a completely normal, closed algebra.

Indeed, it is possible to form a normal algebra as a dual of the complex numbers in the form $a + bj$, whose matrix form is:

$$s = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

This gives the norm of s as:

$$\|s\| = \sqrt{a^2 - b^2}$$

Numbers of the form $a + bj$ are called hyperbolic or split-complex numbers, which can be seen most clearly in Euler's representation of such numbers, corresponding to the unit circle representation of ordinary complex numbers:

$$s = e^{\varphi j} = \cosh\varphi + j\sinh\varphi$$

As Mathoma gives an excellent overview of split-complex numbers on his YouTube channel, there is no need to explore them further here.[570] All I wish to point out is that there is dual hierarchy of hypercomplex numbers that can be generated from a modification of the standard form of the Cayley–Dickson construction. In other words, we see here an instance of the underlying structure of the Universe, as a dual, generative structure of hierarchical relationships. So, once I see the simple general pattern, there is no need to explore the complexities of the particular patterns further.

Growth of numerical operators

Having overviewed the principal classes of number, it is time to look at the basic operators between them. These are like functions in the fundamental data-processing structure, depicted on page 4, taking inputs and producing outputs, a ubiquitous process, long before the invention of the stored-program computer.

In Integral Relational Logic, numbers are entities as instances of classes, like the way they are handled in the Smalltalk programming environment. Operators on numbers are thus relationships between them, transforming one or more numbers into one or more other numbers, sometimes in the same class and sometimes in other classes.

The basic operations between numbers are, of course, the arithmetical ones of addition and multiplication, with their inverses of subtraction and division. However, I've discovered in recent years that mathematicians look at arithmetical operators in a hierarchical manner, as hyperoperations, from level 0 upwards, the first few after multiplication being exponentiation, tetration, pentation, and hexation. This is in keeping with the central theme of this book: to view mathematics as a generative science of patterns and relationships in many different directions.

In Chapter 5 on 'Universal Algebra', I look at the way that mathematicians have generalized the patterns between basic arithmetical operators since the 1800s to operate on types of entity that are not necessarily numbers. These turn out to be special cases of more general principles, which IRL takes to the utmost level of abstraction with the primal concepts of **data element** and **being**.

Looking here at the operators between numbers hierarchically, we can see that addition is repeated counting, multiplication is repeated addition, and exponentiation is repeated multiplication. This is yet another example in mathematics that is in keeping with the hierarchical nature of the Universe, with quite mind-boggling results.

A recently discovered popular science book *One Two Three … Infinity* by the distinguished Soviet-American physicist George Gamow (1904–1968), first published in 1947, before the invention of the stored-program computer, has given me some insights into how to present this topic in line with my own inner experiences. In the frontispiece to the second edition of this book is this little ditty:

There was a young man from Trinity
Who took √∞
But the number of digits
Gave him the fidgets;
He dropped Math and took Divinity.

As Gamow was an atheist, by *divinity* he clearly meant 'theology' or 'the intellectual study of religion', not Divinity, as the brilliant coherent light constantly radiating through us all. This mystical worldview greatly helps me to make sense of both very large finite numbers and the infinity of infinities, which otherwise are far beyond our comprehension. At the beginning of the first chapter, titled 'Big Numbers', Gamow says that the Hottentots, a derogatory term that Dutch settlers in the 1600s in southern Africa gave to the supposedly inferior Khoikhoi race, used a numbering system 'one, two, three, … many'.[571] *Hottentot* means 'stutterer' from the sound of the Khoikhoin language, whereas *Khoikhoi* derives from *khoe-khoe*, from *khoe* 'human being'. So, the Khoikhoins saw themselves as 'men of men, proper humans, men *par excellence*'.[572]

As ever, we in Western civilization can learn much from such indigenous peoples, closer to Nature than the so-called natural sciences. For myself, I have long preferred the numbering system 'one, two, three, …

many', which is especially important when considering large finite numbers, which can lead into a great intellectual quagmire. Such an approach to large numbers does not bring us closer to Transfinite Divinity, which is necessary if we are to gain insights into humanity's place within the overall scheme of things, especially our prospects as a species in the coming years.

At the heart of this issue is the distinction between humans and machines, like computers. As we have seen, computers generically operate through functions, sometimes recursively feeding the results of one operation into the next level of function, entirely in keeping with the underlying structure of the Universe, defined in the boxed sentence on page 60. Such structures lead rapidly into ubiquitous graphs of rapidly growing complexity, much, much faster than the growth of the Internet in recent years. So, if I am not to be overwhelmed by such complexity, I prefer the counting system 'one, two, three, … many', engaging in the meditation practice I describe in the Prologue on page xxv, thereby unifying mysticism and mathematics.

Counting

The most basic operation in the hierarchy of operators in the successor function, as the basis of counting, defined on page 90. This begins with *zero*, related to *shūnya* 'empty' in Sanskrit, successively adding 1 to each previous natural number. Counting or successorship is operation 0 in the hierarchy, and is therefore sometimes called *zeration* in this context. So we have:

$$7 = 0 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$

Recursively, given a starting point of 0—as zero or emptiness—this operation can be expressed as:

$$n \to n + 1$$

Addition

Addition, at level 1, is repeated counting by 1. Having found a count of 3, let us say, we can then count another 5 to give:

$$3 + 5 = (1 + 1 + 1) + (1 + 1 + 1 + 1 + 1) = 8$$

Alternatively, we can start with 5 and add 3, to give:

$$5 + 3 = (1 + 1 + 1 + 1 + 1) + (1 + 1 + 1) = \ 8$$

So, addition is commutative, for

$$a + b = b + a$$

for all classes of number from the natural numbers to the octonians. Furthermore, addition is associative for all classes of number. That is:

$$(a + b) + c = a + (b + c)$$

In recursive terms, if we have found a natural number m by counting—that is through the successor function—we can begin the iterative process with the following, where 0 is the identity for addition:

$$m + 0 = m$$

We can then define inductive addition of a natural number n as:

$$m + (n + 1) = (m + n) + 1$$

This looks like the associative law for addition, but in this recursive context, it is much deeper, as just the second step in a recursive sequence of recursive relationships.[573]

Multiplication

Multiplication, at level 2, is repeated addition of the same number by itself. So we have:

$$3{\times}5 = 3 + 3 + 3 + 3 + 3 = 15$$

or

$$5\times3 = 5 + 5 + 5 = 15$$

So, multiplication is commutative, for

$$a\times b = b\times a$$

for all classes of number from the natural numbers to the complex ones. And while multiplication is associative for quaternions, this property is lost with the octonians, which means that they do not form a group with respect to multiplication.

Now having the two basic operations of addition and multiplication gives the possibility of combining them in the distributive property of two binary operations. There are two distributive operations, a left and right, both of which are valid for numbers up to the complex numbers, but are not necessarily the same for more general objects in abstract algebra, including quaternions and octonians. Here are the two distributive laws for the reals:

$$x\times(y + z) = (x\times y) + (x\times z)$$
$$(y + z)\times x = (y\times x) + (z\times x)$$

In recursive terms, if we have found a natural number m by repeated addition, we can begin the iterative process with the following, where 1 is the identity for multiplication:

$$m\times1 = m$$

We can then define inductive multiplication of a natural number n as:

$$m\times(n + 1) = (m\times n) + m$$

Exponentiation

Exponentiation, at level 3, is repeated multiplication of the same number by itself. So we have:

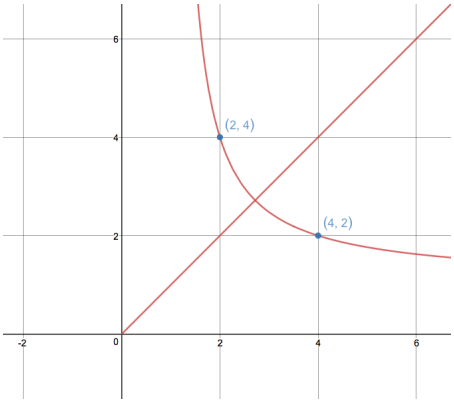
$$3^5 = 3\text{^}5 = 3\times3\times3\times3\times3 = 243$$

where ^ is a symbol for exponentiation in some programming languages, which do not support the superscript notation. But

$$5^3 = 5\text{^}3 = 5\times5\times5 = 125$$

So, exponentiation is not commutative, even for the natural numbers, with different bases and exponents. In general:

$$a^b \neq b^a$$



Nevertheless, there are some situations when $x^y = y^x$, as this diagram illustrates. Fairly obviously, this is the case when $x = y$ and also for a continuous set of values, with $x = 2$ and $y = 4$ and $x = 4$ and $y = 2$, as the only integer values on this curve, which intersects the diagonal line at $x = y = e$.[574]

Furthermore, if the exponent is negative or a fraction, like -1 and ½, the results for a base of 2 are ½ and $\sqrt{2}$, taking exponentiation into another class of number. As another example, what is a complex number to the power of a complex number? The simplest example is i^i. Surprisingly, this has an infinite number of real solutions, which Steve Chow and Matt Parker entertainingly showed on their YouTube channels in 2017:[575]

$$i^i = e^{-\frac{\pi}{2}-n\pi}$$

for $n = 0 \ldots \infty$. So if $n = 0$, $i^i \approx 0.2788$, amazing!

In recursive terms, we have:

$$m^1 = m$$

So, how can we relate exponentiation to our everyday lives? In recent years, we have seen the Internet expand at hyperexponential rates of change, in terms of both apps and people connected to each other in social media. And in 2000, John Templeton pointed out, "More new books are published each month than were written in the entire historical period before the birth of Columbus."[576]

As I explain in my 2016 book titled *Through Evolution's Accumulation Point: Towards Its Glorious Culmination*, I show that what we are witnessing today is the tail-end of some fourteen billion years of evolution since the most recent big bang in the physical universe. Having spent nearly forty years studying, through introspective self-inquiry, what is causing us scientists and technologists to drive the pace of change ever-faster, I define evolution in all its forms in this way: *Evolution is an accumulative process of divergence and convergence, proceeding in an accelerating, exponential fashion by synergistically creating wholes that are greater than the sum of the immediately preceding wholes through the new forms and relationships that emerge, apparently out of nothing.*

What is accelerating is the complexity of structure. However, there is, as yet, very little understanding of what is happening to the human race at the present time, despite all my efforts to communicate an integrative science of causality and holistic theory of evolution. As the physicist Albert A. Bartlett has said, "The greatest shortcoming of the human race is our inability to understand the exponential function."[577]

So let me repeat what I have said in my other books. We have some sense of what 100 years is like or even 1,000, measured from our own lifespan of threescore years and ten, as the Psalmist put it.[578] But what is a billion years in our experience or a quintillion years? Our lack of understanding of large numbers is well illustrated by a story that is said to have originated from the invention of chess.

> According to an old tale, the Grand Vizier Sissa Ben Dahir was granted a boon for having invented chess for the Indian King, Shirham. Since this game is played on a board with 64 squares, Sissa addressed the king: 'Majesty, give me a grain of wheat to place on the first square, and two grains of wheat to place on the second square, and four grains of wheat to place on the third square, and eight grains of wheat to place on the fourth square, and so, Oh King, let me cover each of the 64 squares on the board.' 'And is that all you wish, Sissa, you fool?' exclaimed the astonished King. 'Oh, Sire,' Sissa replied, 'I have asked for more wheat than you have in your entire kingdom, nay, for more wheat than there is in the whole world, verily, for enough to cover the whole surface of the earth to a depth of the twentieth part of a cubit.'[579]

The reason for this is that the number of grains of wheat on the *n*th square is 2^{n-1}. And $1 + 2 + 4 + \ldots + 2^{63}$ is $2^{64} - 1$, equal to 18,446,744,073,709,551,615, about 18 quintillion. This is one less than the theoretical storage capacity of modern 64-bit processors, such as the iMac I am using to write this book. But this has a storage capacity of just 16 gigabytes, 2^{34} and the most powerful Mac that Apple sells, the Mac Pro, has a maximum memory of 64 gigabytes, 2^{36}. These numbers are far smaller than the theoretical maximum for such processors, 2^{64}, which also appears in another story from antiquity:

> In the great temple at Benares, beneath the dome which marks the centre of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four discs of pure gold, the largest disc resting on the brass plate and the others getting smaller and smaller up to the top one. This is the tower of Brahma. Day and night unceasingly, the priests transfer the discs from one diamond needle to another, according to fixed and immutable laws of Brahma, which require that the priest on duty must not move more than one disc at a time and that he must place this disc on a needle so that there is no smaller disc below it. When the sixty-four discs shall have been thus transferred from the needle on which, at the creation, God placed them, to one of the other needles, tower, temple, and Brahmans alike will crumble into dust, and with a thunderclap, the world will vanish.[580]

Now it turns out that the number of transfers that the priests would need to make is again $2^{64} - 1$. If the priests were to make one transfer every second, and work 24 hours a day for every day of the year, it would

take them 584,542,046,091 years to perform this feat, 11 orders of magnitude or about 40 times longer than the time since the most recent big bang. And if we measure time in yoctoseconds or septillionths of a second, (10^{-24}), the shortest unit of temporal measure that I am aware of, the most recent big bang in the physical universe happened about 4×10^{41} yoctoseconds ago, just yesterday in the cosmic scale of things.

As another illustration of exponential power from the world of technology, when IP addresses for Internet Protocol were first set up, they consisted of four groups of 2^8 bits (256) giving 2^{32} possible values. Since then, the Internet has expanded so fast that this IPv4 address space has become exhausted. The Internet Engineering Task Force (IETF) has accordingly defined an IPv6 address space of 2^{128} possible values (about 3.403×10^{38} or 340 undecillion), which should last quite a while, even if only the square root (2^{64}) or half the power seems to be defined as four groups of four hexadecimal digits.[581]

Similarly, in spiritual circles, monks, who wrote the *Avatamsaka Sutra* '*Flower Ornament Scripture*' in Huayan Buddhism, began Chapter Thirty when they contemplated the Incalculable with these words:

> At that time, the enlightening being Mind King said to the Buddha, "World Honoured One, the buddhas speak of incalculable, measureless, boundless, incomparable, innumerable, unaccountable, unthinkable, immeasurable, unspeakable, untold numbers—what are these?"
>
> The Buddha said, "It is good that you ask the Buddha, the Truly Enlightened One, in order to have the beings of the world penetrate the meaning of the numbers known to the Buddha. Listen carefully and think well about this; I will explain for you."

The Buddha then delineated an exponential series, whose first three terms are, "Ten to the tenth power times ten to the tenth power equals ten to the twentieth power; ten to the twentieth power times ten to the twentieth power equals ten to the fortieth power; ten to the fortieth power times ten to the fortieth power equals ten to the eightieth power." In other words, the first term in the series is 10^{20} or 100 quintillion, the fifth root of a googol, and each succeeding number is the square of the previous one. In mathematical terms:

$$a_n = a_{n-1}^2 = 10^{10 \times 2^n}$$

However, the Buddha stopped when $n = 103$, which is 10 to the power of 100 nonillion approximately. The book says that this is ten to the power of 101,493,292,610,318,652,755,325,638,410,240, although there is an error in the calculation. It should be 101,412,048,018,258,352,119,736,256,430,080. According to the Sutra, the Buddha then said that a_{103}^2 is 'incalculable', calling the next few terms 'measureless', 'boundless', 'incomparable', 'innumerable', 'unaccountable', 'unthinkable', 'immeasurable', 'unspeakable', and 'unspeakably unspeakable', proceeding in fourth powers rather than second for he was now in a hurry. He ended by saying, "an untold, which is unspeakably unspeakable, … multiplied by itself, is a square untold".[582] This is a_{123}, nevertheless comprehensible to the human mind.

In temporal terms, periods in the ancient Hindu calendar represent the cyclic changes that society as a whole goes through over time, within the time-cycle of the creation and destruction of the Universe, denoted by the life span of Brahma, the god of creation. The basic unit of these cycles is the *mahayuga*, consisting of four *yugas* of diminishing time periods, measured in divine years, each of which is 360 human years, although some writers refer to the four cycles collectively as a yuga.[88] This table shows these time periods, which diminish in arithmetic progression.

Name	Characteristic	Years	Divine years
Krita- or Satya-Yuga	Golden age	1,728,000	4,800
Treta-Yuga	Sacrifices begin	1,296,000	3,600
Dvarpara-Yuga	Spiritual decline	864,000	2,400
Kali-Yuga	War, fear, and despair	432,000	1,200
Mahayuga		**4,320,000**	**12,000**

In Hindu attempts to capture the vastness of time before the discoveries of modern mathematics, Brahma is deemed to live 100 Brahma-years, to denote the creation and death of the universe, each Brahma-year consisting of 360 Brahma-days and nights or 720 *kalpas*. So in the Hindu calendar, the life and death cycle of the Universe is 311 trillion years, or 14 orders of magnitude.

This is quite small compared to the lifespan of our closest physical universe presented by Brian Cox in the 'Destiny' episode of his BBC documentary series *The Wonders of the Universe* in 2011. Apparently drawing on Fred Adams and Gregory Laughlin's *The Five Ages of the Universe*,[583] he said that the physical universe has a lifespan of "10,000 trillion trillion trillion trillion trillion trillion trillion trillion years", which is 10^{100}, just one googol, defined below.

We can see from these examples that we can much better understand the power of exponential numbers if we think in terms of orders of magnitude or powers of ten, rather than the numbers themselves.

However, we can see from the *Flower Garden Sutra* that Buddhists do not actually have a limited view of time, as do Hindus, astrophysicists, and Christian fundamentalists. In Buddhism, *kalpa* is a "term for an endlessly long period of time, which is the basis of Buddhist time reckoning. The length of a *kalpa* is illustrated by the following metaphor: suppose every hundred years a piece of silk is rubbed once on a solid rock one cubic mile in size; when the rock is worn away by this, one *kalpa* will still not have passed away."[584]

Although Buddhists did not have a mathematical way of representing such vast periods of time, mathematicians have created notations for unbelievably large exponential numbers. For instance, in the 1930s, Edward Kasner tried to explain exponential numbers to his nine-year-old nephew Milton Sirotta by asking him to create a name for a very big number.[585] Milton showed that he had more wisdom than his mathematician uncle imagined. For he coined the word *googol* for 10^{100},[586] which is just 100 orders of magnitude, still quite manageable by the mind. In 1997, Google adapted this term, a misspelling of *googol*, for its search engine, to denote its mission to organize a seemingly infinite amount of information on the web.[587] However, Milton went even further, also defining a googolplex as 10^{googol}, a name that Google has given to its headquarters. But what on earth is a googol orders of magnitude? Or any of Gödel's proof numbers in his incompleteness theorem?[588] Or a googolplex to the power of a googolplex three times, like this, a number that is quite beyond our imagination:

$$googolplex^{googolplex^{googolplex^{googolplex}}}$$

Tetration

This was as far as I had got by 2016 in my study of what David Metzler calls 'Ridiculously Huge Numbers' in a playlist of 47 videos on the subject from 2012 on his YouTube channel.[589] Then I discovered hyperoperations, which make simple sense when viewing mathematics as an abstract generative science, but way beyond anything we can relate to in terms of our everyday lives.

As you might expect, tetration, at level 4, is repeated exponentiation of the same number by itself, from Greek *tetra-*, combining form of *tettares* 'the numeral four'. My googolplex number is an example of tetration, with exponentiation applied to itself, written ^{4}goolgoplex, with a prefix exponent notifying tetration.

Using the same small numbers as in the previous operations, which Derrick Taylor uses in his introductory YouTube video on 'Tetration',[590] we have:

$$^35 = 5\text{^^}3 = 5\text{^}5\text{^}5 = 5^{(5^5)} = 5^{3125} \approx 1.9\times10^{2185}$$

where ^^ is another notation for this superpower. For tetration is right associative. If the calculation had been left associative, this would have given:

$$^3 5 = 5\text{^^}3 = 5\text{^}5\text{^}5 = (5^5)^5 = 3125^5 = 298{,}023{,}223{,}876{,}953{,}125$$

So, in tetration and higher-level operations, the numbers grow much faster with right association. In this example, Wolfram Alpha tells me that $^3 5$ is pretty big, with 2185 digits, just about manageable:

1,911,012,597,945,477,520,356,404,559,703,964,599,198,081,048,990,094,337,139,512,789,246,520,530,242,615,803,012,059,386,519,
739,850,265,586,440,155,794,462,235,359,212,788,673,806,972,288,410,146,915,986,602,087,961,896,757,195,701,839,281,660,338,
047,611,225,975,533,626,101,001,482,651,123,413,147,768,252,411,493,094,447,176,965,282,756,285,196,737,514,395,357,542,479,
093,219,206,641,883,011,787,169,122,552,421,070,050,709,064,674,382,870,851,449,950,256,586,194,461,543,183,511,379,849,133,
691,779,928,127,433,840,431,549,236,855,526,783,596,374,102,105,331,546,031,353,725,325,748,636,909,159,778,690,328,266,459,
182,983,815,230,286,936,572,873,691,422,648,131,291,743,762,136,325,730,321,645,282,979,486,862,576,245,362,218,017,673,224,
940,567,642,819,360,078,720,713,837,072,355,305,446,356,153,946,401,185,348,493,792,719,514,594,505,508,232,749,221,605,848,
912,910,945,189,959,948,686,199,543,147,666,938,013,037,176,163,592,594,479,746,164,220,050,885,079,469,804,487,133,205,133,
160,739,134,230,540,198,872,570,038,329,801,246,050,197,013,467,397,175,909,027,389,493,923,817,315,786,996,845,899,794,781,
068,042,822,436,093,783,946,335,265,422,815,704,302,832,442,385,515,082,316,490,967,285,712,171,708,123,232,790,481,817,268,
327,510,112,746,782,317,410,985,888,683,708,522,000,711,733,492,253,913,322,300,756,147,180,429,007,527,677,793,352,306,200,
618,286,012,455,254,243,061,006,894,805,446,584,704,820,650,982,664,319,360,960,388,736,258,510,747,074,340,636,286,976,576,
702,699,258,649,953,557,976,318,173,902,550,891,331,223,294,743,930,343,956,161,328,334,072,831,663,498,258,145,226,862,004,
307,799,084,688,103,804,187,368,324,800,903,873,596,212,919,633,602,583,120,781,673,673,742,533,322,879,296,907,205,490,595,
621,406,888,825,991,244,581,842,379,597,863,476,484,315,673,760,923,625,090,371,511,798,941,424,262,270,220,066,286,486,867,
868,710,182,980,872,802,560,693,101,949,280,830,825,044,198,424,796,792,058,908,817,112,327,192,301,455,582,916,746,795,197,
430,548,026,404,646,854,002,733,993,860,798,594,465,961,501,752,586,965,811,447,568,510,041,568,687,730,903,712,482,535,343,
839,285,397,598,749,458,497,050,038,225,012,489,284,001,826,590,056,251,286,187,629,938,044,407,340,142,347,062,055,785,305,
325,034,918,189,589,707,199,305,662,188,512,963,187,501,743,535,960,282,201,038,211,616,048,545,121,039,313,312,256,332,260,
766,436,236,688,296,850,208,839,496,142,830,484,739,113,991,669,622,649,948,563,685,234,712,873,294,796,680,884,509,405,893,
951,104,650,944,137,909,502,276,545,653,133,018,670,633,521,323,028,460,519,434,381,399,810,561,400,652,595,300,731,790,772,
711,065,783,494,174,642,684,720,956,134,647,327,748,584,238,274,899,668,755,052,504,394,218,232,191,357,223,054,066,715,373,
374,248,543,645,663,782,045,701,654,593,218,154,053,548,393,614,250,664,498,585,403,307,466,468,541,890,148,134,347,714,650,
315,037,954,175,778,622,811,776,585,876,941,680,908,203,125

However, interchanging the 3 and the 5 gives:

$$^5 3 = 3\text{^}3\text{^}3\text{^}3\text{^}3 = 3^{3^{3^{3}}}$$

This apparently reasonably small number is too large to comprehend. For working from the right, the first exponentiation gives $3^3 = 27$ and the second gives $3^{27} = 7{,}625{,}597{,}484{,}987$, around 7.6 trillion. However, the third is $3^{7{,}625{,}597{,}484{,}987}$, a number that is absurdly large, X say. But we haven't finished yet. We still need to calculate 3^X. If $^5 3$ were to be denoted in place value notation, like that for $^3 5$, there are not enough particles in the observable universe to hold one digit on each. For Tony Padilla estimates that there about 3.28×10^{80} such particles, as electrons and quarks in baryons (protons and neutrons).[591] And we are just dealing with single-digit numbers.

To see how tetration can be represented recursively, in preparation for further hyperoperations, we need Donald E. Knuth's up-arrow notation, which he introduced in a paper in 1976 in *Science* titled 'Coping with Finiteness'.[592] Seeking to extend finite hyperoperations indefinitely, he defined exponentiation with this notation:

$$m^n \coloneqq m \uparrow n$$

which is m multiplied by itself n times. So the recursive exponentiation expression on page 126 can be written as:

$$m \uparrow (n+1) = (m \uparrow n) \times m$$

In the example he gave in his paper,

$$10 \uparrow 10 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000{,}000{,}000$$

which is 10 billion. Similarly, for tetration:

$$m \uparrow\uparrow (n+1) = (m \uparrow\uparrow n) \uparrow m$$

However, 10↑↑10 is much more difficult to represent as an expression. We need a tower of 10's, like this:

$$10 \uparrow\uparrow 10 = 10^{10^{10^{10^{10^{10^{10^{10^{10^{10}}}}}}}}}$$

This is 1 followed by $10^{10^{10^{10^{10^{10^{10^{10^{10}}}}}}}}$ zeros, a number that is far beyond our comprehension, even with comparatively small numbers. In general, the tower of *m*'s is *n* high

Further hyperoperations

But this finite number is puny within the overall scheme of ridiculously huge numbers. As mathematics is a generative science, apparently with no limits, mathematicians do not stop with tetration. The next step is pentation, defined recursively, without ellipses (…) and multiple parentheses, as such numbers are often presented in the literature and on the Internet:

$$m \uparrow\uparrow\uparrow (n+1) = (m \uparrow\uparrow\uparrow n) \uparrow\uparrow m$$

And hexation is defined as:

$$m \uparrow\uparrow\uparrow\uparrow (n+1) = (m \uparrow\uparrow\uparrow\uparrow n) \uparrow\uparrow\uparrow m$$

As far as I am aware, this is as far as the naming of hyperoperations goes, but we can see a pattern here. As Knuth says, "one arrow is defined in terms of none, two in terms of one, three in terms of two, and so on." Written out, we have towers of towers of towers of exponents and so on.

At this point, mathematicians introduce a further level of recursiveness, with the number of up-arrows defined as a power *p*. So we have:

$$m \uparrow^{p+1} (1) = m$$
$$m \uparrow^{p+1} (n+1) = (m \uparrow^{p+1} n) \uparrow^{p} m$$

But *p* here is just a normal counting natural number, increasing by one each time. So what would happen if increasing *p*'s were replaced by a sequence of hyperoperations, whose number of arrows is also increasing by a sequence of hyperoperations, and so on? Well, at this point, I say to myself that there is need to go on any further, bearing in mind that I prefer a counting system of 1, 2, 3, … many. These complex finite expanding structures are simply a manifestation of the underlying structure of the evolving Universe, as I feel in the utmost depth and breadth of my being. So I don't need to explore alternative notations of representing hyperoperations, such as the arrow notation of John Conway, or the fast-growing hierarchy of functions, which is an attempt to capture these recursive processes mathematically.

However, I would just like to spend a moment to explore one example of a super-large number, which arises from graph theory and combinatorics. As I explain on page 45, I view the Totality of Existence as a multidimensional graph of nodes and relationships between them, merging into the undivided Continuum, which is Absolute Reality, called in Sanskrit *Satchitānanda* 'Bliss of Absolute Truth and Consciousness'.

Fairly obviously, this exquisite experience, when the sense of a separate experiencer disappears, is all that really matters in life. As previously mentioned, this all-inclusive worldview is the basis of my meditation practice, helping me to stay comparatively sane within a war-mongering society that has mostly taken leave of its senses, ignorant of what is causing scientists and technologists to drive the pace of change in society at unprecedented exponential rates of acceleration.

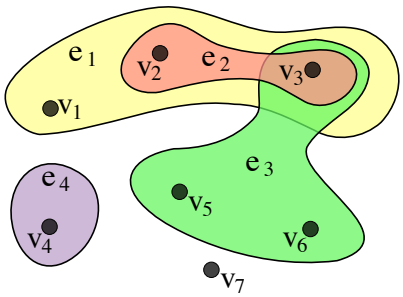
Within this context, the enormously large number I am referring to is Graham's number, made famous in 1977, when the popular recreational-mathematics writer Martin Gardner (1914–2010) introduced it to the

general public in *Scientific American* after talking to the mathematician Ronald L. Graham. At the time, it was claimed to be the largest specific positive integer ever to have been used in a mathematical proof, even being published in the 1980 *Guinness Book of World Records*.

However, the number was not actually used directly in the published proof. Rather, it was used as a simplified explanation of the upper bound of a problem that Graham had been working on that arose from his studies of Ramsey theory. As Graham's number is finite, it is a less stringent upper bound than the number actually used in the proof, but still a valid finite upper bound.

Frank P. Ramsey (1903–1930)[593] introduced the theorem that led to the eponymous theory in a posthumously published paper in 1930 titled 'On a Problem of Formal Logic'. Ramsey was seeking to find a "regular procedure to determine the truth or falsity of any given logical formula", known in German as the *Entscheidungsproblem* 'decision problem',[594] which Turing proved to be unsolvable six years later, as mentioned on page 72. For every attempt to free mathematics of self-contradictions leads back to paradoxes at the heart of the Universe, requiring a non-mechanistic system of thought to resolve, which is Integral Relational Logic, with the Principle of Unity as its only axiom.

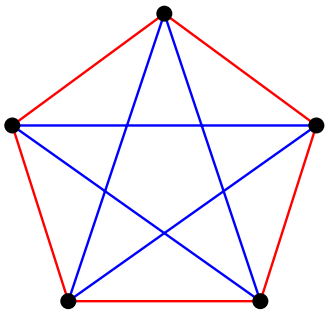
Although Ramsey was primarily concerned about a particular case in mathematical logic, Ramsey theory has been regarded as a cohesive subdiscipline of combinatorial analysis since the 1970s, embracing a number of other theorems in which numbers increase rapidly, even from small natural numbers. A construct that unifies all these theorems is that of a hypergraph,[595] with hyperedges, sometimes confusingly called edges.



As we see in the diagram on page 45, a basic simple graph consists of a number of arcs between nodes, also called edges between vertices. However, in a hypergraph, a hyperedge can join any number of vertices, not just two, defined formally as $H = (V, E)$, where V is a set of vertices and E is a set of nonempty subsets of V. In this example from Wikipedia, a set $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{e_1, e_2, e_3, e_4\} = \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_5, v_6,\}, \{v_4\}\}$.[596] In other words, V is looked at as a complete graph, in which each vertex is connected to all other vertices. Then E is a set of particular subsets of the power set $\mathcal{P}(V)$, which consists of all subsets of V, 2^n of them, where n is the size of V.

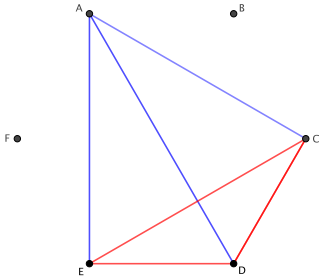
Just to check that I understand this concept, which is new to me, as I write these paragraphs, an icosahedron consists of twelve vertices, with thirty external edges, whose twenty triangular faces can be considered as hyperedges. Also, we can view an icosahedron as a pentagonal antiprism, with five pentagonal pyramids on the top and bottom, in six different ways. So there is a twelve-member set of five coplanar vertices as hyperedges, which are also called subgraphs of a complete graph with n vertices, denoted by K_n.

In other words, as the basic concepts in Ramsey theory are sets, graphs, and the relationships between their members or vertices, this theory is a special case of Integral Relational Logic, as are all theories in mathematics and all other disciplines of knowledge. However, in another sense, Ramsey theory is the exact opposite to IRL in that it is concerned with partitioning a set of vertices in order to determine how many elements of some structure there must be to guarantee that a particular property will hold.



The simplest example of Ramsey theory is well illustrated with the millions of members of Facebook, any two of whom are either friends or not friends.[597] Then Ramsey's theorem shows that any three members are all friends with each other or are not friends. This is not about probabilities, but about the properties of the relationships between them. We can illustrate this by colouring the edges

between the vertices of a graph either red or blue. The above diagram shows that no subgraph of three vertices contains a set that is all blue or all red. So, this graph shows a minimum of five in which the property does not hold.



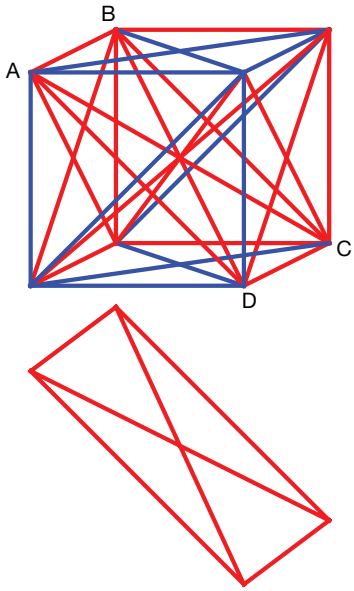
On the other hand, when we move to six vertices, we cannot avoid a subgroup of three vertices that all have the same coloured edges. For, by the pigeon-hole principle, from any one vertex, there must be at least three edges all with the same colour, as in this diagram, connecting A to C, D, and E. However, to avoid ACD, ADE, and ACE having the same colour, the edges CD, DE, and EC must all be coloured differently from the edges AC, AD, and AE, creating a subgroup CDE having edges all the same colour.

This example is an instance of the two-colour version of Ramsey's theorem, defined as "Any graph G on $n = (a, b)$ vertices contains either a clique on a vertices or an independent set of b vertices."[598] Rather surprisingly, as of 2017, exact Ramsey numbers are known only for these nine values of a and b:[599]

$R(3,3)$	$R(3,4)$	$R(3,5)$	$R(3,6)$	$R(3,7)$	$R(3,8)$	$R(3,9)$	$R(4,4)$	$R(4,5)$
6	9	14	18	23	28	36	18	25

For other values of a and b, upper and lower bounds are known for Ramsey numbers, but no constructive ways of determining these other values have been found. Brute-force methods or exhaustion proofs do not work, even with computers, for the number of enumerations is too large.[600] And only one exact value has been found for three colours, namely $R(3,3,3) = 17$.[601]

This brings us to Graham's number, which concerns this problem in Ramsey theory: "Connect each pair of geometric vertices of an n-dimensional hypercube to obtain a complete graph on 2^n vertices. Colour each of the edges of this graph either red or blue. What is the smallest value of n for which every such colouring contains at least one single-coloured complete subgraph on four coplanar vertices?" Wikipedia provides this diagram of a cube to help us understand the problem.[602] In the upper diagram, ABCD is a coplanar subgraph, with all edges coloured red, as the lower diagram indicates. However, if CD is coloured blue, there is no longer a single-coloured coplanar subgraph.



In the original paper that Graham wrote with Bruce Lee Rothschild in 1971, they gave a lower bound for n as 6 and a far greater upper bound, a huge gap between the bounds.[603] Since then the lower bound has been improved to 13 and the upper bound has been revised somewhat.[604]

However, in order to simplify the explanation of this upper bound, Graham defined a much larger number G in 1977 in this recursive manner, which Graham, himself, described in a Numberphile YouTube video in 2014:[605]

$$g_1 = 3 \uparrow\uparrow\uparrow\uparrow 3$$
$$g_n = 3 \uparrow^{g_{n-1}} 3$$

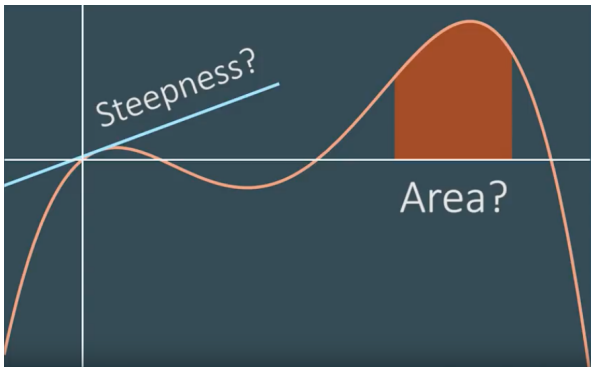
for $2 \le n \le 64$. In other words, starting with a hexation operator to establish an initial number of up-arrows, g_1 defines the number of up-arrows in g_2, which defines the number of up-arrows in g_3, and so on successively until g_{64} is reached, which is G. So this ginormous finite number is defined by just counting to 64, not very far.

Graham's number is far from being the largest finite number today. In tree theory, the study of graphs in which any two vertices are connected by exactly one path, it is possible to generate far larger enormous numbers. The best known example is the tree theorem proven by Joseph Kruskal (1928–2010) in 1960, giving TREE(1) = 1, TREE(2) = 3, and TREE(3), a number that explodes to a value so enormously large that many other 'large' combinatorial constants are extremely small by comparison.[606]

Intuitively visualizing these ever-growing hierarchical structures, what I find most fascinating about them is that they remain finite without ever reaching infinity, which is in a completely different, opposite category, balancing the finite and infinite in Wholeness, as the Transfinite Divine, beyond the categories. Just as there are an infinite number of finite primes, paradoxically, there is an infinite hierarchical sequence of ever-increasing finite numbers, which, themselves, never reach infinity. And don't forget that these numbers denote the count of a set, whose members have similar attributes, although what this attribute they all share in common is, is far from clear.

The infinitesimal calculus

Having taken the growth of finite numbers as far as is reasonable, it is time to move on to infinitely large numbers, in the next section, and infinitely small ones, in this. But rather beginning with the former, I need to return to the beginning, at zero, before moving forward to its opposite, infinity, and thence to Transfinity, as the Divine.



What we are concerned with here is differentiation and integration in the differential and integral calculus, intended to find the steepness or slope of a curve and the area under a curve, illustrated in this diagram, which Michael Harrison used in his excellent introduction to calculus on his Socratica YouTube channel.[607] The reason why zero creeps in here is that we are seeking the rate of change of the curve at a particular point, like the instantaneous velocity of a moving body. Yet speed, without considering direction, is defined as distance/time. So what is 0/0 at an instant, at zero time?

This diagram indicates that what we are considering are values of the slope at a particular point and the area under the curve between two specific points. Yet what is more significant is to find functions that indicate the way that the tangent to the curve or area under the curve change as the independent variable changes. Thus, to resolve the central dilemma of the calculus, let us first denote the curve as a function, where $y = f(x)$, where $x \in \mathbb{R}$. Then, using the standard notations of Gottfried Leibniz (1646–1716) and Joseph-Louis Lagrange (1736–1813), the differential of $f(x)$ is:

$$\frac{dy}{dx} \quad \text{or} \quad f'(x)$$

pronounced *f* prime of *x*. Conversely, the integral of $f(x)$ is given by this expression, in Leibniz's notion:

$$F(x) = \int f(x)dx$$

where ∫ is a long *s* to denote Latin *summa* 'sum, total'. Now, as we see on page 163,

$$F'(x) = f(x)$$

So integration is anti-differentiation, the opposite of differentiation, in conformity with the fundamental law of the Universe, known as the fundamental law of the calculus, not immediately obvious from the diagram.

Although James Gregory (1638–1675) had some understanding that finding areas, as quadratures, and tangents are complementary,[608] and although Pierre de Fermat (1607–1665) knew how to find tangents to curves in the form $y = x^n$,[609] Newton was the first to show that he fully understood the fundamental law of the calculus, as applied to all 'crooked lines' in whatever form. We know this from a video that Jeremy J. Gray of the Open University in the UK made in 1986 on 'The Birth of the Calculus', now on YouTube.[610]

By studying the original notebooks of Newton and Leibniz held in the University of Cambridge Library and in the Leibniz-Archiv at Niedersächsische Landesbibliothek in Hannover, Gray gives us a fascinating glimpse into mathematical discovery. On 20th May 1665, Newton showed that tangent and area problems are inverses of one another and on 13th November 1665 he developed a new method for finding tangents over an infinitely small period of time, viewing curves as depicting motion, simplifying a method that Johannes Hudde (1628–1704) had developed using the norm of circles touching curves, at right angles to the tangent.

Ten years later, Leibniz took a quite different approach, having belatedly learnt mathematics from Christiaan Huygens (1629–1695), when Newton was already Lucasian Professor of Mathematics at Cambridge. All his life, Leibniz worked to mechanize all reasoning processes and formalize the rules of logic, an endeavour that was to lead to the invention of the stored-program computer some three hundred years later. Indeed, Leibniz was made a foreign member of the Royal Society in 1673 for demonstrating an albeit incomplete calculating machine (this one is the sixth version, not completed until after his death), just as Newton had become a Fellow in 1672 for presenting a reflecting telescope that could magnify 150 times, overcoming the problem of chromatic aberration of refraction telescopes.[611]

Seeking to mechanize reasoning, Leibniz paid particular attention to developing a powerful and flexible notation in which to express his insights. Beginning with finding areas under curves, rather than finding tangents, Leibniz saw these as an infinite sum of infinitesimal ordinates, in the manner of Bonaventura Francesco Cavalieri (1598–1647), which Cavalieri called 'omn *l*', from Latin *omnia* 'all', *l* standing for the ordinates. Thinking that the term *omn* got in the way, on 29th October 1675, Leibniz replaced it with ∫, as the long script *s*. He thereby saw that areas are summations and tangents are differences, which he denoted with *dx* twelve days later, as the difference between two neighbouring *x*'s, notations that we use to this day. Using this notation, he even found the product rule for differentials:

$$d(xy) = xdy + ydx$$

Then, in mid-July 1677, Leibniz sought to develop rules for his calculus that would enable them to be amenable to an automatic process of reasoning, rules that were very similar to those of Newton. However, while Leibniz's rules were universal and general, Newton was always wedded to the language of motion, introducing the term *fluxion*—as an infinitesimal speed—in 1671, which he denoted by $\dot{x}$, calling *x*, itself, a *fluent*, a variable that changes with time.[612] For *fluxion* derives from Latin *fluxus* 'flowing', past participle of *fluere* 'to flow', also root of *fluent*.

Regarding publishing the methods of the calculus, Newton wrote three tracts on his discoveries during the ten years after they were made, but which were not published until much later. These were *De Analysi: Per Æquationes Numero Terminorum Infinitas*, written in 1669, but not published until 1711, *De Methodus Fluxionum et Serierum Infinitorum*, written in 1671, not published in English translation until 1736, with the original Latin in 1746, and *De Quadratura Curvarum*, written in 1676, but not published until 1704, as an appendix to *Optiks*.[613]

As Newton mentioned in a scholium to the first edition of *Principia* in 1687, he had begun a correspondence with Leibniz some ten years earlier, showing that the latter had developed a method "which hardly differed from mine except in the form of words and notations".[614] As well as this correspondence, Leibniz could well have been aware of some of Newton's tracts, as news of them were circulating around Europe, but as they did not tell him anything that he did not already know through his own creativity he felt free to publish his discoveries on his own.

This he did in 1684, in *Acta Eruditorum* 'Reports of the Scholars', a journal he had cofounded two years earlier in Leipzig with Otto Mencke,[615] as *Nova methodus pro maximis et minimis, itemque tangentibus, quæ nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus*, translated as 'New method for maxima and minima, and for tangents, that is not hindered by fractional or irrational quantities, and a singular kind of calculus for the above mentioned', which is how the calculus came to be so named. This first paper on the differential calculus was followed two years later with one on the integral calculus, which is the first time ∫ appeared in print.[616]

There is no need to dwell on the priority dispute between Newton and Leibniz, which broke out in 1711,[617] for it is quite clear that when we look at evolution as a whole since the most recent big bang, we see that we are all the products of these fourteen billion years of creativity, defying the second law of thermodynamics. So, in Reality, there are no separate individuals who can be said to have created anything.

When the time is right, similar ideas emerge apparently independently, as we also see with Charles Darwin (1809–1882) and Alfred Russel Wallace (1823–1913), who in the mid 1800s independently viewed evolution as the result of a process of natural selection and with Paul Baran (1926–2011) at RAND in the USA and Donald Watts Davies (1924–2000) at the National Physical Laboratory in the UK, who independently developed packet switching in the 1960s, on which the Internet is based.[618]

For as Advaita sages, like Ramesh S. Balsekar and Vijai Shankar have said, there is no doership. As none of us is ever separate from any other being for an instant—as just waves and currents abstracted from the Ocean of Consciousness, never separate from the Immortal Ground of Being—we cannot say that any of us actually does anything as egoic beings. We are simply channels for the creative power of the Divine, never separate from the Origin of the Cosmos, learning whatever we are meant to learn, as Vijai, a former medical practitioner, said in his December 2018 newsletter.[619]

But what can the differential calculus—as the mathematical science of change—tell us about what is causing the pace of evolutionary change to accelerate exponentially? Could we develop the laws of motion of society, just as Newton developed the laws of motion of lumps of matter? Well, Karl Marx believed that he had discovered such laws in Georg Hegel's dialectic,[620] even though he changed Hegel's *Geist*, as spirit or mind, into dialectical materialism, where matter is primary and consciousness is secondary.[621]

However, Karl Popper was critical of such a claim, as he was of Marx and Plato's authoritarian vision of a utopian society, described in the two-volume *The Open Society and Its Enemies*. Specifically, in *The Poverty of Historicism*, he said, "The hope … that we may some day find the 'laws of motion of society', just as

Newton found the laws of motion of physical bodies, is nothing but the result of these [holistic] misunderstandings. Since there is no motion of society in any sense similar or analogous to the motion of physical bodies, there can be no such laws."[622]

Yet holism is the very essence of evolutionary development, as we see on page 50, because wholes emerge that are greater than the sum of their preceding wholes from the new relationships that are formed. As these apparently arise out of nothing, it is not possible to model them quantitatively with conventional mathematics. To develop a mathematical model of such evolutionary processes, we need a holistic system of thought that embraces meaningful relationships in a semantic fashion.

'Holistic misunderstandings' arise from the fragmented, divergent manner in which evolution has unfolded over the years. For, what Popper is effectively saying is that we cannot develop the much-needed science of humanity within the materialistic, mechanistic worldview that has dominated scientific research since the early 1660s, when the Royal Society rejected Comenius' attempt to establish a Pansophic Academy of Divine Light, twenty years earlier, as mentioned on page 80.

Neither can Leibniz's attempt to automate human reasoning help us directly, for we humans are not machines and nothing but machines. Rather, we can turn to the semantic modelling methods of information systems architects in business to help us in this endeavour, as I outline in the first two chapters of this book. For such modelling methods potentially map all psychological activities in society, whether these are performed by humans or machines.

Because evolutionary divergence has become convergent within me, I have been able to develop a holotropic, nonaxiomatic science of reason, enabling me to embrace paradoxes and self-contradictions in a thoroughly coherent worldview, resolving some of the confusions that arose in mathematics and its philosophy in the nineteenth century, still prevailing to this day. For although the calculus had emerged in consciousness at the end of the seventeenth century, it had not become fully assimilated into mathematics because of people's psychological difficulties relating to the notions of infinity and its opposite the infinitesimal, used as a noun. A number of mathematicians sought to resolve these difficulties with the rigorous definition of a limit, both of the convergence of an infinite series, which we look at in the next chapter, and in the infinitesimal calculus.

Yet, as Benjamin Lee Buckley indicated in *The Continuity Debate*, the notion of a limit in mathematics is dependent on the concept of the continuity of ordered sequences of numbers. As Fredrick F. Schmitt, a professor of philosophy, said in his endorsement, this book is "a highly original and insightful study of one of the most puzzling subjects in the world, the continuity of numbers in real analysis." In Buckley's words: "Some have viewed actual infinity as simply impossible, others have viewed it as simply factual, and still others as mystical, mysterious, or even divine," like Spinoza.[623]

So, to what extent is it possible to say that there is a genuine concept of numerical continuity in mathematics? Well, as this is a book on *Unifying Mysticism and Mathematics*, I endeavour to show how we can answer this question within the overall scheme of things, regarding the mystical Weltanschauung as primary. At the heart of this resolution is the transformation of one-sided, either-or reasoning into an all-embracing, both-and approach to life, caring for everyone with compassion and tolerance.

To illustrate, the infinitesimal calculus has evolved into analysis, as the branch of mathematics concerned with "the study of *dependencies among variable quantities*". Nothing else has influenced the development of modern scientific thinking as deeply, for as Hans Niels Jahnke writes in the Introduction to *A History of Analysis*, "The basic idea of using differential equations to gain insight into the global behaviour of varying

quantities from their (infinitesimal) changes has proved fundamental and fruitful far beyond mathematics and physics and has shaped our overall scientific view of the world, especially our notion of causality."[624]

Yet, mathematical analysis cannot tell me what is causing me to write this coherent book on *Unifying Mysicism and Matematics*, for *analysis* derives from Medieval Latin *analysis*, from Greek *analusis* 'a breaking up, dissolution, end', from *analūein* 'to dissolve, detach, unloose', from *ana-* 'throughout' and *lusis* 'a loosening, separation', from *lūein* 'to loosen', from PIE root **leu-* 'to loosen, divide, cut apart'.

Mathematical analysis thus has a similar approach to human learning as reductionist science: by cutting everything up into pieces. For the root of *science* is Latin *scīre* 'to know', from PIE base **skei-* 'to cut, split', meaning 'to separate one thing from another, to discern', also the root of *schizophrenia* 'split mind' and *consciousness* 'knowing together'. Yet when we conduct science solely in this analytical manner, supposedly the key to systemic, logical reasoning, we throw the interesting parts away, as shown on page 23.

We see a similar divisiveness in the root of the word *fractal*, which Benoît B. Mandelbrot (1924–2010) coined in 1975, like me, an employee of IBM for a time. As he says, *fractal* derives from Latin *frāctus*, past participle of *frangere* 'to break, create irregular patterns', also root of *fragment*, *fracture*, and *fraction*. Interestingly, Mandelbrot points out that *fractal* is the etymological opposite of *algebra*, which derives from Arabic *al-jabr* 'the reunion of broken parts, bone-setting, the surgical treatment of fractures', from *jabara* 'reunite, restore'.[625] The mathematical sense comes from the title of a book *ilm al-jabr wa'l-muqābala* 'the science of restoring what is missing and equating like with like', by the mathematician al-Ḵwārizmī around 820.[626]

Another opposite to *science* is *art*, which is a synthesizing activity, putting back together what science has divided, for *art* derives from Latin *ars* 'skill, way, method', from PIE base **ar-* 'to fit together', also root of *coordinate*, *reason*, *harmony*, and *order*. And *synthesis* derives from Greek *sunthesis* 'composition, logical and mathematical synthesis', from *suntithenai* 'to place together, construct, comprehend', from *sun-* 'together' and *tithenai* 'to put', from PIE base **dhē-* 'to set, put'.

How we view the opposites of *analysis* and *synthesis*—as typifying science and art, respectively—depends very much in the direction we are coming from, on whether we begin with the superficial or the profound. In Reality, when we transcend the categories, no being is ever separate from any other, for distinct beings don't actually exist; they are merely illusions, called *māyā* in the East, as appearances in or abstractions from Consciousness.

I can best illustrate this by starting with Newton's notion of a fluxion, as an infinitesimal speed, indicating something that is flowing with time. However, we can hardly say that a body in motion, whose speed we are endeavouring to calculate, is flowing. A more meaningful metaphor for an undivided flowing movement in the physical universe is that of a stream, from Old English *strēam*, from Proto-Germanic **strauma-*, from PIE base **sreu-* 'to flow', also root of Greek *rheos* 'stream', from *rhein* 'to flow', from which we get *rheostat* 'a continuously variable electrical resistor used to regulate current', literally stopping the flow.

This notion of a stream led Bohm to visualize reality as a holomovement underlying the ever-changing world of form, inspired by the process philosophy of Heraclitus and Whitehead, as we see on page 85. Bohm also saw thought or intelligent thinking as an undivided process, as I experience, contrary to the structure of English and other European languages, which regard the noun, acting as an agent, as the primary part of speech, rather than the verb, typically denoting an action. Syntactically, Bohm attempted to resolve this problem with his linguistic rheomode—the flowing mode of language—putting the verb first,[627] like in

Native American languages, which don't even have nouns.[628] Not surprisingly, this initiative did not take off.

Nevertheless, to understand this, we need to remember that mathematical objects, as the territory being mapped, lie in the psyche, not in the physical universe. The diagram on page 133, which you and I can see in our external worlds, is actually a projection or expression of what we can see within, as the etymology of these words indicates.

However, such a picture does not represent thought as an undivided flowing movement. To visualize and experience what this means, I need to apply Aurobindo's vast, undivided Supermind, at the heart of my meditation practice, as I mention on page 75. From this Holoramic perspective, everything makes sense. However, the Abrahamic religions discourage living in union with the Divine, historically accusing mystics who are awarely doing so of heresy, such as Meister Eckhart in 1329, who Eckhart Tolle is named after. In the event, Eckhart died of natural causes before he could be burnt at the stake,[629] unlike Giordano Bruno, who the inquisitors brutally executed in 1600 for saying, "that law of love that is spread far and wide … which derives … from God the father of all things so that it is in harmony with all nature," "is the religion that I observe".[630]

For myself, I have needed to defy the most fundamental taboo of the culture I was born in in order to heal my fragmented mind and split psyche and to solve the most critical unsolved problem in science, explaining what is causing scientists and technologists, like myself, to drive the pace of change at an unprecedented exponential rate of acceleration. I do not need the differential calculus to explain this, as society is not a body in motion. Rather, I have paradoxically needed the discrete mathematics of the logistic map in chaos theory, related to fractals, as I describe in my book *Through Evolution's Accumulation Point: Towards Its Glorious Culmination*. Viewing the whole of evolution as a dripping tap, since 2004, give or take a couple of years, the evolutionary tap has been turned full on, as an undivided flowing movement, with no more discrete evolutionary turning points to be discerned.

At heart, this is the most vital existential issue that we humans have ever faced during our comparatively brief sojourn on our beautiful planet Earth, while the climate has been amenable to human societies, as they have become ever more domesticated and technological. However, this short window in time is now coming to an end, as we enter the end times of our species, as visionaries have long foreseen. It is thus from the perspective of the Absolute Continuum—transcending the categories, including egoic time—that I look at the notion of numerical continuity.

To begin with simplicity, seeking conceptual clarity and coherence, I see zero and infinity as inverses of each other. For:

$$\frac{1}{0} = \infty \quad \text{and} \quad \frac{1}{\infty} = 0$$

It thus seems that zero and infinity are the extremes of the spectrum of the domain of numbers. However, we should not forget that in Integral Relational Logic, the concept of number first emerges in consciousness as a secondary attribute of sets, as interpretations of the meaningless data patterns of experience, outlined on page 51. Initially, numbers enumerate sets, gathering those with the same count together into a set of sets with the attribute of three, let us say.

But then numbers take on a life of their own, becoming members of sets themselves. So {1, 2, 3} has the same count as {red, green, blue}. So in what sense is the set of numbers, of whatever class, continuous? I

am particularly concerned here with the meaning of *continuity*, as it applies both to the function $f(x)$ and to the independent variable x.

Continuity derives from Latin *continuāre* 'to connect up, unite, make continuous', from *continuus* 'connected up, continuous, unbroken, uninterrupted', from *continēre* 'to hold or keep together', from *com-* 'together with' and *tenēre* 'to hold', from PIE base **ten-* 'to stretch', also root of *contain* and *tantra*, from Sanskrit *tantram* 'loom', an instrument for unifying the opposites of warp and weft in a whole piece of cloth.

Yet, as we see on page 107, Cantor showed that even a finite section of the number line consists of an uncountable number of distinct values in the class **Real number** or $\mathbb{R}$. This he called the continuum ($\mathfrak{c}$), which he proved is the power set of $\aleph_0$, as the countable set of integers, rationals, and real algebraic numbers in one-to-one correspondence with the natural numbers $\mathbb{N}$, as we see in the next section. But is the continuum of real numbers, today also denoted by $\beth_1$, really unbroken and undivided?

Clearly, $\aleph_0$ or $\beth_0$, as the count of the set of the countable numbers, is not, for it contains an uncountable number of gaps, filled with transcendental numbers ($\mathbb{T}$). But neither is $\beth_2$, as the power set of $\beth_1$. Yet, what are the elements of the set whose count is $\beth_2$? Are they numbers? And what does the set whose cardinality is $\beth_\infty$ consist of?

This infinite sequence of infinite power sets has an inverse sequence of what we can call infinitesimals, although I have not found many mathematicians or philosophers of mathematics who view these opposites in this way. For this is how mathematicians define *infinitesimal*, as James Grimes tells us in a Numberphile YouTube video:[631]

$$0 < \varepsilon < r$$

where r is any real number, no matter how small. So ε, as a putative infinitesimal, lies strictly less than r, but not actually zero. In this definition, *infinitesimal* is thus the inverse of a finite number, which can increase indefinitely, without ever reaching infinity, as we see on page 133.

Yet, it makes more semantic sense to me to regard zero as the opposite of infinity. So what are infinitesimals in Panosophy, as a coherent body of knowledge? Well, the word *infinitesimal*—meaning 'indefinitely small value approaching zero' or 'smaller than any assignable fraction or magnitude'—derives from Modern Latin *infīnītesimus*, from Latin *infīnītus* 'unbounded, infinite', in the style of *centesimus* 'hundredth' or *millesimus* 'thousandth'. The OED tells us that when *infīnītesimus* was formed in New Latin in the 1600s, it was originally meant as an ordinal, viz. the 'infiniteth' in order, as the hundredth (part) 1/100 or thousandth (part) 1/1000.

So we can also view the infinite sequences of infinitesimals as the inverse of the sequence of ordinals, $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, …. I don't find it necessary that these infinitesimals should have any existence in so-called physical reality. For mathematical objects exist as concepts in the psyche, so their study is a branch of psychology, formed in exactly the same way as all other concepts. All I am concerned with is that my cognitive map of the Totality of Existence maintains conceptual clarity and coherence, for this holistic approach to learning gives me great joy and satisfaction, leading to Love, Peace, Wholeness, and the Truth.

Nevertheless, as this psychospiritual way of looking at mathematical structures seems to be quite new, it is perhaps not surprising that the related concepts of continuity, infinitesimal, and infinity have troubled the Western mind ever since Aristotle and Archimedes. So, let us briefly review what has been going on since Newton and Leibniz formulated the infinitesimal calculus in the second half of the seventeenth century.

The rationalist theologian Henry More (1614–1687) first used *infinitesimal*, as an infinite ordinal, in 1655 in the Appendix to the second edition of *An Antidote against Atheism*, writing: "An infinite mind might well comprehend an infinite number. But for us whose capacities are finite, if we would venture to name a *first* in infinite succession, we should call it the *first infinitesimal*, and acknowledge ourselves unable to go through, our understandings being finite."[632]

Charles Hayes (1678–1760) first used *infinitesimal* in a mathematical sense, albeit in New Latin form, in *Treatise on Fluxions* in 1704, the first English work explaining Newton's method of infinitesimals: "These infinitely little Parts of an infinitely little Part of a given Quantity are by Geometers called *Infinitesimæ Infinitesimarum* or Fluxions of Fluxions. … Because this doctrine may seem hard to most readers at first, I shall endeavour to prove that there are Quantities infinitely less than a given Quantity, which are also infinitely greater than another Quantity; and consequently, if there be Quantities infinitely little, there are others infinitely less than they."[633]

Then, six years later, the idealist philosopher George Berkeley (1685–1753) wrote in 1710 in *The Principles of Human Knowledge*:

> Of late, the speculations about Infinities have run so high, and grown to such strange notions, as have occasioned no small scruples and disputes among the geometers of the present age. Some … not content with holding that finite lines may be divided into an infinite number of parts, do yet farther maintain that each of these infinitesimals is itself subdivisible into an infinity of other parts or infinitesimals of a second order, and so on *ad infinitum*. These, I say, assert there are infinitesimals of infinitesimals of infinitesimals, etc., without ever coming to an end.

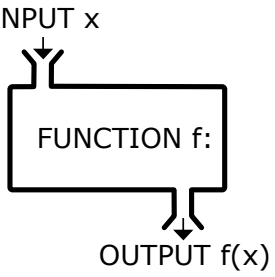
Berkeley at first thought that reducing measures to 'nothing at all' was absurd, leading to the destruction of the very foundations of geometry, writing, "those great men who have raised that science to so astonishing a height have been all the while building a castle in the air." Yet, he added, "To this it may be replied that whatever is useful in geometry, and promotes the benefit of human life, does still remain firm and unshaken by our principles—that science considered practical will rather receive advantage than any prejudice from what has been said."[634] Nevertheless, he was to write in 1734 in *The Analyst* that fluxions are "ghosts of departed quantities".[635]

These 'ghosts of departed quantities' greatly disturbed mathematicians and philosophers of mathematics in the 1800s, who set out arithmetize analysis, banishing infinitesimals from mathematics and shifting the emphasis from geometric to algebraic reasoning. As I read this evolutionary process, by freeing analysis of dependency on space and time, mathematics became more generalised. This led to the emergence of universal algebra, including abstract, linear, and Boolean algebra, freeing mathematics of the tyranny of number, outlined in Chapter 5.

These generalizing processes led to all-powerful Integral Relational Logic, as the apotheosis of human reason, outlined in Chapter 2, which could have freed business from its attachment to money, as a lowest common denominator, giving more meaning to our lives. Sadly, however, it seems that taking such a leap is too great for many to take, despite such visionaries as Peter Russell, Jean Houston, and Steve Taylor urging us to do so in their books, *The Global Brain Awakens: Our Next Evolutionary Leap*, *Jump Time: Shaping Your Future in a World of Radical Change*, and *The Leap: The Psychology of Spiritual Awakening*, respectively. Now, with abrupt climate change about to make our beautiful planet Earth uninhabitable, it seems that it is too late to jump out of the dark ages into the eschatological Age of Light.

Nevertheless, let us continue our exploration of how mathematics has led us to where we are today. At the core of the arithmetization of analysis lay the concept of function, today at the heart of business procedures, whether performed by humans, acting as machines, or computers. In the 1700s, Euler denoted a function as $f(x)$, illustrated with the following diagram from Wikipedia, like the fundamental data-

processing structure in computer science on page 4. This exercise was not only about the development of new concepts in mathematics. It was just as much about ensuring conceptual clarity, expressed in articulate, meaningful definitions, albeit within the conventional axiomatic, linear structure of mathematics, which tends to distort and limit reasoning. For concept formation is dependent on the assumptions contained within the context in which meaningful concepts are formed.



For instance, Victor J. Katz tells us that between 1718 and 1888 mathematicians formulated several definitions for *function*. Perhaps the simplest and clearest is this definition from Eduard Heine (1821–1881) from 1872, who was principally a university teacher, helping to bring clarity to analysis: "A single-valued function of a variable x is an expression which for every single rational or irrational value of x is uniquely defined."[636]

But what is x, as input to a function, in real analysis? The input to a computer is generally in the form of characters in Unicode or binary images, interpreted as active or passive data in the machine. But an input variable in analysis can be any number from the real domain, most frequently denoted in an infinite string of digits, in some defined base. And how can mathematicians avoid Leibniz's infinitesimals when functions are differentiated or integrated?

Well, I have spent some weeks consulting several books on the history of mathematics attempting to make sense of this evolutionary process. I am reminded here of Arthur Koestler's *The Sleepwalkers*, which narrates scientists' strivings over thousands of years to understand just the dynamics of moving bodies, whether on Earth or in the sky. Kepler tells us about his own struggles in *New Astronomy*, not just presenting the results of his calculations from Tycho's many measurements. He compared his presentation process to the journeys of the great explorers, saying, "in telling of Christopher Columbus, Magellan, and of the Portuguese, we do not simply ignore the errors by which the first opened up America, the second, the China Sea, and the last, the coast of America; rather we would not wish them omitted, which would indeed be to deprive ourselves of an enormous pleasure in reading."[637]

To a great extent, we are still sleepwalkers, or rather sleeprunners or sleepdrivers, as synergistic evolutionary effects drive change in society ever faster, for reasons that few are yet aware of, despite the efforts of mathematicians and philosophers of mathematics to establish mathematics on a sound foundation during the past two hundred years. Some mathematicians think that they have cast infinitesimals out of mathematics by developing a precise definition of a limit, which presupposes a clear definition of numerical continuity, in contrast to Undivided Wholeness, as the Absolute Continuum, at the superficial and profound levels of existence.

Yet, when I take a both-and perspective, I don't see any contradictions between limits and infinitesimals, which perhaps I can explain by outlining the attempts of some of major players to clearly define these concepts up to the end of the 1800s, just as the major crisis in the foundations of mathematics became apparent, still not resolved satisfactorily: Bernard Bolzano (1781–1848), Augustin-Louis Cauchy (1789–1857), Karl Weierstrass (1815–1897), Richard Dedekind (1831–1916), Paul du Bois-Reymond (1831–1889), Charles Sanders Peirce (1839–1914), and Georg Cantor (1845–1918). So here are a few thoughts on these gentlemen that have emerged in consciousness during the last few weeks of 2018 and the first of 2019, focusing more on the psychohistory of the infinitesimal calculus than on the technical details, which, for me, are rather distracting, taking me away from the Wholeness that is my True Nature, like everyone else's.

Bolzano is a particularly fascinating figure in the history of ideas, whose life and work I have had little opportunity to study,[638] unlike some other pioneering outsiders, like Kepler, Comenius, and Peirce, who I rank among my kindred spirits, much inspiring my own endeavours. From what I have been able to learn so far, Bolzano took an all-encompassing, foundational approach to human learning, albeit constrained by his cultural conditioning, not unlike Peirce. Both were dismissed from their posts, subsequently writing voluminously, leaving much unpublished in manuscript form.[639]

Bolzano's grandfather moved from Lombardy in Italy, when his father was in childhood, his mother being a German-speaking Bohemian. At fifteen, he went to the Charles University of Prague to begin 'philosophical studies', including philosophy, history, languages, biology, mathematics, and physics. Then, after studying theology for four years, against his merchant father's wishes, he was ordained as a Roman Catholic priest on 7th April 1805.[640]

A few days later, he received his doctorate of philosophy for a treatise on geometry, much influenced by *Mathematische Anfangsgründe* by Abraham Gotthelf Kästner (1719–1800), who took great care to prove many results which were thought 'obvious', so not requiring proof, by other mathematicians of his day. Bolzano adopted a similar approach to philosophical questions in mathematics, writing, "My special pleasure in mathematics rested therefore particularly on its purely speculative parts, in other words I prized only that part of mathematics which was at the same time philosophy." Then in the Preface to his thesis, he wrote, "I could not be satisfied with a completely strict proof if it were not derived from concepts which the thesis to be proved contained, but rather made use of some fortuitous, alien, intermediate concept, which is always an erroneous transition to another kind."[641]

Being an outstanding student, the Austrian Emperor, Kaiser Franz (Francis I),[642] appointed Bolzano to a newly established chair for religious doctrine at the University, Bolzano becoming *professor ordinarius* the following year. However, rather than giving homilies for 'improving religious instruction', "Bolzano alienated many faculty and church leaders with his teachings of the social waste of militarism and the needlessness of war. He urged a total reform of the educational, social and economic systems that would direct the nation's interests toward peace rather than toward armed conflict between nations."[643]

Bolzano also put his ethics and political philosophy into practice "by his activities for the benefit of the poor, subjugated, discriminated and disadvantaged people. Together with his friends and pupils he supported activities in favour of such things as poorhouses, homes for the blind, loan banks for the working-class, and libraries and elementary schools in the countryside."[644]

However, urging Christians to seek peace, not make war, did not go down too well with the Austrian Emperor, whose State Chancellor Klemens von Metternich had created a police state, seeking to censor and suppress progressive political ideas and activities, like those of Bolzano's. So, Emperor Franz dismissed Bolzano from his post in 1820, when he was thirty-nine, henceforth being forbidden to teach, preach, or publish.[645] This was a blessing in disguise, for being free of teaching duties, Bolzano was able to pursue his inquiries in his own way, albeit it in exile in the countryside with friends Josef and Anna Hoffmann,[646] only able to publish his work, if at all, in obscure Eastern European journals.[647]

This freedom led Bolzano to somehow publish the four-volume *Wissenschaftslehre* (*Theory of Science*) in 1837,[648] regarding logic, as the science of reason, as a composite of the *theory of foundations*, the *theory of elements*, the *theory of knowledge*, the *art of discovery* (*heuristics*), and the *theory of science proper*. However, this book is unmistakably antipsychologistic, maintaining a strict separation of logic from psychology, like Peirce, Frege, and Russell were later to do. Some thus say that Bolzano was one of the greatest logicians in

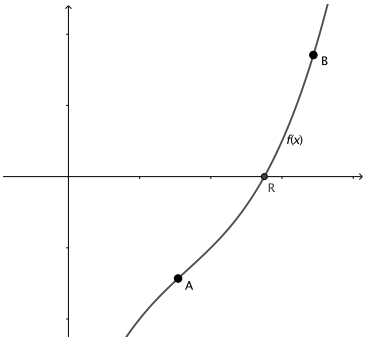
the period between Leibniz and Frege, anticipating, almost exactly 100 years before Tarski and Carnap, their semantic definitions of logical truth and logical consequence,[649] which need not concern us.

Regarding Bolzano's mathematical works, Steve Russ has translated nine pieces, which I don't have time to study in detail. Nevertheless, Bolzano tells us of his general approach in 1804 in the Preface to his D. Phil. thesis titled *Considerations of Some Objects of Elementary Geometry*, where he writes that while studying mathematics is useful in its *application* to practical life, such a pursuit has the secondary benefit of sharpening the mind through the beneficial promotion of a *thorough way of thinking*.[650] I resonate with this very much, as it has led to Integral Relational Logic.

Then, in 1810, in *Contributions to a Better-Grounded Presentation of Mathematics*, Bolzano says that of all the sciences, mathematics "comes nearest to the ideal of perfection", having "more precision and clarity in the concepts, and more certainty and convictions in the judgements, than will be found at present in the most perfect textbook of metaphysics".[651] Little did he know that later attempts to give mathematics a sound foundation were to lead to a loss of certainty, from which it has still not recovered.

In 1816 and 1817, while still working as a Professor of Religious Doctrine, Bolzano then wrote and published in Prague and Leipzig three fundamental pieces on the foundations of analysis, particularly related to the continuity of functions, eschewing the need to consider the infinitely small. His titles, carefully chosen to attract attention from specialist mathematicians—not very easy, as he reflected in the second in 1817—had these imposing titles: *The Binomial Theorem and as a Consequence from it the Polynomial Theorem and the Series which serve for the Calculation of Logarithmic and Exponential Quantities proved more strictly than before*; *Purely Analytic Proof of the Theorem that between any two Values, which give Results of Opposite Sign, there lies at least one real Root of the Equation*; and *The Three Problems of Rectification, Complanation and Cubature, solved without the consideration of Archimedes and without any assumption which is not strictly provable*. To the last of these, he added, "This is also being presented for the scrutiny of all mathematicians as a sample of a complete reorganization of the science of space," quite an undertaking!

The most significant of these treatises was the second, addressing what is today known as the Intermediate Value Theorem. As Bolzano said, this is often obvious geometrically, as we can see from this diagram. Here $f(a) < 0 < f(b)$, so there must be at least one value $f(r)$ between $f(a)$ and $f(b)$ which is zero, denoting r as a root of $f(x)$. However, mathematicians are constantly creating functions where such geometric relationships are not so obvious. So in the interests of bringing rigour to analysis, a movement arose at the beginning of the 1800s to find a definition of the continuity of a function that is not dependent on geometry.



Here is Bolzano's definition: "A function $f(x)$ varies according to the law of continuity for all values of x inside or outside certain limits means only that if x is some such value, the difference $f(x + \omega) - f(x)$ can be made smaller than any given quantity provided ω can be taken as small as we please." Drawing on his earlier essay on polynomials, Bolzano was thus able to prove analytically this theorem:

> If a function of the form
>
> $$x^n + ax^{n-1} + bx^{n-2} + \cdots + px + q,$$
>
> in which n denotes a positive integer, takes a *positive* value for $x = \alpha$, but a *negative* value for $x = \beta$, then the equation
>
> $$x^n + ax^{n-1} + bx^{n-2} + \cdots + px + q = 0$$
>
> has at least one *real root* lying between α and β.[652]

But more than this. He also proved this theorem, as the least-upper-bound property of real numbers, which is probably the first formulation of a result that came to be known as the Bolzano-Weierstrass theorem:

> If a property *M* does not apply to all values of a variable quantity *x* but does apply to *all* values *smaller* than a certain *u*, then there is always a quantity *U* which is the greatest of those of which it can be asserted that all smaller *x* possess the property *M*.[653]

However, in the event, specialist mathematicians were not to discover Bolzano's innovative insights for another fifty years.[654] So Cauchy is generally considered as the first mathematician who began to bring rigour to mathematical analysis, perhaps not surprising, for he was a rather conservative gentleman.[655] His master work, *Cours d'analyse* (*Course of Analysis*) from 1821, was designed for students at École Polytechnique, where he was teaching, and was concerned with developing the basic theorems of the calculus as rigorously as possible. In it, he gives this definition of *continuity*, remarkably similar to that of Bolzano: "The function $f(x)$ will be, between two assigned values of the variable x, a continuous function of this variable if for each value of x between these limits, the [absolute] value of the difference $f(x + \alpha) - f(x)$ decreases indefinitely with α."

Without going into too much detail on Cauchy's contribution at the moment, I need to complete this passage on numerical continuity with that of Weierstrass, who became a mathematician against his father's wishes. At university, he was enrolled on a course to study law, finance, and economics with the view of becoming a civil servant in the Prussian administration. To resolve the inner conflict between duty and inclination that thus arose within him, he led a carefree student life, studying neither mathematics nor his planned course, leaving university without a degree. [656]

Weierstrass's experience is a lesson for us all at the present time. No matter what specialist subject that we might choose to study, mathematics, logic, and all the sciences are based on a denial of the truth of the fundamental law of the Universe. So, if we are to peacefully live in harmony with the Hidden Harmony, we are bound to be disloyal to our parents, in particular, and society, in general, inner conflicts that can only be resolved in Nonduality.

In Weierstrass's case, with the assistance of Christoph Gudermann (1798–1852), his only significant mathematics teacher, he eventually qualified as a school teacher, spending many years of 'unending dreariness and boredom', in his words. However, after publishing a significant paper 'On the Theory of Abelian Functions' in *Crelle's Journal* in 1854, mathematicians noticed his genius, with the University of Berlin offering him a professorship in October 1856, at the age of forty-one. Weierstrass was thus "a striking exception to the common notion that a mathematician of first rank must make his mark early in life", eventually becoming known as the 'Father of Modern Analysis'.[657]

As far as I can ascertain, Weierstrass did not earn this designation by writing a learned treatise on analysis. Rather, his lectures were so popular that he attracted students from all over the world, who subsequently spread his methods to others, with some, such as Wilhelm Killing (1847–1923) and Adolf Hurwitz (1859–1919), publishing their lecture notes in 1868 and 1878, respectively.[658]

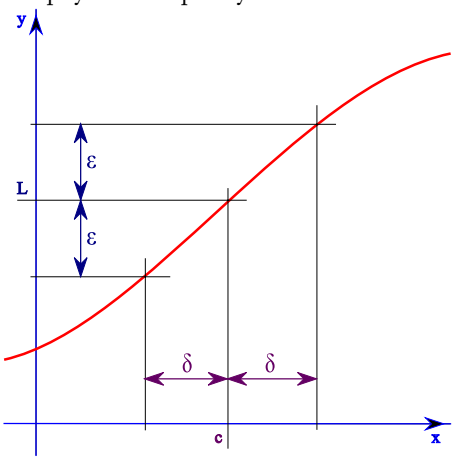
Weierstrass's most notable student was Sofia Kovalsky (1850–1891), née Korvin-Krukovsky, the daughter of a Russian artillery general, of minor nobility.[659] Not being permitted higher education as a woman in Russia, Sofia entered into a marriage of convenience in order to study abroad. She was thus led to Weierstrass, who privately tutored her, as she was not allowed to attend his lectures. During the four years of

Weierstrass's tutelage, Sonya, to use her Russian pet-name, wrote three outstanding papers, which enabled her to be awarded a doctorate in 1874 at the University of Göttingen, which she never actually attended.

After many years not recognized by academia, Sofia Kovalsky's prominence as a professional mathematician reached its peak in 1888, when she received the famous Prix Bordin from the French Académie des Sciences, leading her to be recognized as the most outstanding female mathematician of the 1800s, even by her compatriots. A year later, she was appointed as a full professor of mathematics at the University of Stockholm, where she died in an influenza epidemic aged just forty-one.[660] To summarize her life, in 1895, Anna Carlotta Leffler wrote this in a biographical memoir, with which I naturally resonate very closely:

> It was her great object to find the logical connection between all manifestations of life, as for instance, between the laws of thought and the outward phenomena. She could not satisfy herself with seeing in part, and understanding in part; it was her delight to dream of a more perfect form of life, where, according to the apostle, "we shall see no longer in part, but face to face". To see the unity in the variety was the aim and end of all her philosophy and her poetry.[661]

Regarding analysis itself, despite suffering much physical distress in later life, perhaps because of his psychological difficulties as a student, Weierstrass clarified and made more rigorous Cauchy's 1821 definition of the limit of a function in mathematics: "If the successive values attributed to the same variable approach a fixed value, such that they finally differ from it by as little as one wishes, this latter is called the limit of all the others."[662] In modern terminology, if $f(x)$ is a real-valued function defined on a subset D of the real numbers, this expression



$$\lim_{x \to c} f(x) = L$$

means that for every $\epsilon > 0$ there exists a δ such that, for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.[663] This formal (ε, δ)-definition of limit supposedly freed mathematical analysis of Leibniz's dreaded infinitesimals, whatever they might be. As δ and ε can be as small as one wants, without ever becoming zero, they act like the reciprocals of finite numbers, which never reach infinity, no matter how large. Just as extremely large numbers take a quantum jump to reach infinity, extremely small numbers, like δ and ε, leap into zero as a limit. Infinities and infinitesimals, as reciprocals of infinities, are in a different category from finite numbers, no matter how large or small.

But before we look at the real-number continuum in more detail, we are not quite done with Bolzano's philosophical reflections on the foundations of mathematics. While working on his four-volume *Theory of Science*, about 1830 he began working on a multivolume logical and mathematical work titled *Theory of Quantity* (*Größenlehre*), in which he had a change of heart from his earlier negative attitudes towards the infinite in all its forms.[664]

However, none of these writings were published in Bolzano's lifetime, remaining in manuscript at the time of his death, not unlike Peirce's own endeavours to lay down the logical foundations of mathematics some fifty to seventy-five years later. Wanting his reflections to be published one day, he left his manuscripts to Robert Zimmermann, one of his students. However, Zimmermann, who was twenty-four years old at the time of Bolzano's death, was more interested in philosophy than mathematics, so they did not begin to be published until Martin Jašek discovered them in the National Library in Vienna in the 1920s. The Czech Academy of Sciences then set up the Bolzano Committee to acquire, unify, and publish Bolzano's

manuscripts, belatedly publishing two chapters of *Größenlehre*, as *Functionenlehre* '*Theory of Functions*' and *Zahlentheorie* '*Theory of Number*', in 1930 and 1931, respectively.[665]

The one exception was *Paradoxien des Unendlichen* (*Paradoxes of the Infinite*), which Bolzano wrote during the last few months of his life at the age I am now. Franz Příhonský published this book in 1851, with a quote from Leibniz on the front cover attesting to the actuality of infinity on the front cover, with the Jesuit Donald A. Steele translating it into English in 1950.[666]

As ever, Bolzano began this book with the basics. Using the abstract notion of negation, he said that if we have the finite, we must also have the infinite, implicitly applying the fundamental law of the Universe. Applying this principle to unordered multitudes or sets (German *Menge*) of units of a certain kind—measured by quantities, at the heart of mathematics, as he conceived it—there must be both infinitely great and infinitely small magnitudes,[667] giving rise to many contradictions, which he set out to explore.

For instance, applying the Principle of Unity again, he wrote that it cannot be correct to say that the infinitely small, as mere zero or Nothing, is opposite to the infinitely large. As another example of a contradiction, which Galileo observed, Bolzano wrote:

> Every infinite multitude [set], not only the points in a line, can be divided into parts which themselves contain infinite multitudes, indeed into infinitely many such parts. For if ∞ denotes an infinite magnitude, then $\frac{\infty}{2}, \frac{\infty}{4}, \frac{\infty}{8}, \ldots$ are *infinite multitudes*. Thus it is with the concept of the infinite.[668]

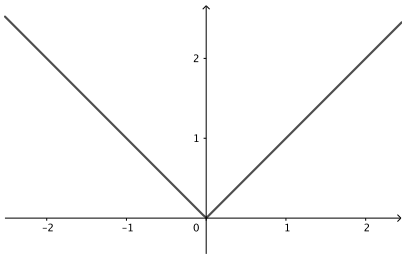
Bolzano also addressed the thorny question of what this alternating series means:

$$a - a + a - a + a - a + \cdots in\ inf$$

He quoted a paper from 1830 in a reputable mathematical journal where a person signified as *M. R. S.* attempted to prove that this sums to *a*/2. Alternatively, by grouping pairs of terms in the sequence odd-even or even-odd, falsely assuming that the associative law of arithmetic holds here, this series would total to either zero or *a*, showing that the series is not an actual quantity,[669] a situation we look at further in the next chapter.

There is no need to go into further detail. Once again, we see that Bolzano was a man living far ahead of his time, prefiguring Cantor's transfinite set theory and the turmoil that arose in mathematics as the result of the paradoxes that were to be discovered in its very foundations, still not satisfactorily resolved to general acceptance.

Of the chapters of the *Theory of Quantity* that Steve Russ has translated from Bolzano's *Collected Works*, his construction of a continuous function that is nowhere differentiable stands out in a chapter titled *Theory of Functions*, not published until 1930 and not translated into English until 2004.



It is well known that $f(x) = |x|$ is not differentiable at $x = 0$, for

$$f'(x) = -1 \text{ if } x < 1$$
$$f'(x) = 1 \text{ if } x > 1$$

So at $x = 0$, $f'(x)$ is both -1 and 1, not permitted in single-valued functions, as this diagram illustrates.

Bolzano took this abrupt change in direction to its ultimate, creating a function that is nowhere monotonically increasing or decreasing over any interval ω, no matter how infinitesimal.[670] He first defined a function, as the segment of a straight line, from points A (a, A) to B (b, B), with this equation, which we can call y_0:

$$y = A + (x - a)\frac{B - A}{b - a}$$

He then added points *C*, *D*, and *E* with these coordinates with respect to the initial segment:

$$C\text{: } \left(\frac{a+b}{2}, \frac{A+B}{2}\right)$$
$$D\text{: } \left(a + \frac{3}{8}(b-a), A + \frac{5}{8}(B-A)\right)$$
$$E\text{: } \left(a + \frac{7}{8}(b-a), A + \frac{9}{8}(B-A)\right)$$

Bolzano then defined y_1 by drawing four line segments from *A* to *D* to *C* to *E* to *B*. Each of these line segments then formed the basis for forming y_2, with the end points being new values for *A*, *B*, *C*, *D*, and *E*. This recursive process can be continued indefinitely, to create a line of infinite magnitude with 4^∞ line segments of ever-diminishing size, but which is bounded finitely, a little like Koch's famous snowflake curve from 1905. If *A* and *B* are (0, 0) and (1, 1), as Steve Russ suggests, the diagram on the next page shows the first three iterations of what is today called the 'Bolzano function', which is presumably the first analytically defined fractal set,[671] formed some forty years before Weierstrass used a Fourier series to present another 'pathological' continuous function that is nowhere differentiable.[672]

So, what is the fractal dimension of the Bolzano fractal function? Geometrically, we normally think in terms of an integer number of dimensions, with points, lines, planes, and cubes and spheres having 0, 1, 2, and 3 dimensions, respectively. However, in the 1900s, as mathematicians created many weird and wonderful functions, it was realized that bounded infinite lines, for instance, could have dimensions lying between 1 and 2. Felix Hausdorff (1868–1942) introduced the basic idea in 1919 as a 'fractional dimension',[673] which Benoît Mandelbrot (1924–2010), a fellow IBM alumnus, said in *The Fractal Geometry of Nature* could be more appropriately called 'fractal dimension'.[674]

Fractal dimension *D* is determined in terms of a scaling factor *s* and *n*, which is a number that denotes the level of detail revealed as a structure is scaled, given by this formula:

$$D = \log_{1/s} n = \frac{\log n}{\log 1/s}$$

Using the Koch curve as a model, described in my book *Through Evolution's Accumulation Point*, in the case of the Bolzano function, $n = 4$ and *s* is:

$$s = \frac{\frac{2}{4}\left(\frac{\sqrt{34}}{8} + \frac{\sqrt{2}}{8}\right)}{\sqrt{2}} = \frac{\sqrt{17}+1}{16} \approx 0.3202$$

So, the fractal dimension is about

$$D \approx \frac{\log 4}{\log 3.1231} \approx 1.2173$$

We can see that the Bolzano fractal has an upper bound *M* by iterating from the initial *y*-value of *E*, as M_1, in relationship to the first *y*-value of *B*, as M_0. Beginning with:

$$A_0 = 0 \quad \text{and} \quad M_0 = B_0 = 1$$

at the first iteration, we have:

$$M_1 = A_0 + \frac{9}{8}(B_0 - A_0) = \frac{9}{8} = 1\text{⅛}$$

Then, in general, from

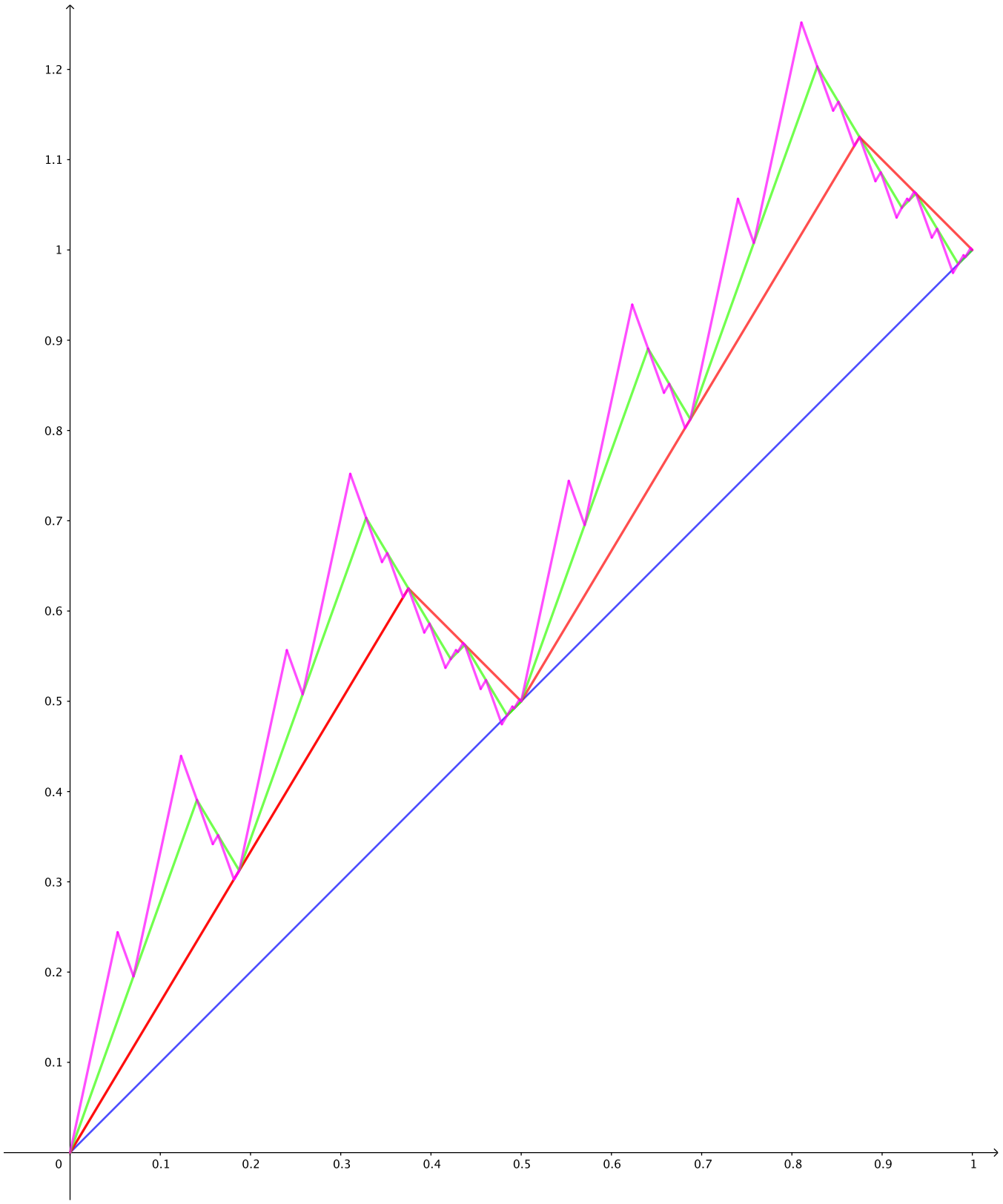
$$A_n = \tfrac{A_{n-1}+B_{n-1}}{2} \quad \text{and} \quad B_n = M_n$$

we have:

$$M_{n+1} = A_n + \frac{9}{8}(B_n - A_n) = \frac{9B_n - A_n}{8}$$

So:

$$M_{n+1} - M_n = \frac{9}{8}B_n - \frac{1}{8}A_n - B_n = \frac{B_n - A_n}{8} = \frac{5(B_{n-1} - A_{n-1})}{64}$$



And:

$$\frac{M_{n+1}-M_n}{M_n-M_{n-1}}=\frac{\frac{5(B_{n-1}-A_{n-1})}{64}}{\frac{B_{n-1}-A_{n-1}}{8}}=5/8$$

From this, we have:

$$M=1+\frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{5}{8}\right)^n=1+\frac{1}{8}\times\frac{1}{1-\frac{5}{8}}=1\frac{1}{3}$$

As a coda, after the Bolzano committee published *Functionenlehre* and *Zahlentheorie*, in 1932 they published a Utopian piece titled *Von dem besten Staate* '*About the Ideal State*', with this summary:

> Bolzano, in this work, argues for an ideal human society based on the absolute equality of all people in the society. He argues that owning large amounts of property does not merit respect but should be viewed as possibly having been acquired through immoral and/or illegal means. Even if legally acquired, he believes, such wealth is merely the 'quest of riches blessed by law'. He regrets that many people have to work so hard to earn enough to feed their families that they damage their health.[675]

Bolzano was writing when the Industrial Revolution was gathering pace and labourers were moving off the land into factories with often mind-numbing long hours. He was addressing perhaps the most intractable challenge that has faced human societies for thousands of years, from Plato and Confucius onwards. I wonder how he saw the possibilities in the middle of the 1800s. As a reflection of these social tensions, Karl Marx and Friedrich Engels' *Communist Manifesto* was published in 1848, the year of Bolzano's death. Today, the gap between the financially wealthy and poor is getting ever wider, with women often paid less than men for the same work, as workers have become cogs in a vast technological, economic machine, called *The Matrix* in the movie, a popular allegory of our times.

Having been one of the architects of such an oppressive regime in the late 1970s, governed by leading companies in the information technology industry, since then I have adopted a creative, life-enhancing work ethic to explore whether it would ever be possible to organize our business affairs in harmony with the fundamental laws of the Universe so that everyone has the opportunity to realize their fullest potential as human beings. Sadly, given the cultural, psychosocial constraints on our lives, it now looks most unlikely that materialistic, mechanistic science will ever pass through these twilight times and collectively awaken in the Age of Light, fulfilling our potential as a superhuman species, transcending our machines. Nevertheless, I continue to work on the necessary science of humanity as if such a wonderful transformation could happen, stretching out to my own fullest potential in order to stay as joyfully sane and healthy as possible.

To this end, we now come to the four philosopher-mathematicians whose studies of continuity Benjamin Lee Buckley highlighted in *The Continuity Debate*, in which he sought to establish infinitesimals as valid mathematical objects, inspired by *Non-standard Analysis*, which Abraham Robinson introduced in 1960, based on axiomatic linear mathematical logic, rather than Integral Relational Logic, which has awarely evolved from the Divine Origin of the Universe.

One example that Buckley gives from probability theory is particularly appealing. He says that if we toss a coin an infinite number of times, then the probability that it comes down heads each time, or in any other predetermined pattern, such as alternate heads and tails, is $1/2^{\infty}$. This infinitesimal quantity denotes something that is infinitely improbable, rather than logically impossible, a subtle distinction.[676]

What is key here is how can mathematicians ensure the continuity of $f(x)$ without first establishing the continuity of x, as the input to the function, even one as simple as $y = x$? One of the first to address this question was Dedekind, as one of the great pioneers in the logical and philosophical analysis of the foundations of mathematics.[677] He thus assumed that Aristotle's Law of Contradiction is absolutely and universally true, denying the truth of Heraclitus' Hidden Harmony, of which there are countless examples in the dualistic world we live in, not least within the psyche. Also, Dedekind did not like infinitesimals, beginning with an atomistic view of the continuum, like Archimedes of Syracuse (281–212 BCE), who "assumed that continuous things were in fact composed of indivisible parts; and more importantly, that a continuum could be divided into indivisibles."[678]

To establish the continuity of the real number line, Dedekind divided it into two parts at an arbitrary point! This he did in 1858, but he was doubtful about whether to publish "because, in the first place, the presentation did not seem altogether simple, and further, the theory itself had little promise". However, by 1872, he was getting ready to publish when he was further encouraged to do so by Eduard Heine's paper on functions, mentioned on page 141, and Cantor's forthcoming paper on the same subject. So Dedekind wrote a short monograph titled *Stetigkeit und irrationale Zahlen* (*Continuity and irrational numbers*), in which he sought to show how a continuum of reals, including both rationals and irrationals, could be created from just the rationals by making a cut (*Schnitt*) between them.[679]

What is now called a Dedekind cut divides the rational numbers into two classes A and B at a point c, in which there are two possibilities. Either every number a in A is less than every number b in B or the number c is either the greatest number of the class A or the least number of the class B. In the former case, c is an irrational number, $\sqrt{2}$, let us say, created in a gap between the rationals. And in the latter case, c is a rational number, ⅔, let us say. In this way, a Dedekind cut defines a real number[680] and we have:

$$\mathbb{R} = \mathbb{Q} + \overline{\mathbb{Q}} = \mathbb{Q} + \mathbb{A} + \mathbb{T}$$

Dedekind thus simply characterizes continuity in terms of completeness, with no connections between the indivisible real numbers other than that they are ordered.[681] There is thus no room for infinitesimals in this construction of all numbers before they are combined in more than one dimension as complex numbers, quaternions, octonians, etc.

The 1872 Cantor paper that Dedekind referred to in the introduction to his own was titled '*Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen*',[682] which seems to be as much the culmination of Cantor's earlier work on the uniqueness of trigonometric series as the beginning of a revolution in the understanding of number in mathematics. For while this paper includes an initial axiom of continuity, it led Cantor into a lifetime search into the essential nature of the continuum, which had eluded thinkers for thousands of years. It was an enormous challenge, for Cantor thought that finding a sound definition of the mathematical continuum would provide insights into fundamental metaphysical and theological aspects of human existence.[683]

However, pushing the boundaries of mathematical knowledge beyond what was generally considered acceptable at the time meant that throughout his life he remained Professor of Mathematics at Halle University, generally considered second class, midway between the more prestigious universities of Berlin, where Weierstrass and Kronecker were professors, and Göttingen, where Gauss and Reimann held sway. Cantor worked very much in solitude, not even being able to persuade his friend Dedekind to move to Halle University from Brunswick Polytechnic in 1882, leading them to become estranged for seventeen years.[684]

Before Cantor's transfinite theory of sets blew up the foundations of mathematics, showing that the entire mathematical enterprise is not based on the Truth, he felt the need to explore further the essential difference between the rationals and irrationals, as we see from this letter that Cantor wrote to Dedekind on 29th November 1873, a year after their first meeting in Switzerland:

> Take the collection of all positive whole numbers n and denote it by (n); then think of the collection of all real numbers x and denote it by (x); the question is simply whether (n) and (x) may be corresponded so that each individual of one collection corresponds to one and only one of the other? At first glance one might say no, it is not possible, for (n) consists of discrete parts while (x) builds a continuum; but nothing is won by this objection, and as much as I am inclined to the opinion that (n) and (x) permit no such unique correspondence, I cannot find the reason, and while I attach great importance to it, the reason may be a very simple one.[685]

Having posed the question, Cantor found the solution to his problem the very next month, publishing his results in a paper in early 1874, 'On a Property of the Collection of All Real Algebraic Numbers', implying that "the major accomplishment of his new methods was the proof that the set of all algebraic numbers was countable", summarized on page 106. However, inspired by Liouville's proof that in any given interval (α, β) there are infinitely many transcendental numbers, he was able to show that $\mathbb{R}$ is nondenumerable using his now famous diagonal method,[686] as we see on page 107.

As the existence of more than one order of infinity was so revolutionary at the time, Cantor did not mention this in the title of his paper, hoping that the editor of the prestigious *Crelle's Journal* in Berlin would publish it without closely investigating its controversial contents. The ploy worked and the paper was duly published, despite Kronecker's reservations about the validity of Cantor's work.[687]

Having shown that there are infinitely many more transcendental numbers than natural ones, Cantor then set out to explore the mathematical continuum in more depth, arriving at an even more staggering result. The cardinality of the whole of n-dimensional space is no greater than that of a finite section of the real number line, as already mentioned on page 107.

Cantor was so surprised by this discovery that in the letter he wrote to Dedekind on 29th June 1877 he uncharacteristically expressed his astonishment in French: "*Je le vois, mais je ne le crois pas*." Yet, it is very easy to prove this relationship using Cartesian coordinates. If a point in the unit square is given as $(0.a_1a_2a_3\ldots, 0.b_1b_2b_3\ldots)$, then $0.a_1b_1a_2b_2a_3b_3\ldots$ denotes a point on the number line between 0 and 1. Such a construction can be extended to any number of dimensions. Dedekind was most cautious in his response, advising Cantor not to polemicize too much even when congratulating him on his ingenuity.[688]

It was at this point that Kronecker became actively hostile to Cantor's work, attempting to prevent the publication of his paper on the irrelevance of dimension in *Crelle's Journal*. Their antagonism existed on more than one level. In terms of mathematics itself, Kronecker was an algebraist, thinking in discrete terms, while analysts, like Cantor, and indeed Weierstrass, tended to visualize numbers and other mathematical entities on a continuum.[689] Kronecker was to famously say in a lecture at a congress of scientists in Berlin in 1886, "God made the integers, and all the rest is the work of man."[690]

By bringing God into the picture, Kronecker became personal, saying on occasion that Cantor was 'a corrupter of youth', using a similar phrase to that which led the democracy in Athens to condemn Socrates to death, saying at his trial, "An unexamined life is not worth living." By pushing back the frontiers of mathematical knowledge, Cantor's work became existential, for people's sense of a separate identity is so often tied up with their religious beliefs. Any potential conflicts that thus arise can best be satisfactorily resolved through a genuinely innate, mystical approach to life, realizing that Love is the Divine Essence we all share.

To see how Cantor dealt with these theological issues, it is necessary to understand his family backround, about which there is some uncertainty. While Cantor's parents were Christians, there is much evidence that they had a Jewish background, quite possibly arising from the diaspora from the Iberian Peninsula, which most notably led Baruch Spinoza's family to move to the Netherlands in the sixteenth century.

In Cantor's case, his father, Georg Waldemar, grandfather Jacob Cantor, and maternal great grandfather Abraham Meier were merchants born in Copenhagen, later moving to St. Petersburg, where Cantor was born in 1845, his father being baptized a Lutheran. Cantor's mother Maria Böhm was a Roman Catholic, born in St. Petersburg, but from a Jewish family from Pest, Hungary. Both sides of Cantor's family were highly musical, with two of Abraham's sons being court musicians to the Tsar and Maria being a violinist,

like her father Franz and uncle Joseph Böhm, one in St Petersburg and the other the founder of the violin school at the Vienna Conservatory in 1814, having a working relationship with Ludwig van Beethoven and teaching many eminent violinists, like Joseph Joachim, a close collaborator of Johannes Brahms.

Cantor moved to Frankfurt in Germany with his family in 1856 when he was eleven because his father suffered from tuberculosis, his distinguished cousin Dmitry Ivanovich Meyer, a professor of law, who taught Leo Tolstoy at the University of Kazan and who was influential in the emancipation of serfs in Russia, having just died from the disease aged thirty-seven.[691]

Now Amir D. Aczel tells us in *The Mystery of the Aleph: Mathematics, the Kabbalah and the Search for Infinity* that Jewish families who publically converted to Christianity maintained many of their Jewish traditions within the family, even marrying Jews, as Cantor did, marrying Vally Guttmann in 1874, having met her through his sister. We know that some of these traditions were maintained in Cantor's family from two letters he wrote in later life to Paul Tannery and Bertrand Russell, stating that his paternal grandparents were members of the Sephardic Jewish community of Copenhagen.[692]

However, what we don't know explicitly is to what extent Cantor was aware of the Kabbalistic view of God and infinity, this mystical school of Judaism having originated in Iberia. In Kabbalah, the term for God prior to manifestation in the relativistic world of form is *Ein Sof* (אין סוף), translated as 'infinity', literally that which has no end or limit. From *Ein Sof*, as the Datum of the Universe, emerges *Ohr Ein Sof*, as Divine Light, as a metaphor for the infinite light of pure awareness that "illuminates every corner of the universe and all the wisdom of creation".[693] This is what I call the coherent Light of Consciousness, which enables us to view the Cosmos holographically with Self-reflective Intelligence. Hence, even though Cantor may not have had detailed knowledge of Kabbalah, "the concept of infinity was known to anyone with a Jewish background, including the converted Jews of Iberia."[694]

Yet, in investigating the mathematical continuum and the continuity of the real numbers, Cantor was venturing into the unknown, into what many consider unknowable, either cognitively or experientially, as Gnosis. It is thus perhaps not surprising that Cantor's studies of infinite sets provoked such antipathy and that Cantor was severely psychologically disturbed as a result.

In the event, Cantor's paper on the independence of dimension to the continuum was published in *Crelle's Journal* in 1878, the last of his papers to be published in Berlin. Then, from 1879 to 1884 Cantor wrote six papers on an analysis of infinite linear point sets under the general heading '*Über unendliche lineare Punktmannigfaltigkeiten*' (On Infinite Linear Point Manifolds). Five of these focused mainly on mathematics, with the fifth titled '*Grundlagen einer allgemeinen Mannigfaltigkeitslehre*' (Foundations of a General Theory of Manifolds) in 1883 addressing the deeper philosophical issues, for, to Cantor, "the metaphysics and the mathematics went hand in hand". B. G. Teubner published these in Leipzig in *Mathematische Annalen*, with Gösta Mittag-Leffler publishing the purely mathematical aspects of these papers in French in Stockholm in his newly founded journal *Acta Mathematica* in 1883 and 1884.[695]

Cantor explained what he meant by *manifold* in an endnote to *Grundlagen*, which Uwe Parpart translated into English in 1976 from the 1932 edition of Cantor's *Collected Treatises*, edited by Ernst Zermelo. "By a 'manifold' or 'aggregate' [*Menge*]," Cantor wrote, "I generally understand every multiplicity which can be thought of as one, i.e., any totality of definite elements which by means of a law can be bound up into a whole." He likened this conception to Plato's *eidos* or *idea*, as a supposedly eternal universal, which appears as class in object-oriented modelling and programming methods, distinct from object, as a particular.

Cantor also found points of contact with the Christian mystics Nicolaus Cusanus and Giordano Bruno, in contradistinction to Aristotle's aversion to the existence of the infinite.[696]

From a mathematical perspective, *Grundlagen* introduced two fundamental concepts, as a development of ones that Cantor had been working on for the previous eleven years. First, he introduced a quite new type of number, which he denoted with ω, as omega, the last letter in the Greek alphabet, for ω comes after all the finite natural numbers in a well-ordered sequence. But ω is also the first number in an infinite sequence of infinite sequences of well-ordered ordinals, the next being ω + 1. These he called transfinite numbers, with ω, as an ordinal, denoting the cardinality of the natural numbers, described in a little more detail on page 168.

In a review of *Grundlagen*, the Austrian philosopher and mathematician Benno Kerry (né Kohn) (1858–1889) pointed out that if an infinity of ordinals exists, then so must their reciprocals as ordered infinitesimals, even if they have no empirical applicability, such as:[697]

$$\frac{1}{\omega}, \frac{1}{\omega+1}, \cdots \frac{1}{2\omega}, \cdots \frac{1}{\omega^2}, \cdots$$

Now, as you can see from the way that these infinitesimals are formed, their denominators are countable, with gaps within them. So we must also consider the sequence of infinitesimal ordinals to be countable, falling short of the continuum of the real numbers. To define a sequence of infinitesimal ordinals that is consistent with the reals, we can do so with ω_1, as the cardinality of the set of the first level of infinite ordinals:

$$\frac{1}{\omega_1}, \frac{1}{\omega_1+1}, \cdots \frac{1}{2\omega_1}, \cdots \frac{1}{\omega_1^2}, \cdots$$

What I am doing at this stage in my reasoning is to use the Principle of Unity, as the fundamental law of the Universe, to build a coherent picture of the number system, as just one tiny instance of the multidimensional jigsaw puzzle that is the Theory of Everything, which I call Panosophy or the Unified Relationship Theory. So just as there is no limit to the number of infinities that can be formed, there is no limit to the number of zeroes. In other words, as I see the situation, there is an infinite sequence of infinitesimals that culminate in Absolute Zero, called *shūnyatā*, as 'emptiness' in Buddhism, a notion that some quantum physicists have been considering to make sense of their observations.

However, this is not how Cantor viewed infinitesimals, which he believed were contradictory to transfinite numbers, denying that opposites are always balanced in wholeness, not the least in the fundamental theorem of the calculus. What is important to remember here, if I have not previously made this crystal clear, is that there are two meanings of *infinitesimal*, as opposites.

As mentioned on page 139, the conventional meaning is that infinitesimals are nonzero numbers that are less than any particular real number, no matter how small. Such infinitesimals are thus the opposite of finite numbers that can get larger and larger, without ever reaching infinity. However, such enormously large numbers only emerged in mathematical consciousness in the second half of the twentieth century, quite unknown in Cantor's time, as we see in subsection 'Further hyperoperations' on page 130. The second meaning, as we have just seen, is that infinitesimals are actually an infinite sequence of zeroes, as the opposite of the collection of all the alephs, described on page 169.

However, Cantor was not willing to contemplate infinitesimals with either meaning in 1883, writing in *Grundlagen*:

> The infinitely small magnitudes, to my knowledge, have so far been worked out for useful purposes only in the form of the non-genuine-infinite, and as such are capable of all those variations, modifications, and relationships which find

> application and expression in infinitesimal analysis as well as in function theory, in order to there form the basis of a rich abundance of analytical truths. On the other hand, all attempts to transform the infinitely small by force into a genuinely infinitely small magnitude should finally be abandoned as purposeless. If genuinely infinitely small magnitudes exist in any other form at all, i.e., are definable, still they surely do not have any immediate connection with the ordinary magnitudes that become infinitely small.[698]

However, he felt that he had been "too reserved in the *Grundlagen* by failing to deny explicitly the existence of actually infinitesimal quantities". So, in May 1887, he sought to 'prove' the nonexistence of infinitesimals in a letter to Franz Goldscheider, copied to his former teacher Weierstrass,[699] which was published as a section of a collection of letters titled *Mitteilungen zur Lehre vom Transfiniten* (*Communications on Transfinite Theory*).[700]

Regarding the way that Cantor used the concept of infinite linear point sets to define what he meant by the mathematical continuum and the continuity of the real numbers, denoting points on the real number line, I must admit that I have not fully grasped his reasoning, for this does not help me to unify mysticism and mathematics. All I know is that he built on his notion of derived sets, originally defined in his 1872 paper on trigonometric series, writing in *Grundlagen*,

> I also believe, however, that with these *two* predicates, 'perfect' and 'connected', I have come across the necessary and *sufficient* characteristics of a point continuum, and I therefore define a point continuum within G_n [n-dimensional space] as a *perfectly-connected aggregate*. Here 'perfect' and 'connected' are not mere words, but entirely general predicates of the *continuum*, conceptually characterized in the sharpest possible manner by the preceding definitions.[701]

As I wish to move on with completing this chapter in my book before the end of January 2019, I'll pause here, returning later if it should prove necessary to do so. For such an understanding requires me to penetrate deeper into Cantor's psyche in the context of the evolution of Western thought during the past few thousand years. And as I am seeking to be free of such constraints, maybe there is no need to explore further.

Nevertheless, let us return to how some other philosophers of mathematics saw the situation in the 1800s. In contrast to Dedekind and Cantor, Paul du Bois-Reymond considered infinitesimals to be essential to an understanding of continuity. He spent much of his career creating his *Infinitärcalcül* 'Infinitary Calculus', a theory whereby functions with infinite domains and ranges are compared and ordered. In an 1877 paper, '*Ueber die Paradoxen des Infinitärcalcüls*' 'On the paradoxes of the infinitary calculus', he sought to show that the continuity of his *Infinitärcalcül* is comparable to the continuity of the real numbers, writing:

> The infinitely small is a mathematical quantity and has all its properties in common with the finite … A belief in the infinitely small does not triumph easily. Yet when one thinks boldly and freely, the initial distrust will soon mellow into a pleasant certainty … A majority of educated people will admit an infinite in space and time, and not just an 'unboundedly large'. But they will only with difficulty believe in the infinitely small, despite the fact that the infinitely small has the same right to existence as the infinitely large …[702]

Implicitly acknowledging the truth of the fundamental law of the Universe, du Bois-Reymond is thus saying that as infinities exist, so must infinitesimals. That is simple logic, once we recognize that the Law of Contradiction is not universally true.

Then, in *Die allgemeine Functiontheorie* (*General Theory of Functions*) in 1882, du Bois-Reymond, as a physicist, sought to show that mathematics is a science like any other, saying in the Introduction that fundamental mathematical concepts are not as rigorously established as key concepts in other sciences, for our mathematical intuitions often conflict.[703]

In this book, du Bois-Reymond stood at an intelligent, both-and vantage point, looking at the claims of the 'Idealist', for whom infinitesimals are a logical necessity, and the 'Empiricist', who does not accept infinitesimals at all. As he says, "Idealism believes that the truth of certain limited forms of our ideas is

required by our understanding, though they may lie outside of all perception and sensory representations. Empiricism is the system of complete abnegation; it admits only as extant that which can be perceived or reduced to perception."[704]

While du Bois-Reymond believed that the Idealist and Empiricist perspectives are "equally authorized to serve as the base of the science [of mathematics]", he thought that it is not possible to find a compromise between them, because the chasm separating them "is too profound and too vast to be able to be filled by reciprocal concessions".[705] This is a situation that we often face in life, faced with contradictory ethical or political ideals, for instance, which can only be reconciled in the Absolute.

For if we look at these supposed contradictions from the perspective of the science of humanity, whose Contextual Foundation and framework is provided by Integral Relational Logic, there is no longer a contradiction and we are able to find deep Inner Peace. As humans, never separate from the Divine for an instant, we can egolessly experience the Absolute Continuum, from which infinitesimals emerge, like any other being in the relativistic world of form.

Finally, in this brief overview of the history of continuity and the mathematical continuum, I come to Charles Sanders Peirce, a close kindred spirit, with his triadic architectonic and principle of synechism, the keys to the unification of mysticism and mathematics, as he pointed out in the 1893 article quoted on page 76, not published until 1958 by Arthur W. Burks, the Ph. D. supervisor to Ted Codd, whose relational model of data has evolved into Integral Relational Logic.

It is also noteworthy that Paul Carus, who had intended to publish Peirce's essay on 'Immortality in the Light of Synechism', was a student of Hermann Grassmann,[706] both interested in the integral foundations of mathematics and in Eastern mysticism. However, even though Cantor was an avid reader of whatever he could get his hands on being published in Europe, I'm not sure to what extent Peirce was aware of Grassmann's work. Also, I've seen no reference in Peirce's writings to those of Bolzano, not even to *Theory of Science* and *Paradoxes of the Infinite*, published in 1837 and 1851, respectively.

These observations are pertinent for Peirce was as much a scientist as a philosopher of science and mathematics, having been educated as a chemist and employed as a geophysicist with the United States Coast Survey for nearly thirty years, of which his father Benjamin, Professor of Mathematics at Harvard, was the Supervisor from 1867 to 1874.

Now while Peirce did not publish any books on his architectonic in his lifetime, we know something of the direction of his thoughts on continuity and the philosophy of mathematics and scientific method from two series of essays that were published in the *Popular Science Monthly* in 1877 and 1878 and in *The Monist* from 1891 to 1893, all republished in the first volume of *The Essential Peirce* in 1992. To see these as a whole, the first group is titled 'The Fixation of Belief', 'How to Make Our Ideas Clear', 'The Doctrine of Chances', 'The Probability of Induction', 'The Order of Nature', and 'Deduction, Induction, and Hypothesis'. And the second group of titles is 'The Architecture of Theories', 'The Doctrine of Necessity Examined', 'The Law of Mind', 'Man's Glassy Essence', and 'Evolutionary Love'. In between, in 1885, he told his friend William James that he may have "found the key to the secret of the universe", as mentioned on page 76. So pervading his entire thought process is an innate sense of wholeness.

In studying these writings, it is important to bear in mind the hostile environment in which they were written. Like Cantor, with Kronecker, Peirce had a nemesis in the figure of Simon Newcomb, a student of Peirce's father Benjamin. As Joseph Brent said in his insightful biography *Charles Sanders Peirce*, "Newcomb was a self-made man who resented Peirce's privileged arrogance and great talent, despised his morals, and

was able to move effectively against him because he was a politically astute scientist whose influence was powerful and pervasive."[707]

Newcomb was particularly appalled that from 1877 Peirce had lived openly with Juliette, his second wife from a mysterious European high-society background, after his first wife Zina had left him in 1875, but before they were officially divorced in 1883. Zina, the granddaughter of an Episcopal bishop and daughter of a friend of his father, said in a letter to his boss that she could not live any longer with 'Charley' because of his 'brilliant but erratic genius', asking for help to tame his reckless and extravagant behaviour,[708] which sounds like exuberance to me. Regarding how conventional society viewed this separation, Brent sums up the situation in this way, "For a sanctimonious man of affairs of the period such as Newcomb, for Peirce to have a mistress was both understandable and acceptable if the affair were carried on discretely, but to marry her after such a public liaison was outrageous because to do so attacked the sanctity of marriage."[709]

As a consequence, both academia and 'polite' society regarded Peirce as an outcast. In practical terms, in 1884, he was sacked from his job as a lecturer in logic at John Hopkins University, rather than being offered tenure, which had previously been on the cards. This is particularly ironic because it was at this newly formed institution that Peirce laid down the foundations of the logic of relatives,[710] which has since evolved into the modelling methods that underlie the Internet and thence into Integral Relational Logic, the science of thought and consciousness that we all implicitly use every day to form concepts and organize our ideas, healing the cultural split between science and spirituality and mysticism and mathematics.

So, when Peirce resigned from the US Coast Survey in 1891 without a pension, despite some thirty years of service, he was without a regular income. Furthermore, Juliette was diagnosed with tuberculosis in 1889 and suffered gynaecological problems for five years, which resulted in a hysterectomy in 1896. In that year, Peirce was sometimes living on the streets of New York City from hand to mouth, often without food or shelter for days at a time, desperate to find some way to make enough money to feed and shelter Juliette and himself, she in urgent need of medical treatment. Eventually, in 1907, William James came to the rescue, and organized some friends and associates to donate fifty dollars a year, to be managed by Mrs. C.S.P., for she was a first-rate economist and Peirce had no notion of the difference between a dollar and a hundred,[711] looking at numbers from a transcendental perspective, as we can from what he wrote in the 1890s.

But returning to the radical transformation of consciousness that Peirce went through during the eight years either side of his fiftieth birthday in 1889, after he introduced his triadic logic in 'The Architecture of Theories', quoted from on page 67, in the other four metaphysical essays that Carus published in *The Monist*, Peirce described the three pillars of his evolutionary cosmology: "*tychism*, the theory that chance is really operative in the universe; *synechism*, the theory that continuity prevails and that the presumption of continuity is of enormous methodological importance for philosophy; and, finally, *agapism*, the thesis that love, or sympathy, has real influence in the world."[712]

By tychism, from Greek *tukhē* 'chance', Peirce emphasized the nonmechanistic, vitalistic quality of evolutionary processes, writing, "there is probably in nature some agency by which the complexity and diversity of things can be increased, … thus admitting pure spontaneity or life as a character of the universe."[713] Jung was later to call this phenomenon synchronicity 'meaningful coincidences with no apparent causal connection', which we sometimes experience as serendipity 'the occurrence and development of events by chance in a happy or beneficial way'.

Peirce described agapism, as the last and most important of his three cosmological principles, in 'Evolutionary Love', from Greek *agapē*, used by Christian writers in the New Testament to mean 'Divine

Love'. For as John said in his first Epistle, as Peirce reminds us: "God is Love; and he that dwelleth in Love dwelleth in God, and God in him,"[714] words that Pope Benedict XVI took as the text for his first encyclical '*Caritas Deus Est*,' published on 25th January 2006.[715] Most significantly, Peirce said that Love "cannot have a contrary, but must embrace what is most opposed to it," noble words, under any circumstances, but especially in the context of the medical and psychosocial challenges he was then facing in life.

Returning to the union of mysticism and mathematics, Peirce introduced the term *synechism* in the 'Law of Mind', taking a psychospiritual approach to scientific method and reason, even though he had said in a lecture in 1865 that psychology and mathematical logic should be kept separate from each other, as we see on page 83. Although Peirce was to reemphasize this split in 1898, in July 1892 he was more in touch with his True Nature, shortly after having had a profound mystical experience, not revealed until the mid 1990s, when a letter to a priest describing a visit to his church was found.[716]

However, there was little in Peirce's social environment that could help him understand his mystical experiences. Peirce mentions the esoteric Transcendentalist movement led by Ralph Waldo Emerson (1803–1882) and Frederic Henry Hedge (1805–1890) in the 1830s and 1840s in New England, but says sardonically that even though "some benignant form of the disease" may have been implanted in his soul, he sought to distance himself from the transcendentalists, influenced by "minds stricken with the monstrous mysticism of the East".[717]

Being aware of such subconscious influences, "modified by mathematical conceptions and by training in physical investigations", he said, "The next step in the study of cosmology must be to examine the general law of mental action." This led him to see that continuity, termed *synechism*, "is an idea of prime importance in philosophy". Peirce was thus effectively beginning the development of his cosmology with what Aurobindo was to call an undivided Supermind some thirty years later, as we see on page 75. In Peirce's words, "Logical analysis applied to mental phenomena shows that there is but one law of mind, namely, that ideas spread continuously and to affect certain others which stand to them in a peculiar relation of affectability."[718]

By watching the way thoughts interact in his mind. Peirce could see the interconnectedness of all beings, a holistic worldview that is gradually becoming accepted today in the most progressive of scientific and spiritual circles.

This statement is a development of one that he had made in 1878 in his essay 'The Doctrine of Chances'. In studying the way that naturalists form concepts—by noting the differences and resemblances in the forms that they observe, the fundamental principle of concept formation, defined on page 51—he said that such attempts at classification vary imperceptibly in a form of continuity, which is "the passage from one form to another by insensible degrees". Peirce said that it is vitally important to emphasize this taxonomic principle in order to put mathematical precision into its proper perspective. As he said, "Number, after all, only serves to pin us down to a precision in our thoughts which, however beneficial, can seldom lead to lofty conceptions and frequently descends to pettiness … [tending] to narrow the powers of the mind."[719]

This perspective of number is very similar to that of David Bohm, who thought when he entered the California Institute of Technology in 1939 that physics needed a deeper philosophical ground than the mathematical techniques that were being used to study the nature of physical reality. Regarding himself more as a natural philosopher, like Newton, Bohm said, "the general practice of physics has indeed become remote from these deeper considerations."[720]

In the fourth essay in the second series, Peirce provided a metaphor for continuity in the physical universe, rather than in the mental domain. The word he used here was *protoplasm*, which he called 'life-slime',[721] as

what we could consider as amorphous form, from Greek *amorphos*, from *a-* 'without' and *morphē* 'shape'. Indeed, *protoplasm* derives from Late Latin *prōtoplasma* 'first thing created', from Greek *prōto-* 'first' and *plasma* 'moulded thing, figure, form', from *plassein* 'to shape', from PIE base *pelə* 'flat, to spread', also root of *field*, *plane*, and *floor*. So, before Jan Evangelista Purkyně (1787–1869) and Hugo von Mohl (1805-1872) coined *protoplasm* in 1839 and 1846, respectively, to denote properties of cells, *Protoplasm* could well have been used in ecclesiastical Greek[722] to denote one way in which the relativistic world of form emerges directly from the Formless Absolute, as the undivided Continuum.

We see from these essays that by the time Peirce was in his early fifties he had made enormous progress in healing the split between science and spirituality, gradually revealing the deeply hidden secrets of the Universe that he had discovered through direct experience by looking inwards. But to what extent could he take this convergent evolutionary process to its glorious culmination during the last twenty years of his life by unifying mysticism and mathematics?

Well, to answer this question, I really need to go beyond the words and feel even deeper into Peirce's psyche than I have done so far to date, quite a task. All I can really do here is to glean what I can from Peirce's experiences as a mirror to my own, trusting that this could help others understand my own unprecedented life story a little better. I wrote something about this in my book *The Theory of Everything* after I discovered his architectonic in 2012. But now I need to feel and look deeper into myself, discovering some similarities in the mathematical arena that I had not found before, arriving at the apotheosis of this unification of mysticism and mathematics.

In this context, the most important of Peirce's mathematical writings are a number of memoirs on 'Multitude and Continuity' that Carolyn Eisele gathered together in the 1970s as part of a 4-volume, 2,500-page set of books titled *The New Elements of Mathematics*.[723] Most, if not all of these, are listed in the catalogue and its supplement of manuscripts that Richard S. Robin prepared in 1967 and 1971 from those that Peirce's widow Juliette deposited in the library at Harvard University after he died in 1914. This catalogue is an extension of the *Bibliography* that Arthur W. Burks prepared in 1958 of Peirce's writings, some published for the first time in the *Collected Papers* in the 1930s and 1950s.

For myself, all I can do, engaged in a similar journey, is to feel into myself, highlighting what seems to be the most relevant at this stage in my life and to this chapter in this book. To this end, I can best look at the three fundamental questions that Peirce asked at the beginning of a 25-page memoir he wrote about 1895: "What is *mathematics*?", "What is *quantity*?", and "What is *continuity*?"[724]

The word that stands out here is *merge*. For while Dedekind placed a cut in the middle of the real numbers and Cantor viewed them as a set of perfectly connected limit points, Peirce said that if the real numbers are to truly have the property of continuity; they must merge into one another. In temporal terms, instants of time "are so crowded as to merge into one another and lose their distinct existence". As he said, what is involved here is the conception of the 'flow' of time,[725] which Bohm was to use in 1980 to unify quantum and relativity theories.

We need to remind ourselves here that these points in the real-number continuum are mostly transcendental numbers, distinct from real algebraic ones, which include the rationals, and, in turn, the integers. As ever, I look at the root of *transcendental* to discover its original meaning. The word derives from Latin *transcendere* 'to climb over or beyond, surmount', from *trans-* 'over, across, beyond' and *scandere* 'to climb', from PIE base **skand-* 'to leap, climb', also root of *scale*, as a measuring device, from Latin *scālæ* 'staircase, ladder'.

In English and the other European languages, Latin *transcendere* has given rise to several words with many different meanings. The first to appear was 'to transcend' around 1340, which Richard Rolle of Hampole introduced in his *Psalter* to mean 'to go beyond the limits of (something immaterial); to exceed'. For me, this meaning lies at the heart of the unification of mysticism and mathematics.

First, Euler said that transcendental numbers were so called because, "They transcend the power of algebraic methods,"[726] related to transcendental functions, such as trigonometric, hyperbolic, exponential, and logarithmic, which do not satisfy a polynomial equation, even though they can be expressed as an infinite power series, as we see in the next chapter.

In terms of Peirce's studies of continuity, in 1897 he introduced the word *supermultitudinous* to mean 'an aggregate collection of a denumerable multitude of collections of distinct individuals'. As he said, such "a supermultitudinous collection … is no longer discrete; but is continuous". Then at the end of these rather rambling deliberations on 'Multitude and Quantity' he wrote, "a collection may be so great that its individuals lose their separate identities", a situation he called paradoxical, whose key "will probably ultimately be discovered to lie in some unnoticed condition in the general hypothesis of a collection which requires this mergency of individuals".[727]

Therein lies the key that unlocks all the innermost secrets of the Universe, applicable to all aggregates, whether they be a set of complex numbers, all physical universes, being born in big bangs and dying in black holes, or whatever. In human terms, we are all both the entire Ocean of Consciousness, inseparable from any other being, and individual waves and currents on and beneath the surface of the Ocean. In my experience, such a 'mergency of individuals' happens in divine love-making with a beloved woman, when two apparently separate beings become united in Stillness, the most popular aspect of Tantric, Yogic mysticism, as the union of all opposites, not generally understood. Throughout my life, I have known nothing more exquisitely beautiful, quite impossible to put into words.

If such a sublime, intimate experience is to be expressed in language, there is no better word than *merge*, which derives from Latin *mergere* 'to dip, plunge into liquid, immerse'. The second edition of the OED records the use of *merge* mostly in legal terms, saying that its literal meaning is more or less obsolete, not mentioning its use in computer science, as the merging of sets of data, or in spiritual practices. Yet, *mergence* simply encapsulates what is happening to me in my meditation practice, as all sets of data and the relationships between them merge into the Ocean of Consciousness, which is the undivided Continuum. For me, being immersed in this vast Ocean, like the Atlantic Ocean in south-west France, is the most effective way of dealing calmly with the turmoil of the dual and dualistic world we all live in.

Of course, the opposite of *merge* is *emerge*, as the entire world of form emerges from the Formless Absolute through the irresistible power of Life and the Logos. It is this universal emergent, generative principle that I am endeavouring to describe in this book, applying it, in particular to mathematics, as the science of beautiful patterns and relationships, many of which are to be found in geometry and the number system.

However, what triggered this holographic, nonaxiomatic, nonlinear view of mathematical logic was an apocalyptic awakening in April 1980, when I realized that synergistic active and passive data in humans and computers are causal and therefore energetic. During the next few years, I passed through what Christina and Stanislav Grof call a 'spiritual emergency', as Spirit emerged faster than my body-mind-soul organism could handle at the time. It has taken many years of mindfulness, called *smriti*, Sanskrit for 'attention' in Buddhism, to find a balanced way of living with all this commotion, as *līlā*, the ever- and never-changing play of the Divine.

If all of this is to make sense in mathematical terms, it is important to remember that the domain that contains the transcendental numbers, which are the vast majority of the reals, is the psyche, as consciousness. And as numerals, expressed in base ten, transcendental numbers consist of an infinite sequence of decimal digits that have no discernible pattern, quite impossible to see with our physical eyes. So, if we are to engage in mathematics, or any other scientific studies, it is vitally important to view them from a psychological perspective, by looking inwards through self-inquiry.

Such introspective practices are not necessarily subjective, valid only for the experimenter, and therefore regarded as unscientific. It is not only transcendent functions that go beyond the limits of algebraic analysis. We can see what we all share in common by standing outside ourselves, transcending our limited identities as apparently separate individuals. It is from this Holoramic 'Whole-seeing' vantage point that subjective and objective perspectives merge in Nonduality. It is through emergence and mergence that the Transcendent and Immanent Divine is revealed.

However, unifying the mystical and mathematical views of the continuum in this manner does not help mathematicians to make calculations, as Buckley points out in *The Continuity Debate*.[728] Nevertheless, for me, Peirce's notion of mergency helps me to put the entire topic of limits and continuity in the infinitesimal calculus into its integral, holistic perspective.

Regarding infinitesimals themselves, Peirce was quite willing to accept their existence even after he discovered Cantor's work in 1884, he tells us in his notes for a lecture to be given in 1903.[729] For instance, in his 1895 essay on Quantity, he wrote, "is [there] any absurdity in the conception of an infinitesimal any more than in the conception of a square root of negative unity?", a sentiment he had expressed in 1892, when introducing the concept of synechism. Then, in an undated 6-page essay titled 'The Question of Infinitesimals', not listed until Robin's supplement to his catalogue in 1971, Peirce wrote, "It is singular that nobody objects to $\sqrt{-1}$ as involving any contradiction, nor, since Cantor, are infinitely great quantities much objected to, but still the antique prejudice against infinitesimally small quantities remains."[730]

Even today, the confusion around the concepts of continuity and infinitesimals is not satisfactorily resolved, as Buckley points out in his concluding comments to *The Continuity Debate*. He suggests, "A non-standard analysis, such as the one presented by Abraham Robinson, may well turn out to be the system of calculus that is maximally useful for thorough analysis of continuous phenomenon, but even this would not prove infinitesimal."[731]

I'm not sure what 'prove infinitesimal' might mean here. For myself, I am seeking to show how the concepts of mathematics fit into a coherent conceptual model of the Totality of Existence, using the four basic principles of concept formation, defined on page 49: clarity, simplicity, integrity, and consistency, guided by the fundamental law of the Universe: opposites are never separate from each other. So, as a numeral, I could represent an infinitesimal as 0.000 ... 0001, where the ellipsis denotes an infinite number of zeroes. So the number that this numeral represents is both zero and not zero, which I am entirely comfortable with.

Regarding the way mathematicians view infinitesimals, Abraham Robinson (1918–1974) said in *Non-standard Analysis* that while Bolzano, Cauchy, and Weierstrass had apparently eliminated the need for infinitesimals in analysis with their concept of limit, this did not resolve the issue of whether it were still possible to create a mathematical theory of infinitesimals, as envisaged by Leibniz. This is what Robinson set out to do, saying:

> Arithmetic starts with the integers and proceeds by successively enlarging the number system by rational and negative numbers, irrational numbers, etc. But the next quite natural step after the reals, namely the introduction of infinitesimals, has simply been omitted. I think, in coming centuries it will be considered a great oddity in the history of mathematics that the first exact theory of infinitesimals was developed 300 years after the invention of the differential calculus.[732]

Robinson tells us that he developed non-standard analysis from mathematical logic, which includes something called 'model theory', of which he was a pioneer. He was particularly inspired by a paper that Thoralf Skolem (1887–1963) wrote in 1934, titled '*Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen*',[733] which Google translates as 'On the non-characterizability of the series of numbers by means of finite or countable infinite statements with exclusively numerical variables'.

I'm getting way out of my depth here. Nevertheless, it is interesting to note that while Frege is generally recognized as the key figure in the development of first-order predicate logic,[734] mathematical logic also evolved from Peirce to Skolem via Ernst Schröder (1841–1902) and Leopold Löwenheim (1878–1957), as Geraldine Brady tells us in *From Peirce to Skolem: A Neglected Chapter in the History of Logic*, a development of a Ph. D. thesis from the University of Oslo, appropriately because Skolem was Norwegian.[735]

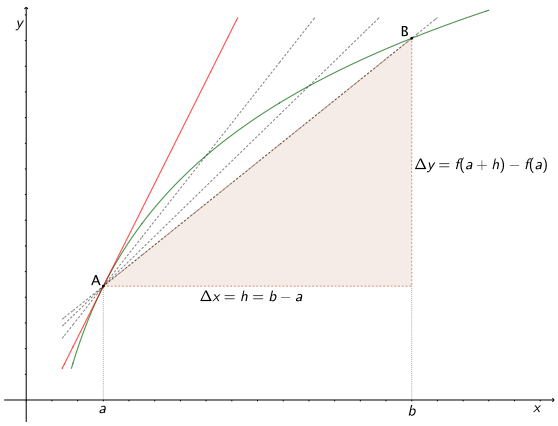
Of course, I have some awareness of the evolutionary history of logic, as the science of reason. However, I haven't studied this in any detail because it is too superficial, focusing attention on language, as signs and symbols, rather than on underlying human experience and cognitive maps before they are externally expressed. This is what turned me away from mathematics as an undergraduate in the early 1960s, when I was introduced to mathematical logic, just as I had previously abandoned physics and monetary economics because they did not make any sense as cogent, well-founded disciplines. For I could intuitively see that I was not being taught how humans actually reason or what it means to be a human being, essential if I were ever to heal the deep wound in the cultural psyche, as the split between science and religion, as I understood them as a teenager.

To resolve this discomfort, the Logos has created Integral Relational Logic within consciousness, looking at set theory, model theory, and category theory in a much more profound way than mathematicians, as specialists, study these subjects. Having said this, what can a cursory glance tell me about the types of number that underlie non-standard analysis?

Well, as far as I can ascertain, non-standard analysis is some form of metamathematical technique generating another type of real number, which Robinson denotes with $^*R$, which maps from $\mathbb{R}$, the standard real domain, in a one-to-one mapping.[736] However, having a more detailed understanding of what this means does not enhance my understanding of what it means to be a human being or increase the chances of completing the final revolution in science before abrupt climate change overwhelms us all in the next five to fifty years. So I'll leave the subject of infinitesimals and continuity here, coming back to it should this be necessary.

Needless to say, none of these metaphysical and mystical perspectives of the continuum were mentioned when I learned the infinitesimal calculus at high school in the late 1950s. Indeed, as far as I can tell from teachers on the Internet, even today, calculus is introduced to students very much along Leibniz's original lines, albeit with far greater conceptual clarity, despite the attempts in the 1800s to separate analysis from geometry. For instance, Grant Sanderson brilliantly helps us understand the fundamental theorem of the calculus in his 10-part graphical view of the calculus on his 3Blue1Brown YouTube channel, much clearer than I learned it at a student. So, for completeness, to end this section on the infinitesimal calculus, let me briefly review how I look at this subject today.

To begin with the differential calculus, what this is concerned with is finding the rate of change of a function, $y = f(x)$, as the slope of a continuous curve at each point. This is determined by finding the tangent at a particular point, such as A in this diagram. However, the difficulty here is that slope is defined as 'rise over run', which requires two points, not just one, to calculate. If a curve represents distance travelled over a period of time, then the slope of the curve at any moment is the instantaneous velocity, for $v = s/t$. So how do the speedometers in our cars resolve this paradoxical situation?



Well, what mathematicians have been doing since before Newton and Leibniz laid down the foundations of the infinitesimal calculus is to first draw a secant through points A and B, whose slope can be calculated, as illustrated by this slope triangle in GeoGebra. Here Δy marks the change in distance, for instance, during a period from a to b, as Δx. So $\Delta y/\Delta x$ marks the first approximation to the slope of the tangent at A.

Then, as B moves ever closer to A, the approximation gets better and better as Δx and Δy get smaller and smaller, passing through the continuum of real numbers as they approach zero from the positive side. The slope triangle thus vanishes into point A, and the slope of $f(x)$ at this point in Lagrange's notation is given by this expression:

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Of course, what scientists, engineers, mathematicians, amongst others, are really interested in is to find the rate of change of a function for any x. So we can modify this formula, adding Leibniz's notation, to give:

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

But what are dx and dy? Well, they look very much like infinitesimals to me, which seems to be how Robinson treats them in non-standard analysis. Yet, as numbers denoting infinitesimal change, they need to be treated carefully, not separating them. For 0/0 only makes sense under the strict rules of limit theory. Nevertheless, under some circumstances, I've seen maths teachers treat dx as if it is variable, as in u-substitution, as an integration technique, which is the inverse of the chain rule in differentiation.[737]

To give the simplest example of differentiation, other than that of a straight line, if $y = x^2$, then

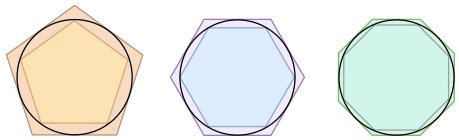
$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h = 2x$$

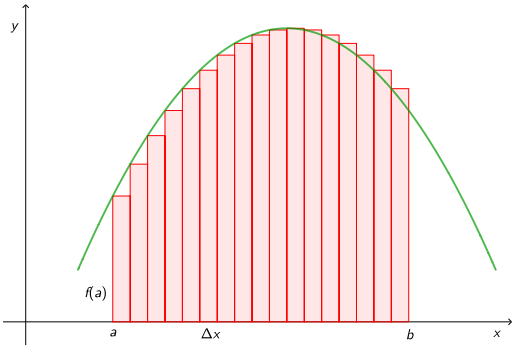
as $h \to 0$. In general, if $y = x^n$, then

$$\frac{dy}{dx} = nx^{n-1}$$

which arises directly from the binomial theorem for $(x + h)^n$, which we look at in the next chapter. And as many functions, like trigonometric and exponential ones, can be expressed as a power series, this is as far as we need to go with the basics of the differential calculus. As mathematics is a generative science emerging directly from the Continuum, mathematicians have great fun in showing how these simple seeds grow into ever-increasing complex structures of quite outstanding beauty.

Seeking to maintain an intuitive sense of wholeness in mathematics, as in all walks of life, having found the derivative of a function by subtraction, I now need to find the integral of a function by addition. Graphically, integration is represented by finding the area under a curve of whatever shape. Like the ancient Greeks, we know how to find the area of simple geometric shapes, like rectangles and triangles. And Archimedes found the area of a circle by approximating it to the area of polygons with an ever-increasing number of sides, known as the method of exhaustion, as this diagram from Wikipedia illustrates.[738]





But how do we find the area under a curve in general? Well, just as we find the derivative of a function by increasingly accurate approximations, we find areas in a similar manner, illustrated in this diagram. Here, I have approximated the area under the curve with rectangles of sides Δx and successive values of $f(x)$. The latter are Leibniz's infinitesimal ordinates as Δx tends towards zero. So the area of the first rectangle is $f(a) \cdot \Delta x$ and that of the second is $f(a + \Delta x) \cdot \Delta x$. The integral $F(x)$ of function $f(x)$ from $x = a$ to b is thus given by this expression:

$$F(a,b) = \lim_{\Delta x \to 0} \sum_{k=0}^{\frac{b-1}{n}} f(a + k\,\Delta x) \cdot \Delta x = \int_a^b f(x)dx$$

where $\Delta x = (b - a)/n$. So, as $n \to \infty$, $\Delta x \to 0$.

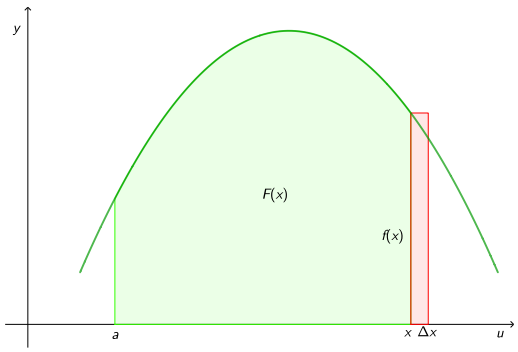
But how does this monstrous expression help us to find the area under the curve, other than in special cases? Well, rather than finding the area between two fixed points, mathematicians first define the function $F(x)$ as an indefinite integral:

$$F(x) = \int_a^x f(u)du$$

Then, as Newton and Leibniz brilliantly discovered, "The derivative of the indefinite integral as a function of x is equal to the value of $f(u)$ at the point x:"

$$F'(x) = f(x)$$

This is the fundamental theorem of the calculus, sometimes not presented very well in textbooks, as Courant and Robbins point out in *What Is Mathematics?* So, to overcome this problem, they give a simple proof of the theorem.[739] As they say, rather than looking at the derivative $F'(x)$ as a slope, we need to look at it in terms of area. But, again, to find the change of area at a point, we need to approximate it between two points, x and $x + \Delta x$, as in this diagram. Then



$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x)$$

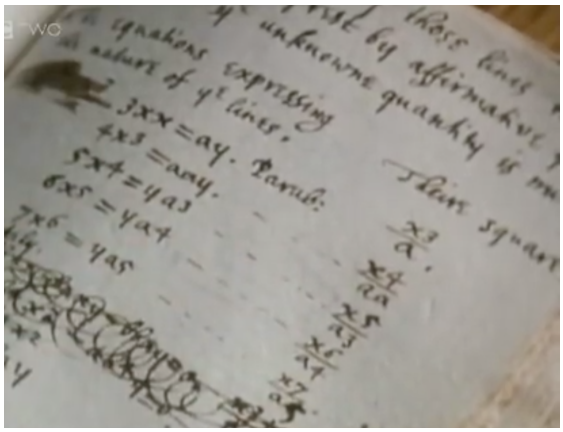
Here, we have the basic definition of the derivative of $F(x)$. Intuitively, this expresses the fact that the rate of change of the area under the curve $y = f(x)$ as x increases is equal to the height of the curve at the point x. So, to find the integral of $f(x)$, we need to find a group of general functions $g(x)$, whose derivatives equal $f(x)$, naturally called its antiderivative. For instance, we know

$$f'(x^3) = 3x^2$$

So, the indefinite integral $g(x)$ is given by this formula:

$$g(x) = \int x^2 = \frac{x^3}{3} + C$$

where C is a constant of integration, whose derivative is zero, disappearing in the integrand. Indeed, there is a whole sequence of one-to-one pairings of such power terms, as Newton showed in his notebook in 1665,[740] depicted here, which must have given him an immense sense of satisfaction. In general,



$$\int x^n = \frac{x^{n+1}}{n+1} + C$$

To find the definite integral between $x = a$ and b, mathematicians simply need to find $g(b) - g(a)$, not always so easy. For instance,

$$\int_1^2 x^2 = \frac{x^3}{3}\Big]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

In summary, the infinitesimal calculus is based on the relationships between three functions: $f(x)$, its derivative $f'(x)$, and its integral $F(x)$, whose derivative is $f(x)$, as the antiderivative of $f(x)$.

Of course, there are countless situations in which mathematical analysis can be applied. At the simplest level, if a function $v(t)$ denotes the velocity of a body over time, then the area under the curve denotes the distance travelled. For the antiderivative of the velocity is the derivative of the distance, giving

$$\frac{ds}{dt} = v$$

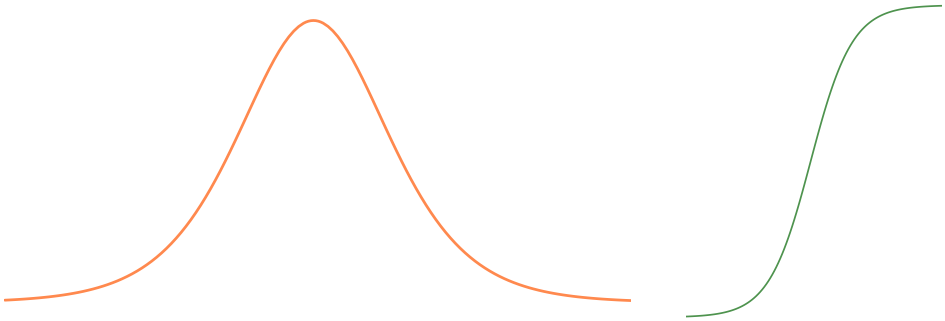
And if $acc(t)$ represents the acceleration of a rocket leaving the earth, then its integral denotes its changing velocity. However, while the differential calculus, as the science of change, can tell us much about the mechanical movements of objects in outer space, it can tell us little about inner space, about why we humans behave as we do, about the creative powers that are causing the rate of change in science and technology to accelerate exponentially, most evident in the way that the Internet has been expanding during the last few decades.

To understand this phenomenon, we need to study the whole of evolution since the most recent big bang by first studying the human phenomenon through introspection, as Teilhard realized in the 1920s. My own contributions in this field are outlined in my books *The Four Spheres: Healing the Split between Mysticism and Science* and *Through Evolution's Accumulation Point: Towards Its Glorious Culmination*, written during the autumn and winter of 2015 and 2016, and hence a little dated.

For not only is my outer world changing rapidly, my own awakening as a human being has taken giant strides in recent years, as I have learnt to see the prospects for Generation Z as they actually are, rather than from the idealized, optimistic perspective that had previously been guiding my studies of our rapidly changing world, reflecting the joy of my inner well-being. For whether we are ready to face this or not, it looks highly probable that the aptly named Gen X, Y, and Z will be the last generations of *Homo sapiens* to be born on Earth.

From the perspective of the infinitesimal calculus, I can perhaps best illustrate this in terms of two universal, schematic curves, which have been increasingly guiding my awakening of intelligence and

consciousness for the past forty years, depicted below. I call the red one on the left the life-and-death curve, as beings emerge from the Source and return there at the end of their lifespans. The green one on the right is the growth curve, showing the cumulative effects of these growth processes at any one point in time. It depicts growth under constraint, in contrast to the unbridled exponential function, whose every derivative is the function itself, quite amazing.



For instance, as I outline in *Through Evolution's Accumulation Point*, in 1956, M. King Hubbert, Chief Consultant (General Geology) for Shell, wrote a 56-page paper on what is today called 'peak oil', extending it into a 150-page report on 'Energy Resources' in 1962. He was particularly concerned with the finite limits of fossil fuels, such as oil, gas, and coal, on which our industrial society depends. In the first report, Hubbert used the function represented by the life-and-death schema to denote the rate of production P over time of a resource Q, such as oil, whose source, in this instance, is the Earth, rather than the Absolute:

$$P(t) = \frac{dQ}{dt}$$

Conversely, the area under the curve is the antiderivative of the growth curve at any one point of time t, giving this formula for the cumulative quantity Q that has been extracted from the Earth at any one moment:

$$Q(t) = \int_0^t P(u)du$$

When $t \to \infty$, we have

$$Q_{max} = \int_0^\infty P(u)du$$

Of course, we do not need to wait for infinite time for a finite resource to be exhausted, as the growth curve indicates. The period during which growth actually occurs is a tiny finite fraction of the infinite real-number line. For instance, in the thirteenth century, "the inhabitants of northeast England discovered that certain black rocks found along the seashore, and thereafter known as 'sea coles', would burn. Thus began the mining of coal and the first systematic exploitation of the earth's supply of fossil fuels," as Hubbert points out.[741] However, such exploitation cannot continue indefinitely, which became crystal clear in December 2015, when the last coalmine in the UK closed, bringing an 800-year industry to an end, in that country, at least.

One reason why this critical situation is not headline news is that traditionally human learning has been an evolutionary growth process under constraint. We are constrained by our personal conditioning, such as pre-, peri-, and postnatal traumatic events, by our cultural conditioning, as the seven pillars of unwisdom, in particular the Law of Contradiction, and collective archetypes, such as the supercilious way that men have been treating women during the five or six thousand years of the patriarchal epoch.

This mechanistic conditioning acts like a filter or veil masking our inner eyes, as Self-reflective Intelligence, inhibiting us from understanding just what is happening to our species at the present moment.

It lies deep in our sub- and unconscious psyches, and thus needs to be brought to the surface and examined in the brilliant light of day if we are to become fully awakened before our inevitable demise as a species. In my experience, this light is provided by Consciousness, which, as Absolute Reality, contains no constraints. So, it is quite possible for individual consciousness to expand and deepen to such an extent that it becomes coterminous with Consciousness itself.

Infinity of infinities

Having looked briefly at the way that continuity and infinity have historically been viewed in the infinitesimal calculus, in the final section of this chapter we look at what Cantor called transfinite numbers, completing the unification of mysticism and mathematics. In a sense, the first is opposite to both finite numbers, no matter how large, and infinitesimals, as their reciprocals. However, implicitly invoking the underlying hierarchical structure of the Cosmos, Cantor created an infinity of transfinite numbers, notions that are not easy to relate to human experience, busy, as we are, with the minutiae of life. Nevertheless, it has been vitally important for me to learn to experience the infinities in the psyche directly, establishing them as sound scientific concepts.

Ever since April 1982, when I consciously reached the Omega Point of evolution when working as a computer consultant in Kuwait in the middle of the Falklands War, I have realized that *Homo sapiens* is not Immortal. Because of the many existential risks facing humanity—not the least machines with so-called artificial intelligence and abrupt climate change—one day a generation of children will be born who will not grow old enough to have children of their own. As I write these words in December 2018, it appears that this final generation of humans on Earth has already been born.

To see this, I apply *infinity* to time, which we call *Eternity*, from Old French *eternité* 'eternity, perpetuity' or directly from Late Latin *æternālis*, from Latin *æternus* 'enduring, permanent', contraction of *æviternus* 'of great age, eternal, everlasting', from *ævum* or *ætās* 'age, lifetime, period of time', from PIE base **aiw-* or **ayu-* 'vital force, life; long life, eternity', root of many other words, such as *age*, *ever*, *utopia*, Greek *aiōn* 'age, vital force; a period of existence, a lifetime, a generation; a long space of time', and *Ayurveda* 'The ancient Hindu science of health and medicine', from Sanskrit *āyus* 'life' and *veda* '(sacred) knowledge'.

From this brief glimpse at the archaeology of language, we see a rich etymological history, as humans have sought to understand the relationship of humanity to Divinity and the meanings of their lives, faced with the certainty that everyone is born to die. *Eternity* entered the English language about 1374, through Chaucer's translation of Boethius' *Consolation of Philosophy*, written in jail in 524 CE, just before he was executed. At the time, *eternity* meant 'quality of being eternal', *eternal* meaning 'infinite in past and future duration; without beginning or end; that always has existed and always will exist', especially with reference to the Divine Being, as God, the Eternal. It was not until the 1580s that *eternity* came to mean 'infinite time'.

Why this is important for all of us today is that evolution passed through the most momentous turning point in its fourteen billion-year history around 2004, give or take a couple of years, as I explain in my 2016 book *Through Evolution's Accumulation Point: Towards Its Glorious Culmination*. By viewing the entire history of evolution as a single dynamical system, that book uses the nonlinear, discrete mathematics of chaos theory to explain what is causing scientists and technologists to drive the pace of scientific discovery and technological invention at unprecedented exponential rates of acceleration.

But we cannot intelligently adapt to our rapidly changing times by holding on to anything from the past. As I have discovered in my own experience, the only option is to turn the mechanistic horizontal dimension

of time into the life-giving vertical dimension in the Eternal Now, enabling us to realize our fullest potential as humans before the inevitable demise of our species.

What this means is that to generate the infinity of infinities, I begin with the intuitive concept of set, as Cantor defined it in very first sentence of the first part of his great synoptic work of 1895 and 1897, titled *Beiträge zur Begründung der transfiniten Mengenlehre*, *Beiträge* for short: "*Unter einer „Menge" verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die „Elemente" von M genannt werden) zu einem Ganzen.*"[742]

As this sentence is the first attempt to define the foundation stone for the whole of mathematics and human reason, it is vitally important that we understand what it means. In 1915, in *Contributions to the Founding of the Theory of Transfinite Numbers*, Philip E. B. Jourdain translated this sentence as "By an 'aggregate' (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the 'elements' of M." In the Preface, he explained why he had translated the title in the way he did:

> Since these memoirs are chiefly occupied with the various transfinite cardinal and ordinal numbers and not with investigations belonging to what is usually described as 'the theory of aggregates' or 'theory of sets' (*Mengenlehre*, *théorie des ensembles*),—the elements of the sets being real or complex numbers that are imaged as geometric 'points' in space in one or more dimensions,—the title given to them in this translation is more suitable.[743]

However, by this time, it was already realized that the concept of *Menge* is so general that elements could themselves be *Mengen*, as well, of course, as finite numbers. So the title of this book is rather misleading, not recognizing that Cantor was attempting a more abstract definition of *Menge*, going beyond the particular character of point sets, as Dauben points out. Saying that *Beiträge* set the tone for all that was to follow, Dauben translated the opening sentence in this way: "By a 'set' [*Menge*] we mean any collection *M* into a whole of definite, distinct objects *m* (which are called the 'elements' of *M*) of our perception [*Anschauung*] or of our thought."[744]

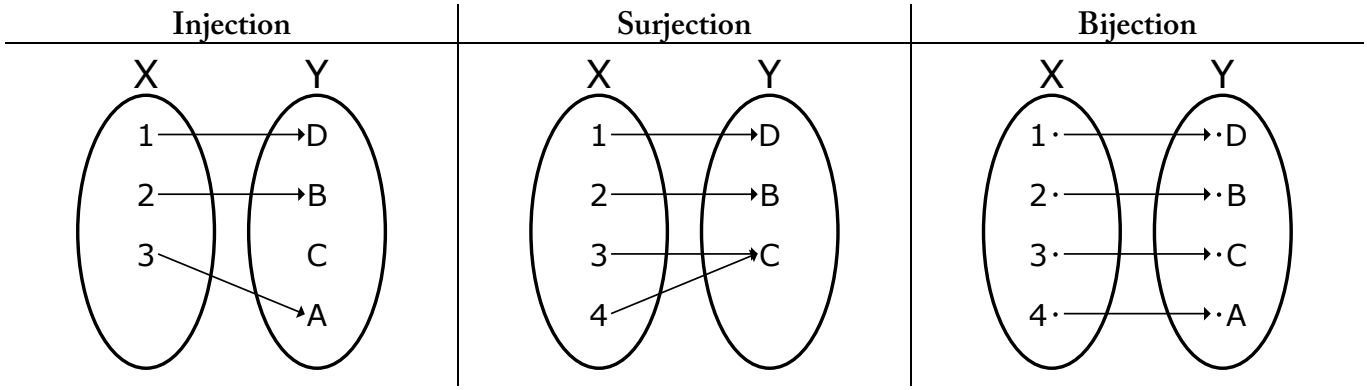
However, I feel that 'perception', as a translation of *Anschauung*, is rather ambiguous. For, as we see on page 75, giving the etymology of *Weltanschauung*, *Anschauung* has a root meaning 'mystical contemplation', as intuitive insight. So I feel that David M. Burton's translation of Cantor's definition of *set* is more appropriate: "By a set we are to understand any collection into a whole *M* of definite and distinguishable objects of our intuition or our thought. These objects are called the elements of *M*."[745] This is close to the definition of *set*, given on page 35, which I discovered in 1980, when setting out to develop a mathematical explanation for the underlying cause of the accelerating rate of exponential change in society, accepting paradoxes and self-contradictions in a nonaxiomatic, evolutionary approach to mathematics.[746]

This natural definition of set—which I use as the basis for meaningful conceptual formation from the meaningless data patterns of experience, as explained on page 50—leads to the definition of natural number on page 90, which is not based on the mechanistic ZFC axioms of set theory and the Peano axioms of arithmetic.

Nevertheless, I still need the mechanistic concept of function, which arises from the fundamental data-processing structure on page 4. The way that infinite sets are enumerated is through a mapping function between the members of a set, which Nicolas Bourbaki, the collective pseudonym of a group of (mainly French) mathematicians, called *injection*, *surjection*, and *bijection* in their fundamental book on set theory.[747]

As these diagrams from Wikipedia illustrate, an injective mapping between sets *X* and *Y* occurs when

each member in the domain *X* maps to a *unique* member in codomain *Y*. A surjective mapping, also called *onto*, happens when *all* members in *Y* are mapped from the members in *X*. A bijective mapping is both injective and surjective, called one-to-one correspondence or one-to-one onto mapping, distinct from the injective function, which is confusingly called a one-to-one mapping or function.



This means that an injection that is not a surjection leads to a strictly larger set when the sets are infinite. We have seen a few examples of bijective functions between the infinite number of natural numbers and the even ones, integers, rationals, and algebraic irrationals, leading to some rather strange results, which Galileo Galilei (1564–1642) was one of the first to notice, known as Galileo's paradox.[748] In *Dialogue Concerning the Two Chief Systems of the World*, published in 1632, he made this telling observation, "There are as many squares as there are numbers because they are just as numerous as their roots."[749] In other words, $n^2 \rightarrow n$, like Stifel's mapping in 1544 on page 180, which led to the discovery of logarithms. George Gamow illustrated this paradox in his book *One Two Three … Infinity* with an anecdote about David Hilbert, which has since become famous. Gamow quotes from an "unpublished, and even never written, but widely circulating volume: 'The Complete Collection of Hilbert Stories' by R. Courant", saying that Hilbert said this in his lectures on infinity:

> Let us imagine a hotel with a finite number of rooms, and assume that all the rooms are occupied. A new guest arrives and asks for a room. "Sorry—says the proprietor—but all the rooms are occupied." Now let us imagine a hotel with an *infinite* number of rooms, and all the rooms are occupied. To this hotel, too, comes a new guest and asks for a room.
>
> "But of course!"—exclaims the proprietor, and he moves the person previously occupying room N1 into room N2, the person from room N2 into room N3, the person from room N3 into room N4, and so on. … And the new customer receives room N1, which became free as the result of these transpositions.
>
> Let us imagine now a hotel with an infinite number of rooms, all taken up, and an infinite number of new guests who come in and ask for rooms.
>
> "Certainly, gentlemen," says the proprietor, "just wait a minute."
>
> He moves the occupant of N1 into N2, the occupant of N2 into N4, the occupant of N3 into N6, and so on, and so on …
>
> Now all odd-numbered rooms become free and the infinity of new guests can easily be accommodated in them.

This ordering of the natural numbers leads to two ways of denoting fundamental properties of sets. The cardinality of a set, denoted by its cardinal number, marks the *size* of the set. This is the attribute value of all sets that share the same cardinality. Cardinals, if I may use this word as a noun, are not concerned about the way that the members of a set are ordered. They are simply a measure of how many members there are in the set.

In contrast, ordinal numbers mark the *position* of a member in the sequence of a well-ordered set, such as *first*, *second*, *third*, and so on, distinct from the cardinal numbers *one*, *two*, *three*, etc. An ordinal is normally defined as the set of all ordinals that precede it. For instance, first is {}, the empty set, second is {1st}, third is {1st, 2nd}, and so on. With finite sets, there is a direct correspondence between ordinal and cardinal numbers.

However, this is not the case with the ordinals of infinite sets. Considering all the natural numbers, we have a sequence, first, second, third, … last, the 'last' ordinal denoting all the ordinals that precede it, which Cantor aptly denoted by ω (lower-case omega), the last letter in the Greek alphabet. To Cantor, ω was an *actual* infinity, called transfinite, in contrast to ∞, as a *potential* infinity.[750] The ordinal ω thus corresponds to the cardinality of the natural numbers, which he later denoted with $\aleph_0$, sometimes pronounced 'aleph-null', which could more appropriately be called 'aleph-nought', often spelt 'aleph-naught' in the USA, or 'aleph-zero', for *null* is not a number in English, although it is zero in German.

But now something quite extraordinary happens. Cantor defined an ordinal ω + 1 to denote the set of all the ordinals that precedes it, distinct from ω. This is called the order type (*ordnungtypus*) of the set rather than its ordinality. Yet, this set can be placed in a one-to-one correspondence with the natural numbers in an order isomorphism and so has the same cardinality as that of the natural numbers. So the arithmetic of the ordinals is quite different from that of the cardinals. With the latter, ∞ + ∞ = ∞, as we have seen. Cardinals do not change when they are added or multiplied together.

This is not the case with the ordinals, which can be added, multiplied, and even exponentiated to give ever-increasing ordinals, whose sets nevertheless biject onto the natural numbers. Although Cantor was not willing to admit to the existence of infinitesimals, he said in *Grundlagen* that the transfinite numbers are as real as any other mathematical object, like the natural numbers, using the term *reellen Zahlen* to distinguish them from *realen Zahlen*, distinct from the complex and hypercomplex numbers, such as quaternions.[751]

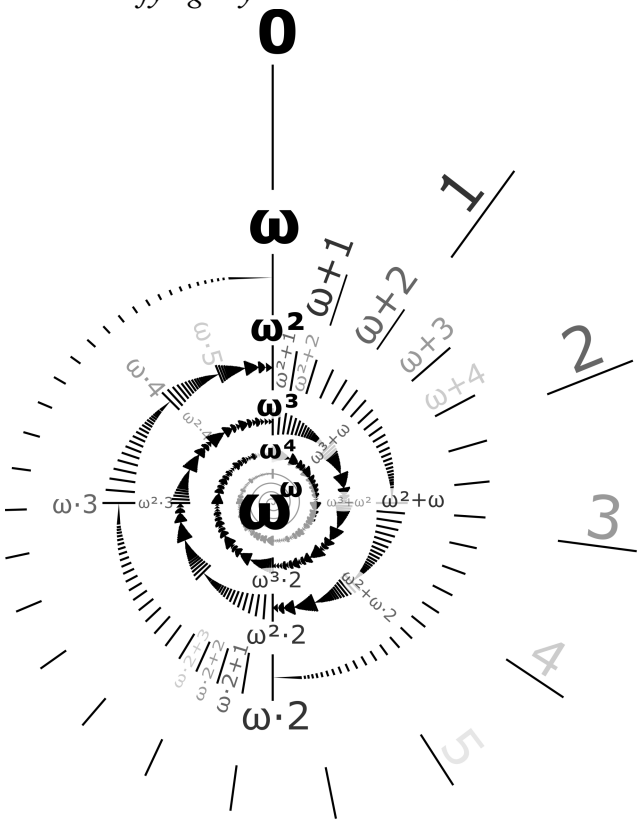
However, it is important to note that ordinal arithmetic is not commutative. For instance, while 3 + ω has a bijective relationship with ω, giving 3 + ω = ω, ω + 3 is a quite distinct ordinal or order type, as a Wikipedia article explains quite well.[752] This ordering of the ordinals leads to some quite extraordinary results.

As a starter, after all the natural numbers are added to ω, from the right, denoted by the outer spiral in the Wikipedia diagram on the next page, Cantor added another infinite sequence of ordinals to them, giving ω + ω = 2ω. The second ω is a subsequence that follows the first infinite subsequence. This leads to the next inner spiral of infinite subsequences, which terminates at ω^2. But now the fun begins. What is now generated is an infinite sequence of diminishing spirals of infinite subsequences, which terminate at ω^ω, at the centre of the diagram.

But we are not finished at the exponentiation of the ordinals. As with the ever-increasing finite numbers, mathematicians continue with tetration, giving a tower of ordinals ω high:

$$^{\omega}\omega = \omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}}}}}$$

After this, we enter a world of numbers that Cantor aptly called *Giganten* in a letter to Franz Goldscheider dated 11th October 1886.[753] Then, at the end of Part II of *Beiträge* in 1897, one of the last of his published writings, he introduced a new notation to denote what he called the second number-class of transfinite ordinals. Rather than embarking on pentation, hexation, and so on, the monstrosity that we have just formed is followed by a sequence of epsilon numbers, the first of which is ε_0 pronounced 'epsilon-nought' or 'epsilon-zero'.[754] However, exploring epsilon arithmetic further does not increase my understanding of what is happening here. As I understand the situation, ordinals can be added to ordinals in this manner indefinitely without changing the cardinality of the sets that they represent. They all have the cardinality of ω or $\aleph_0$.



I say $\aleph_0$ here, although in *Grundlagen* in 1883, when Cantor used the ω notation, he had not yet introduced the alephs—as an infinite sequence of infinities with cardinality $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, … $\aleph_\nu$, … . He did this in Part I of *Beiträge* in 1895. Amir D. Aczel suggests that Cantor chose this notation because *aleph* (א), the first letter of the Hebrew alphabet, is also the initial letter of *Ein Sof* (אין סוף), meaning 'infinity' or 'God'. As he says, "The letter aleph represents the infinite nature, and the oneness, of God."[755]

The aleph notation is an alternative to that which Cantor introduced in his 1874 and 1891 papers, when giving alternative proofs of the nondenumerability of the real numbers. In these, he refers to an infinite sequence of omegas, as ordinals: ω, ω_1, ω_2, ω_3, … ω_ν, …, having cardinality $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, … $\aleph_\nu$, ….[756]

However, in *Grundlagen*, Cantor focuses attention on just two number-classes (*Zahlenklasse*), saying that these can then be extended to a third, fourth, and so on. As Uwe Parpart tells us in his English translation, "the first number-class (I) is the aggregate of the finite whole numbers, … which [are] followed by the second number-class (II) consisting of certain infinite whole numbers following each other in determinate succession." Then in §13, he proves that the powers (*Mächtigkeiten*) of (I) and (II) immediately follow each other so that no powers lie in between.[757]

But before this, Cantor introduces the mysterious third number-class (III), today denoted by ω_1, as "the totality of all 'analytic' functions, i.e. functions of one or several variables generated through continuation of convergent power series, or the set of all functions of one or several real variables which are representable by trigonometric series".[758] This statement has led to some confusion in the literature, which Aczel clarifies with this endnote:

> The discontinuous functions are the ones giving this set its higher order of infinity. A popular book about mathematics describes this order of infinity as: "all the curves you can draw on the back of a stamp". This statement is incorrect not because of the small size of the stamp (we already know that size as we know it doesn't affect infinity), but rather because curves alone—continuous drawings on the stamp—will not do. Continuous functions have the order of infinity of the real numbers. The discontinuous functions are of a higher order.[759]

In *Grundlagen*, Cantor denoted the first ordinal in this third number-class with Ω, as if this were the last. However, in a letter to Dedekind dated 28th July 1899, Cantor denoted the entire system of alephs with

Ω, denoting its cardinality with tav/taf (ת), the last letter of the Hebrew alphabet, to denote finality. Here is a Google translate of a property of Ω: "The system Ω of all numbers is an inconsistent, an absolutely infinite plurality." So, it seems that Cantor eventually acknowledged that his great mathematical edifice was not based on the fundamental principle of mathematical proof and deduction, a situation that must have disturbed him greatly.[760]

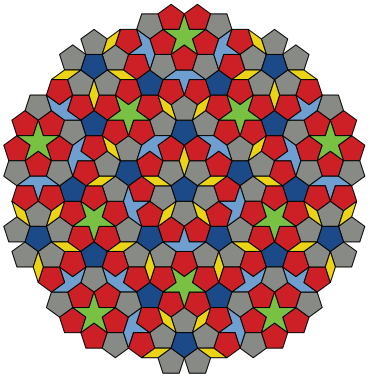
However, the infinite sequence of ordinals is not the only way that an infinite sequence of infinities can be generated. Almost as an afterthought, in 1891, Cantor introduced his famous diagonal argument to prove that the power set of a set is larger than the set itself, and hence is also uncountable, although he did not use the term *power set*.[761] This is how he described this important proof, which I have not found in English translation:

> This proof seems remarkable not only because of its great simplicity, but also because the principle which it follows can be extended directly to the general theorem, that the powers of well-defined sets have no maximum or, what is the same, that in place of any given set *L* another set *M* can be placed which is of greater power than *L*.[762]

The word *power* here is a translation of *Mächtigkeiten*, which can also be translated as 'cardinality', used frequently in his earlier papers, causing me some confusion from the English translations of the development of his thoughts. Cantor introduced this simpler proof for the nondenumerability of some sets because he was dissatisfied with his 1874 proof in *Grundlagen* (for which I have not found an English translation), where he used *Mächtigkeiten* for 'power' or 'cardinality'.

It is interesting to note here that the *Oxford English Dictionary* tells us that English *power* could mean in Charles Dickens' time 'a large number, multitude, quantity, or abundance', sometimes as a counting word. Indeed, in the dialect of the Upper Southern United States today, the words *powerful* and *mighty* are intensives used frequently in the same way as *very* and *power* can mean 'a large number or amount'.[763] Of course, *Mächtigkeiten* is cognate with *mighty*, literally 'mightiness', from PIE-base **magh-* 'to be able, have power', also root of *magic* and *machine*.

To explain further, the power set of {a b c} has eight members (2^3): {{a b c} {a b} {b c} {c a} {a} {b} {c} {}}. In general, the cardinality of the power set with n members is 2^n. This relationship between sets also applies to infinite sets, such as countable ones, whose cardinality Cantor denoted by $\aleph_0$. "In particular, by considering all the possible subsets of the set of whole numbers, Cantor was able to show that $2^{\aleph_0} = \mathfrak{c}$."[764]



It is not only the set of real numbers that is uncountable. For instance, Roger Penrose has pointed out that there are $2^{\aleph_0}$ different patterns of nonperiodic tilings, such as this one, which is self-similar and nonperiodic (it lacks translational symmetry), yet covers the entire infinite plane.[765] As he said, "Different arrangements are, in a certain 'finite' sense, all indistinguishable from each other. Thus, no matter how large a finite portion is selected in one such pattern, this finite portion will appear somewhere in every completed pattern (infinitely many times, in fact)."[766]

In turn, the cardinality of the power set of a set with $\mathfrak{c}$ members is strictly greater that the cardinality of $\mathfrak{c}$, a sequence that can be continued indefinitely, like the ordinals, which map to the infinite sequence of alephs.

To illustrate, an element in the power set of the nondenumerable real numbers is {7, -4, ⅗, $\sqrt{2}$, π}, which is not actually a number, and so does not have a reciprocal, as a numerical inverse. Another element is {7, φ, δ}, where φ is the golden ratio and δ is the Feigenbaum bifurcation velocity constant, which I have

used to explain mathematically why our political systems are degenerating into chaos at the moment. There is a mystical and psychospiritual way of bringing universal order to this chaos, not yet widely known. As an illustration, an element in the power set of the power set of the reals is {{7, -4, ⅗, $\sqrt{2}$, π}, {7, φ, δ}}. So reals can be stacked up in subsets of subsets of subsets indefinitely, culminating in Transfinity.

However, Cantor left this infinite sequence of power sets unnamed, as far as I can tell. So, to help us understand this infinite sequence, Gabe Perez-Giz tells us on the Infinite Series YouTube channel that mathematicians have named this sequence with beth numbers (ב), a variation of the second Hebrew letter spelled *bet*. So if $\mathcal{P}(X)$ denotes a power set and $|X|$ denotes the size of a set, we have an infinite sequence of strictly increasing power sets named $\beth_0$, $\beth_1$, $\beth_2$, $\beth_3$, … $\beth_\omega$, …:

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < \left|\mathcal{P}\left(\mathcal{P}(\mathbb{N})\right)\right| < \left|\mathcal{P}\left(\mathcal{P}\left(\mathcal{P}(\mathbb{N})\right)\right)\right| < \cdots$$

Now, $\aleph_0 = \beth_0 = |\mathbb{N}|$ from the way that these countable infinities are constructed. However, as we have just seen, Cantor also proved that $\beth_1 = 2^{\aleph_0} = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \mathfrak{c}$. It was thus natural to ask, "is $\aleph_1$ equal to $\beth_1$?"[767] He conjectured that these two infinite cardinals are equal, known as the *continuum hypothesis*, which asserts that there are no sets whose cardinality is strictly between $\aleph_0$ and $\mathfrak{c}$, with $\aleph_1$ being a candidate. None have been found, so such a hypothesis makes complete sense. However, proving it is a little trickier.

Proving that the real number continuum is the smallest noncountable set is so important for mathematicians that it was the first of 23 unresolved mathematical problems that David Hilbert posed at the International Congress of Mathematicians in Paris in 1900, even more important than proving that the axioms are consistent, the second problem in the list.[768]

Applying this hypothesis to the two ways of creating an infinite sequence of infinite cardinals, the generalized continuum hypothesis asks whether it is true that $2^{\aleph_n} = \beth_{n+1} = \aleph_{n+1}$? In the event, Kurt Gödel (1906–1978) in 1940[769] and Paul Cohen (1934–2007) in 1963[770] showed that the hypothesis can neither be disproved nor be proved using the ZFC axioms of set theory, provided these axioms are consistent.[771] Or we could say is that the generalized continuum hypothesis is both unprovable and undisprovable through axiomatic, linear reasoning, one of countless instances of the universal Principle of Unity. Nevertheless, for all practical purposes, it is reasonable to assume that the infinite sequence of power sets contains all distinct infinite cardinals.

However, we now come to the central difference of how mathematicians and I look at sets. In Integral Relational Logic, I define a set in the natural way, as a group of entities with a common property. But at the highest reaches of abstraction that we are dealing with here, the sets are so vast that they cannot be said to have a common property. Furthermore, it is possible to create sets of ordinals indefinitely. However, as Cesare Burali-Forti (1861–1931) pointed out in 1897, constructing a set that contains all ordinals leads to a paradox, as a set that is supposedly larger than itself.[772] For, as Bertrand Russell simply pointed out in *Principles of Mathematics* in 1903, "there cannot be a largest ordinal number, because every ordinal is increased by the addition of 1." Similarly, the power set of all sets, supposedly the largest set, leads to a set that is larger than the supposed maximum.[773]

In terms of the cardinals, the power set of the set of the so-called largest set is strictly larger than this set, a semantic contradiction that has led mathematicians into contortions in an attempt to expel paradoxes from the foundations of mathematics. This cannot be done through axiomatic, linear reasoning, as Kurt Gödel, Alan Turing, and Alonzo Church demonstrated in the 1930s, to much distress.

Contemplating the Totality of Existence in this orderly manner can paradoxically lead to severe psychological disturbances, when we live in denial of the fundamental law of the Universe: *Wholeness is the union of all opposites*. Mathematical logicians, such as Cantor, are particularly prone to suffer in this way when they explore the shaky foundations of mathematics and hence all knowledge without the necessary mystical experience, as Amir D. Aczel reminds us in *The Mystery of the Aleph*. It is thus perhaps not surprising that many avoid such investigations in their lives, inhibiting us from discovering the roots cause of our collective, cultural, and personal malaise and hence its remedy.

For myself, what I do at this stage in my reasoning is to drop into the meditation practice I began in October 1983, transcending the categories, as I describe on page 72. I denote the 'largest' infinity with $\aleph_\infty$, where ∞ is $\aleph_\infty$, defined recursively, *ad infinitum*, allowing it to dissolve into Transfinity, merging into the Continuum, which is the Absolute or Ultimate Reality in my experience. I denote the Transfinite Absolute, with capital omega Ω, corresponding to the Omega Point, which is inseparable from the Alpha Point, corresponding to zero or *Shūnya* 'Emptiness', as we see at the beginning of this chapter on the number system.

By unifying mysticism and mathematics in this way, I have found Love, Peace, Wholeness, and the Truth, which I set out to discover as a seven-year old seventy years ago, when I realized that I had been born into a world at war with itself, reiterating an excruciatingly painful prenatal trauma. As my fragmented mind and split psyche have now been healed in Wholeness, nearly as much as possible, I feel that my life's work is complete. I could die today knowing that I have been successful far beyond my wildest dreams.

However, even though there is nothing and nobody outside me, I am still active in this body-mind-soul organism as a human being, seeking to assist with the awakening of Intelligence and evolutionary convergence to the best of my ability. So, if nothing else happens in the meantime, I plan to write Chapters 4 and 5 on 'Sequences and Series' and 'Universal Algebra' during 2019, clarifying some mathematical constructs that I don't yet fully understand.

Epilogue

Although this book illustrates how Integral Relational Logic maps pure mathematics as an emergent, generative science of patterns and relationships—just as it can model all other specialist disciplines of learning—it is not actually my most important mathematical treatise from an anthropological, evolutionary perspective.

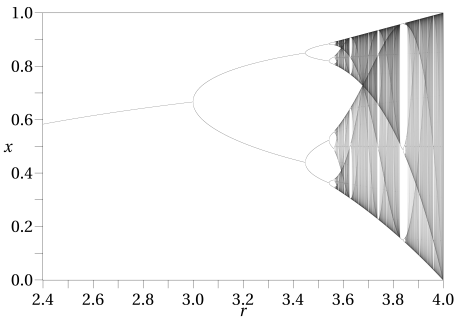
What I am most concerned with here is our inability to see that what is happening to humanity at the present time is the culmination of some fourteen billion years of evolution since the most recent big bang, illustrated in the diagram on pages xiii and 29. This year, 2019, it is fifteen years, give or take a couple, since evolution passed through its Accumulation Point into chaos, when there are no longer any further discrete evolutionary turning points to be discerned. Like a dripping tap turned full on, the flow is now continuous and undivided. In mathematical terms, the period between the most recent big bang and evolution's Accumulation Point, explained in my book *Through Evolution's Accumulation Point*, is given by this formula:

$$AP = a\sum_{n=0}^{\infty}\left(\frac{1}{\delta}\right)^{n} = \frac{a\delta}{\delta - 1}$$

where a is about ten and a half billion years, the period between the most recent big bang and the emergence of the first self-reproducing forms of life on Earth, and δ is the Feigenbaum bifurcation velocity constant, about 4.6692. The value of a might seem rather rough and ready. However, we can obtain a reasonably accurate year for evolution's Accumulation Point (like the Singularity on page 12) from the 11th, 12th, and 13th evolutionary turning points: the beginning of the Industrial Revolution around 1750, the invention of the stored-program computer in 1948/1949, and the introduction of the World Wide Web in 1991. Furthermore, δ is a limiting value of nonlinear dynamical systems as they near their accumulation point. So even though the first few terms in the series are approximate, it is quite legitimate to use 4.6692 as the bifurcation velocity constant as the periods between the evolutionary points approach zero.

This evolutionary model is generated from this nonlinear difference equation, known as the logistic map,[816] where the term 1 - x_n keeps the growth within bounds, since as x_n rises, 1 - x_n falls.[817]

$$x_{n+1} = ax_n(1 - x_n)$$



The logistic map surprisingly leads to chaotic patterns, as Robert May discovered when he used it to study the growth of populations under constraint in the 1970s, such as fish in a pond. He went on to become the Chief Scientist to the UK government and President of the Royal Society. So the logistic map has an impeccable scientific pedigree, leading to this diagram, showing how bifurcating systems, like evolution, degenerate into chaos if the participants in the system are not aware of what is happening, as a whole, like politics today.

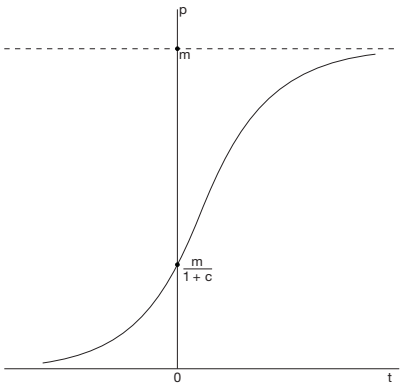
The logistic map is the discrete version of a differential equation, whose solution is the logistic function, which Pierre François Verhulst introduced in 1844, when studying the potential population growth $p(t)$ of the newly formed nation of Belgium:[818]

$$p(t) = \frac{m}{1 + ce^{-rt}}$$

where r is the underlying rate of growth, m is the maximum population, and

$$c = \frac{m - p_0}{p_0} = \frac{m}{p_0} - 1$$

where p_0 is the population at time zero.

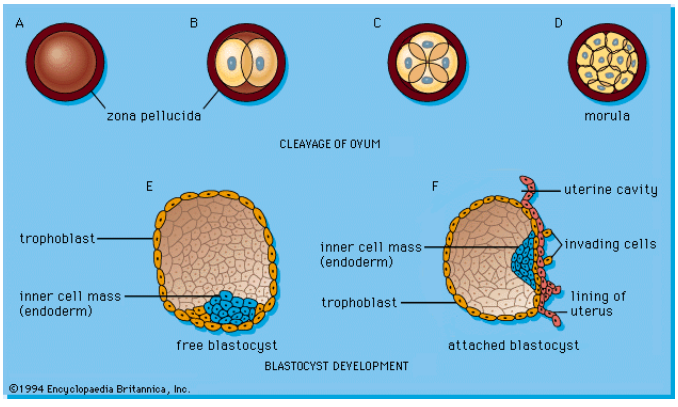


Here is a diagram illustrating the familiar S-shape of the growth curve, which the mathematical biologist D'Arcy Wentworth Thompson much studied in his classic book *On Growth and Form*, from 1917 and 1942. As he pointed out, this one curve recurs in endless shapes and circumstances, for mathematics generalizes and "is fond of giving the same name to different things".[819]

We often know the growth curve as the learning curve, where its steep section illustrates the way that learning can accelerate once the comparatively slow initial coordinating phase is complete, as we learn to see a picture of the learning process, as a whole. But learning, like any other evolutionary growth process, does not continue indefinitely. It eventually reaches a maximum at the top of the curve, at its saturation point, where the curve turns towards its maximum, which the total human population is approaching today.

The fact that the logistic map and function can be applied to so many different situations arises from the fractal-like, holographic nature of the Cosmos, whose underlying structure is a multidimensional network of hierarchical relationships, as the chapter on Integral Relational Logic explains.

So is it inevitable for our global society to degenerate into chaos at the culmination of fourteen billion years of bifurcating evolutionary development? Well, our bodies are bifurcating systems, formed at conception when female and male haploid gametes, each with one of a pair of a set of chromosomes, become unified in a zygote, a diploid cell, cognate with *yoga* and *join*, containing genetic information from both parents. The zygote then splits into two, four, and eight similar cells to form a morula, from the Latin *morum* 'mulberry' rather like a blackberry. Then something magical happens in the human embryo around five days after conception. Although every cell contains the same genetic information, some cells form different characteristics from others to form a blastocyst, as this diagram illustrates.



How is this possible? In general, all cells, whether they be bone cells, brain cells, heart cells, skin cells, or whatever, contain the same genetic information, forming distinct organs, each with its own function, working in harmony with all other organs. As Richard Dawkins says, "Skin cells have the same genes as bone cells, but different genes are switched on in the two tissues. … Genes … behave as if they 'know' where they are. … There are formidable difficulties in working out how they 'know'."[820]

These difficulties arise because, while we are all unique, differentiated cells in the body politic, we lack the innate sense of wholeness that characterizes the cells in our bodies, even though Wholeness is the True Nature of us all. While Integral Relational Logic provides the framework or skeleton for the body of all knowledge that is the elusive Theory of Everything, as we all implicitly use this science of thought and consciousness to form concepts and organize our ideas, it also acts rather like the genes in the specialist cells in our bodies, being used by specialists in a wide variety of different ways.

However, as most of us are not continuously aware of these universal holistic processes within us, we have developed a sense of identity that is a fragment, based on attachment to religion, country, economic ideology, business enterprise, occupation, sports team, family, and so on, separate from the Whole. We are thus not generally aware that the Divine Essence that we all share is Love and that the Cosmic Context for all our lives is Consciousness. If our bodies functioned like society, they would not survive for long, blowing themselves apart, as we are witnessing in society today, with rising isolationism, ultranationalism, and demagogical populism, manifesting in the chaos of Brexit and Trump's America, for instance.

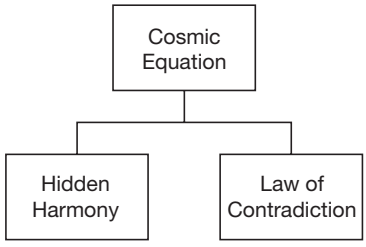
Because we are all taught from an early age that we must fight our fellow human beings for a slice of the finite monetary cake, most civilizations are built on the seven pillars of unwisdom, briefly defined in the Prologue on page xiv—misconceptions of God, Universe, Life, humanity, money, justice, and reason. So, if Life is to bring universal order to our chaotic society, harmonizing evolutionary convergence, it can only do so by rebuilding society on the seven pillars of wisdom, when each of us knows deep in our hearts that nobody is ever separate from any other being, including the Supreme Being, for an instant.

By far the most important of these is the seventh pillar of wisdom, as the fundamental law of the Universe, which few have written about during the history of human learning. Notable exceptions are Laozi's *Tao Te Ching*, Jung's *Commentary* on Richard Wilhelm's translation of *The Secret of the Golden Flower*, and Osho's *Hidden Harmony*, a sequence of discourses on the fragments of Heraclitus. Among my contemporaries, Tim Freke's *The Mystery Experience: A Revolutionary Approach to Spiritual Awakening*, in which he introduced the words *paralogical* and *paradoxity*,[821] is a vitally important step in this awakening process.

There is just one more step to take, which is the purpose of this book on *Unifying Mysticism and Mathematics*. As mathematics is based on the seventh pillar of unwisdom, it is necessary to show how mathematics can be rebuilt on the seventh pillar of wisdom, which I call the Principle of Unity or the Cosmic Equation, the simple, elegant equation that Einstein and Hawking sought at the heart of their attempts to develop their theories of everything within physics:

$$W = A = A \cup \neg A$$

This diagram, a revision of ones on pages 67 and 73, shows how the Cosmic Equation unifies the incompatibilities between Heraclitus' Hidden Harmony and Aristotle's Law of Contradiction, which has sent Western thought into the cul-de-sac it finds itself in today, illogically putting second things first.



However, the seven pillars of unwisdom are so deeply entrenched within the cultural psyche, I have long realized that it would take a tsunami to sweep away these misconceptions, acting like Shiva, the destroyer, before Brahma, the creator, could rebuild society on the seven pillars of wisdom. Such a radical transformation of consciousness would be earth-shattering, but maybe it could happen by using a mathematical model of the entire history of evolution to complete the final revolution in

science, just as Kepler and Newton's *New Astronomy*, *The Harmony of the World*, and *Mathematical Principles of Natural Philosophy* completed the first in the 1600s.

Nevertheless, my optimism in this regard has become somewhat diminished since 2011, when I gave a poster presentation on 'The Two Dimensions of Time' at the Science and Nonduality (SAND) conference, whose theme that year was 'On the Edge of Time'. To demonstrate why it is vitally important to see the primary-secondary relationship between the vertical and horizontal dimensions of time, I put some numbers onto Teilhard's four-stage model of evolution, outlined in this table, which I presented there and in my book *The Four Spheres*, published on the Web in December 2015.

Evolutionary stages, years ago						Transition stages, years ago		
Teilhard	**Type**	**Realm**	**Start**	**End**	**Duration**	**Start**	**End**	**Duration**
Prelife	Physical	Hylosphere	14,000,000,000	4,500,000,000	9,500,000,000	4,500,000,000	3,500,000,000	1,000,000,000
Life	Biological	Biosphere	3,500,000,000	25,000	3,500,000,000	25,000	5,000	20,000
Thought	Mental	Noosphere	5,000	50	5,000	50	-50	100
Superlife	Spiritual	Numinosphere	-50	-300	250			

As you can see, I naively thought that the transition period between the mental and spiritual stages of the global awakening of intelligence and consciousness would take about one hundred years, after which the eschatological Age of Light could perhaps last for a few generations before the inevitable extinction of *Homo sapiens*, and hence *Homo divinus* and *Homo universalis*, as some visionaries have foreseen the future, as mentioned on page 11. This final stage of evolution, which is more involutionary than evolutionary, corresponds to Ken Wilber's third, transpersonal phase of human development, illustrated on pages xiii and 33, with the second tier in his spectrum of consciousness, depicted on pages xv and 68, denoting the transition period between the two.

However, from what I have read and heard, revolutionary scientists are not yet ready to acknowledge that mystical psychology is the primary science on which all the sciences and humanities are built. Even when attempting to bring consciousness into science, they tend to act rather like Tycho Brahe, who thought that the inner planets revolve around the Sun, while the Sun, Moon, and outer planets revolve around the Earth. During the first scientific revolution, he thus found a compromise between the geocentric view of Aristotle and Ptolemy and the heliocentric view of Aristarchus (as the Greek Copernicus)[822] and Copernicus himself.

Similarly, in *Global Mind Change* in 1988, Willis Harman hedged his bets, defining three metaphysical perspectives: M-1, in which matter gives rise to mind (materialistic monism), M-2, in which matter and mind coexist as two fundamentally different kinds of stuff, à la Descartes (dualism), and M-3, in which the ultimate stuff of the Universe is recognized as consciousness, mind thus giving rise to matter (transcendental monism).[823] Nothing much seems to have changed during the thirty years since then. With scientists still holding tenaciously to the materialistic paradigm (M-1), some attempt to find a compromise between the spiritual and materialistic worldviews, even fighting wars between them (M-2), not recognizing that all there is, is Consciousness (M-3).

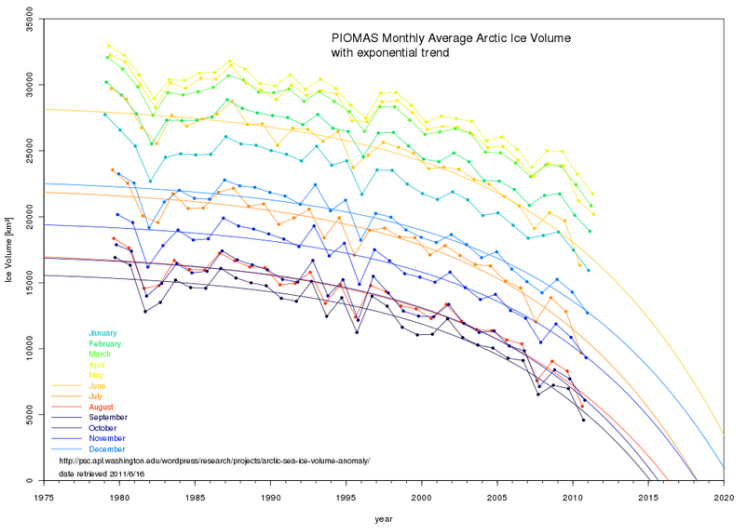
As a consequence, science is not ready to develop a comprehensive model of the psychodynamics of society, explaining why it is vehemently opposed to acknowledging what mystics like J. Krishnamurti, Ramesh S. Balsekar, and Rupert Spira have been saying in such books as *The Awakening of Intelligence*,

Consciousness Speaks, and *The Nature of Consciousness*. And without the holistic science of humanity that Fromm called for in 1976, we have little chance of rebuilding our education and economic systems on the seven pillars of wisdom, or even realizing World Peace, when the long-running war between science and religion would come to an end.

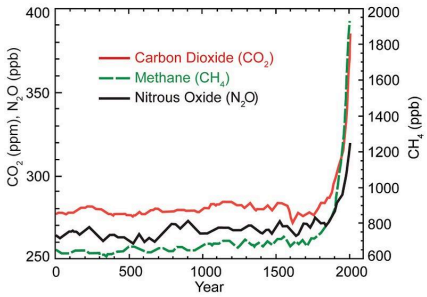
Nor is this all. The many existential risks that humanity faces today, which I reviewed in the autumn of 2015 in *The Four Spheres*, are today much more acute, as I have discovered from reading *Extinction Dialogs: How to Live with Death in Mind* by Carolyn Baker and Guy McPherson in the autumn of 2016. This vitally insightful book reviews the urgent psychological and ecological aspects of human extinction, which the spiritual teacher Andrew Harvey had asked them to write. I met Guy for lunch in Oslo in December 2017 and asked him if the dangers from the accelerating release of methane gas are the most dangerous facing humanity today.

He told me that there is an even greater danger. What is slowing down climate change today is global dimming, caused by pollution from industrial society. Yet, if this were to collapse in order for us humans to realize our fullest potential as intelligent, conscious beings, then global dimming would diminish, causing an accelerating increase in positive feedback loops, such as that of methane.

Following that meeting, in May 2018, I wrote a blog post reviewing this critical situation, which I summarize here. First, the Annual Review of Earth and Planetary Sciences and the Polar Science Centre project that the Arctic will be ice-free by the summer of 2019, as this chart of the exponential trend indicates.



To see what effects these projections could have on climate change in the immediate future, the next chart shows the way that the concentration of three principal greenhouse gases have increased since the beginning of the industrial revolution, around 1750. Now there are between 500 and 5,000 gigatons of methane frozen in the East Siberian Arctic Shelf, north of Russia, compared to just 5 gigatons currently in the atmosphere. And methane is 150 times more potent a greenhouse gas than carbon dioxide during its first ten years in the atmosphere.

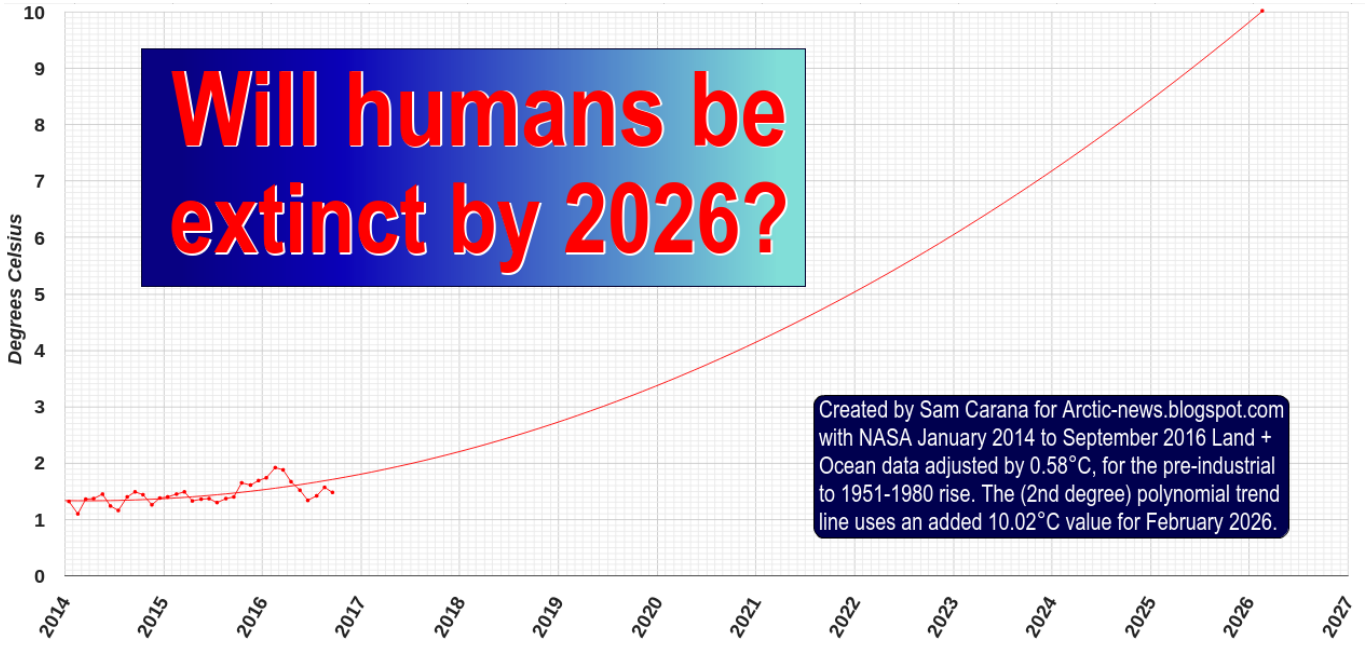


So if just one per cent of the minimum estimate of methane trapped in the Arctic were released into the atmosphere, this would double its parts per billion, having the effect of an increase of 300 parts per million of carbon dioxide, an increase of 75%, far beyond the 410 ppm in April 2018. As planet Earth warms, methane is likely to be released into the atmosphere at ever-increasing exponential rates through a positive feedback loop.

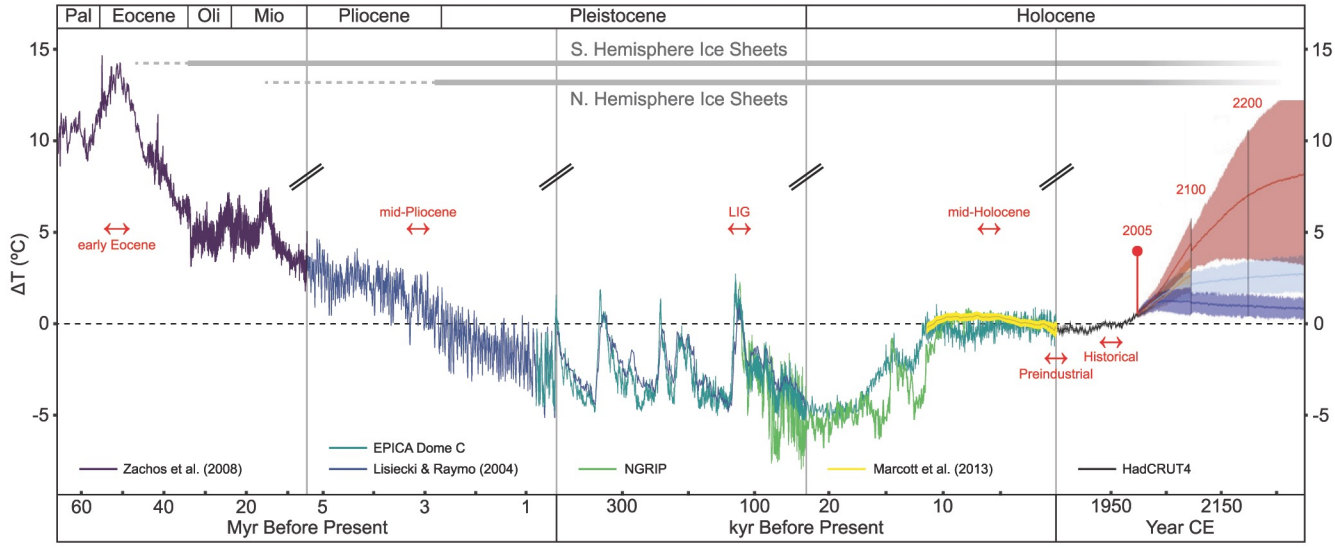
One consequence of this is that governmental attempts to keep the average temperature on Earth below two degrees above pre-industrial levels through the Intergovernmental Panel on Climate Change (IPCC) are way off the mark. Even predictions that temperatures could actually rise by four or five degrees by 2100, making the Earth uninhabitable by then, fall far short of what is likely to happen.

For instance, my friend David Wasdell, a leading researcher into the ecological and psychological implications of rapid climate change, said in his Christmas letter of 2018 that his earlier estimates on global warming, which Guy quoted in his book, are far too conservative. He is now saying that temperatures are likely to rise by 10℃ in the foreseeable future, an estimate with which James E. Hansen, formerly head of the NASA Goddard Institute for Space Studies (GISS), concurs.

Sam Carana, editor of the Arctic News blogspot, is saying much the same thing, as this chart illustrates most vividly. It seems that 'Sam Carana' is a collective moniker for a group of intrepid climate scientists—like that of Nicolas Bourbaki for mathematicians—who are saying what politicians and the general public do not want to hear.



I don't know if temperatures will actually increase this fast. Having a reasonable understanding of exponential growth, I know that this scenario is entirely possible. To set this possibility into a broader context, on 13th December 2018, Paul Beckwith, a well-known climate science educator, presented a review of a paper published on 26th December 2018 in the *Proceedings of the National Academy of Science in the United States of America* on his YouTube channel,[824] also reviewed in *Newsweek* on 10th December 2018.[825] This chart, from the paper,[826] succinctly summarizes the situation we all face at the moment:



As you can see, by conservative estimates, the planet's climate could be comparable to that of the mid-Pliocene era of three million years ago by 2030. And by 2150 it could go back to the climate of the Eocene period, fifty million years ago, not long after the fifth, most recent mass extinction of species on Earth, when temperatures were over 10℃ above base level. The chart also shows that following the five major ice ages, temperatures have been comparatively even during the ten thousand years of the Holocene epoch, giving humans and the flora and fauna we depend on for our survival a tiny window in time in which to thrive before our inevitable demise.

It is vitally important not to blame humans for this critical situation. Even if our lives are awarely (consciously and intelligently) guided by the Principle of Unity, enabling us to live in love, peace, and harmony with each other and our environment, we are all the products of some 13.8 billion years of evolution, never separate from the vast Ocean of Consciousness for an instant. So, we don't have the power or free will to defy the fundamental law of the Universe, through changes in lifestyle, geoengineering, or whatever.

So, as there is absolutely nothing that any of us can do to prevent the extinction of our species, I do not wish to ponder further the how and when of this inevitability any more than I contemplate the how and when of the death of my body. Rather, focusing attention on the positive in the Eternal Now, following the advice of Andrew Harvey and Carolyn Baker, quoted at the end of the Prologue on page xxxii, is it still possible to realize Matthew Fox's vision of scientists and inventors joining hands with mystics and contemplatives "so that our action flows from being and from a deep place of return to the Source"?

Connecting with scientists and mystics as an ordinary human being is something that I have been attempting to manifest for many years, with little success so far, generating not a little inner conflict, which I can only resolve through my meditation practice. Basing my life on the Immortal Ground of Being that we all share has become of paramount importance, faced, as I am, with the imminent extinction of *Homo sapiens*.

Most significantly, I cannot realize Inner Peace by wanting something to happen that is not meant to happen. For as Ramana Maharshi wrote to his mother, as 1898 turned into 1899, when she tried to persuade him to return home from Arunachala as a nineteen-year-old, "What is not meant to happen will not happen, however much you wish it. What is meant to happen will happen, no matter what you do to prevent it. This is certain. Therefore the best path is to remain silent."[827]

In this contemplative space, where I embrace every moment of my life and those of all other beings who have ever lived in Stillness and Wholeness, the sense of a separate self disappears completely. For Wholeness is my True Nature, like that of everyone else, with nothing and nobody outside me. As Eastern mystics have been teaching for millennia, the entire world of form is *māyā* 'deception, illusion, appearance' and *līlā* 'play of the Divine' in Sanskrit.

So is it still possible that we could have some fun during the last few years of human existence, passionately pursuing a life of excellence, knowing that at the edge of extinction only Love remains, mottos on Guy McPherson's website. To this end, I am in the process of making one final change to what I had previously thought was my purpose in life, inspired to do so by Eckhart Tolle's *A New Earth: Awakening to Your Life's Purpose*, being liberated as much as possible from the social constraints on my learning and hence behaviour.

When I was a young adult, I thought that my life's purpose was to get married, have children, and pursue

a successful business career, like the 'American Dream' in the USA. This changed in the late 1970s, when I realized that because of the invention of the stored-program computer thirty years earlier the global economy contains the seeds of its own destruction within it and would most probably collapse within my lifetime. So I set out to use my skills as an information systems architect to design a meaningful, moneyless economy that would give everybody the opportunity to realize their fullest potential as human beings, free, as much as possible, of their mechanistic, cultural conditioning. And, as I could see that my children were not being educated to live in the world that would exist when they came to have children of their own, I also set out to discover how they should have been educated.

However, before I could get very far with this endeavour, my life's purpose further changed in April 1982, when I became aware that evolution had carried me to its Omega Point, inseparable from its Alpha Point, where I had begun my adventurous journey just two years earlier. The Principle of Duality, which shows that opposites are never separate from each other, enabled me to see that *Homo sapiens* is not immortal, that one day in the not too distant future a generation of children would be born who would not grow old enough to have children of their own.

Even technology cannot save us, as R. Buckminster Fuller believed in *Utopia or Oblivion: The Prospects for Humanity*. Indeed, with the unprecedented rate of accelerating evolutionary change, mainly manifested in scientific discovery and technological invention, the only way forward for humanity is to rise above the level of our machines, discovering what it truly means to be a human being. As José Argüelles said in *The Mayan Factor* in 1987, at the end of time, there is a path beyond technology.

I was greatly helped to realize what this path might be in October 1983, when David Bohm's method for bringing order to quantum physics enabled me to use the Principle of Duality to form the concept of the Formless Absolute in exactly the same way as I form all other concepts in the relativistic world of form, as I describe on page 72. So not only did I have the cognitive means of facing death in all its forms, what became the Principle of Unity enabled me to return to my original life's purpose, formulated as a seven-year-old: to live in Love, Peace, and harmony with my fellow human beings by ending the long-running war between science and religion.

Of course, this was not something I could do on my own. However, before I could attract others to join me in our peace-making endeavours, I needed a word to denote what might seem to be an unattainable Utopian vision. To this end, I coined the word *paragonian* on 29th October 1984, following several weeks searching Greek and Latin dictionaries in Wimbledon library in London. The word derives from the Greek words *para* 'beyond' and *agon* 'contest' or 'conflict', a word that is also the root of *agony*, until the 17th century meaning 'mental stress', *antagonist* 'a person who one struggles against', and *protagonist* 'leading person in a contest'.

Paragonian thus means 'beyond conflict and suffering', a healthy, liberated, and awakened way of being that we can realize when we are both unified with the Divine and integrated with the Cosmos; when we base our lives firmly and squarely on our Immortal Ground of Being. *Paragonian* thus denotes the essence of *Advaita* ('not-two') in a word with a Western etymology.

Seeing that collectively realizing that Wholeness is the Genuine Identity of all of us would require us to cocreate quite new social institutions, I spent the autumn and winter of 1986 and 1987 deep in the Scandinavian forest with my Norwegian wife Berit, who I had met at The Other Economic Summit (TOES) in London in 1985, writing a booklet for the Paragonian Institute, with the motto 'Serving the Whole'.

However, this initiative did not take off, essentially because the time was not right for it to do so. So, after Berit and I moved to Stockholm in 1990, when I rejoined IBM at its Nordic Software Development Laboratory, we went our separate ways. But I had not yet abandoned my quest for World Peace. So, after I was invited to join a spiritual and ecological community in western Sweden in 2003, after taking early retirement from IBM when fifty-five, I was inspired to self-publish a book the next year titled *The Paragonian Manifesto: Revealing the Coherent Light of Consciousness*. This short book was intended as a spiritual replacement for Marx and Engels' *Communist Manifesto* from 1848, albeit four times longer. It was ironically funded from fees received from working as a computer consultant in the Stockholm World Trade Centre for a company making advanced software systems for investment banks.

Once again, events did not work out as I had visualized. The holistic community where I was living degenerated into open warfare despite its intention to cocreate an alternative lifestyle based on compassion and tolerance, at great financial loss to the housing association. Also, this revolutionary book did not take off, engendering a great deal of hostility, from which it took me many years to recover.

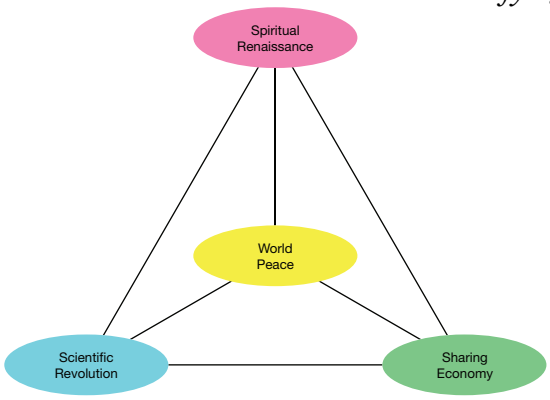
I clearly had much more work to do in solitude if I were to be completely free of the inner conflict that my lifelong search for Peace had ironically caused. After a rather long, turbulent hiatus, as I struggled to reorganize my writings as a coherent expression of my holistic evolutionary vision, my creative energies returned and began to flow unabated once again in the early years of this decade. I was also beginning to get positive responses from my outside world, especially from academics in India, which is far more attuned to the union of all opposites at the heart of the mystical worldview than is the West.

For, as Jung wrote in his *Commentary* to Richard Wilhelm's translation of *The Secret of the Golden Flower*, "The Chinese have never failed to recognize the paradoxes and the polarity inherent in all life. The opposites always balance on the scales—a sign of high culture. Onesideness, though it lends momentum, is a mark of barbarism."[828] And as Jung said in 1935 to his fellow psychotherapists, "The greatest danger that threatens psychology is one-sidedness."[829] As Cary Baynes said in her 1931 English translation of Jung's *Commentary*, "the East creeps in among us by the back door of the unconscious."[830]

So, as well as several books that I have written during the past few years, Ananta Kumar Giri and Vir Singh, who I met or connected with through friends in the West, have invited me to write essays for books of essays that they have been editing. These essays, from 2014 to 2016, are titled 'Mystical Pragmatics: Harmonizing Evolutionary Convergence', 'The Coherent Light of Consciousness: Awakening Self-reflective Intelligence', and 'Revealing the Hidden Harmony: The Heart of Transformative Harmony'.

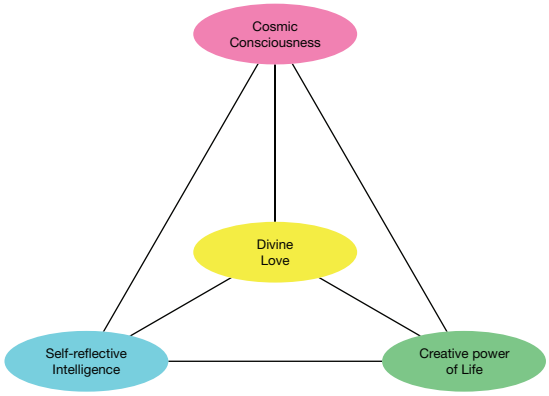
However, I have long known that I cannot make a worthwhile contribution to humanity unless I am able to meet people as a simple, ordinary human being, albeit with some rather unusual experiences and abilities, as a holistic, integral visionary. To this end, in 2013, a friend Pär, living just three hours north of me in Sweden, offered to redesign my website for what had become the Paragonian Foundation with four constituents: Paragonian University, Fellowship, Business Academy, and Publications.

When attempting to relaunch the Foundation, I renamed it the Alliance for Mystical Pragmatics with the motto 'Harmonizing evolutionary convergence', inspired to do so by Ananta's teachings on spiritual pragmatics, similarly inspired by Peirce's philosophy of pragmatics, as the science of business etymologically. As well as the website, in 2016, I wrote a 28-page brochure for the Alliance, as an update of the booklet for the Paragonian Institute that I had written in the winter of 1986. At the time, I felt that this was the best way to end my long sojourn as an outsider, attracting people to join me in synergistically accelerating the pace of evolutionary convergence.



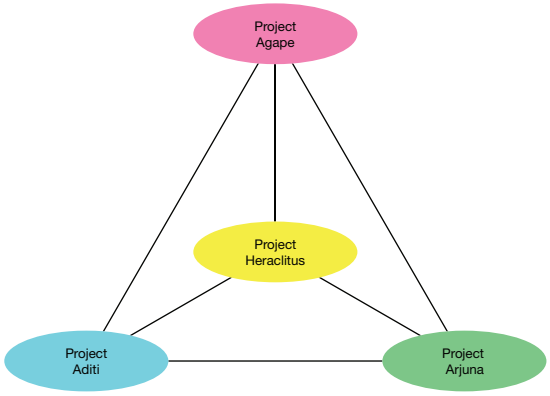
As the booklet indicates, the object of the Alliance is to integrate four major global movements in the world today into a coherent whole: Spiritual Renaissance, Scientific Revolution, Sharing Economy, and World Peace, their relationships being illustrated by the flattened tetrahedron in this diagram. Most significantly, as technological development cannot drive economic growth for very much longer, trade would no longer be the principal driving force of the economy. We would adopt a work ethic in which the awakening of intelligence and consciousness in humans is paramount.

As my 2017 book on *The Psychodynamics of Society* indicates, a major purpose is therefore to complete the final revolution in science, fulfilling William James, Eugen Bleuler, and Carl Gustav Jung's dream, not only of establishing psychology as a coherent science, but also as the primary science, on which all humanities and sciences are built. This could happen when evolution becomes fully conscious of itself, which Barbara Marx Hubbard calls the 'Second Great Event' in the history of the universe, the first being the most recent big bang, which supposedly brought it into existence.[831]



This means, of course, that we cannot bring this miracle about by basing the future of our species on the past. Evolution can only become fully conscious of itself within us human beings when we start afresh at the very beginning, invoking the primal energies of Divine Love, Cosmic Consciousness, Self-reflective Intelligence, and the creative power of Life, arranged in this diagram, corresponding to the four constituents of the Alliance.

With World Peace as the ultimate purpose, the plan was to focus the activities of the Alliance on Project Heraclitus, with the motto 'Revealing the Hidden Harmony', enabling us to live intelligently and consciously in harmony with the fundamental law of the Universe: opposites can never be separated in Reality.



Project Heraclitus would be organized into three subprojects, Project Agape, Project Aditi, and Project Arjuna, with the mottos 'Healing the split', 'Awakening Self-reflective Intelligence', and 'Transcending the Divisiveness of Money', respectively. Project Agape reflects the Greek word *agapē*, used by Christian writers in the New Testament to mean 'Divine Love'. Aditi is the Divine Matrix, a symbol for Consciousness, as the mother of the Universe in the *Rig Veda*. In turn, Arjuna was the spiritual warrior in the Hindu classic *Bhagavad Gita*, invoking time-honoured, both-and, all-inclusive spiritual practices to deal intelligently with conflict-ridden, either-or, polarizing politics.

To ensure optimal communications between different cultures and disciplines, we would need to develop an interlocking glossary of terms for the underlying coherent conceptual model of the Totality of Existence, going back to their roots as much as possible. I have made a start on this project in my writings and on the Alliance website. However, there is still much work to be done. For while a rose is a rose is a rose for most

people, there is no common language for referring to the psychodynamics of our inner worlds. For such an understanding can only arise through introspection, free from any preconceptions of what such self-inquiries might reveal.

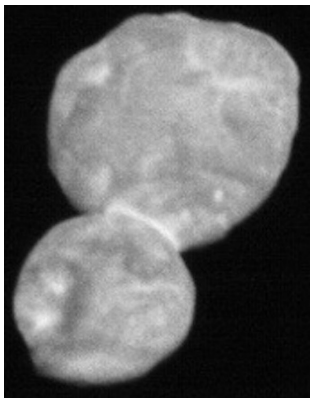
There is just one snag to overcome if I am to make a worthwhile contribution to humanity at these end times we live in. As my life's work—unifying mysticism and mathematics—is transcultural and transdisciplinary, as a human being, I don't fit into any social grouping anywhere in the world. My occupation—as an adviser to the Galileo Commission, which is seeking to promote a spiritually informed science, beyond the materialistic worldview—is recorded as 'systems architect and author'.[832]

Yes, denoting myself as a systems architect is the best way to give my profession in the information technology and data processing industry, as I explain in Section 'Function of information systems architect' on page 14. In this capacity, I see myself as a generalist working with specialists in user departments developing integrated information systems, rather like my local doctor, who calls herself a specialist in general medicine, working with consultants in regional hospitals.

However, in order to heal my fragmented mind and split psyche, my occupation as an information systems architect has evolved into that of a Panosopher, studying Panosophy, where there are no longer any divisions between science, philosophy, and religion, or between science and the humanities. Yet, in this chaotic postmodern world, which is sceptical that a 'grand narrative' can explain all our experiences from the mystical to the mundane,[833] how can I describe to others the nature of this occupation? Even its possibility seems to be beyond the imagination of most people, even that of my closest friends, who enjoy my company as just an ordinary guy, albeit with some rather unusual insights.

So what to do? I have long felt that my life's purpose as a human being is to complete the final revolution in science, going far beyond the revolutions that Kepler, Newton, Darwin, Einstein, and Bohm introduced. After all, these gentlemen were once living and working at the leading edge of evolution. So I thought that it would only be a matter of time before evolution would solve what Bohm identified as the central problem of our times: how to heal the fragmented mind in Wholeness.

However, when I look at the series of events that have led me to where I am today—turning fourteen billion years of evolution in the horizontal dimension of time into the vertical—it looks unlikely that anyone else will pass through a similar apocalyptic life experience, removing the veils that occlude our vision. With 95% and 5% of the population living in the first and second tiers of the spectrum of consciousness, illustrated on pages xv and 68, what are the chances of more than a few making the transition into the transpersonal, mystical third tier, necessary for democracy to be a viable and sustainable system of governance.



We live in a world where the media is more focused on reporting on phenomena in outer space than on the fascinating happenings in inner space—the ninety-nine per cent of the Cosmos that is inaccessible to our physical senses, including mathematics, as both the territory and its map. For instance, the Americans and Chinese have recently landed instruments on Mars and the far side of the Moon, and the New Horizons spacecraft sent back this photo of Ultima Thule, a fused pair of rocks 33 kilometres across and 6.6 billion kilometres from Earth, taken on 1st January 2019.[834]

With such a lack of interest into what is causing us all to behave as we do—at the new horizons of the psyche—it now looks most unlikely that today's Information Society will evolve into the Mystical Society, as the diagram on page 22 illustrates, realizing the awakened society that Julian

Huxley visualized in his essay on 'Transhumanism' and Richard Maurice Bucke foresaw in *Cosmic Consciousness*,[835] in 1957 and 1901, respectively.

Borrowing ideas from mystics and depth psychologists, I have written much about the root cause of our collective malaise, about why it is so difficult to heal the cultural and collective wounds in our psyches, healing the splits between science and religion and East and West, respectively. However, if the chances of applying the remedy are infinitesimal, in either meaning of this word, there is little benefit in describing how our fragmented minds could be healed in Wholeness. The best that could happen is that our split psyches could be healed in Oneness, where the cognitive expression of mystical understanding is more intuitive, poetic, and musical than rational, logical, and mathematical.

Rupert Spira, who I met in Scotland and California in 2010 and 2011, respectively, is one of the world's leading teachers of this time-honoured approach to Self-realization, much influenced by the Vedantic tradition, as he tells us in *The Nature of Consciousness*.[836] As he said in an eloquent talk at the Science and Nonduality (SAND) conference in 2018, whose theme was 'The Mystery of Being Human', to live our humanity, we must first recognize our Divinity.[837] As he continued, such a realization of our Absolute Nature is not some abstract, intellectual idea, but a limitless intimacy that is the Foundation for World Peace, grounded in Absolute Truth and Love, which scientists are seeking.

However, while what Rupert says is necessary to realize the Utopian vision of World Peace, it is not sufficient. In my outer world, there is still a war going on between the spiritual and scientific worldviews, as for example between Deepak Chopra, who wrote the Foreword to Rupert's book on *Consciousness*, and Leonard Mlodinow, co-author with Stephen Hawking of *The Grand Design*, mentioned in the Prologue on page xxv. For myself, after a lifetime of study and self-inquiry, this war has ceased to exist within me, which is perhaps why I still live as an outsider to society, as it is constituted today.

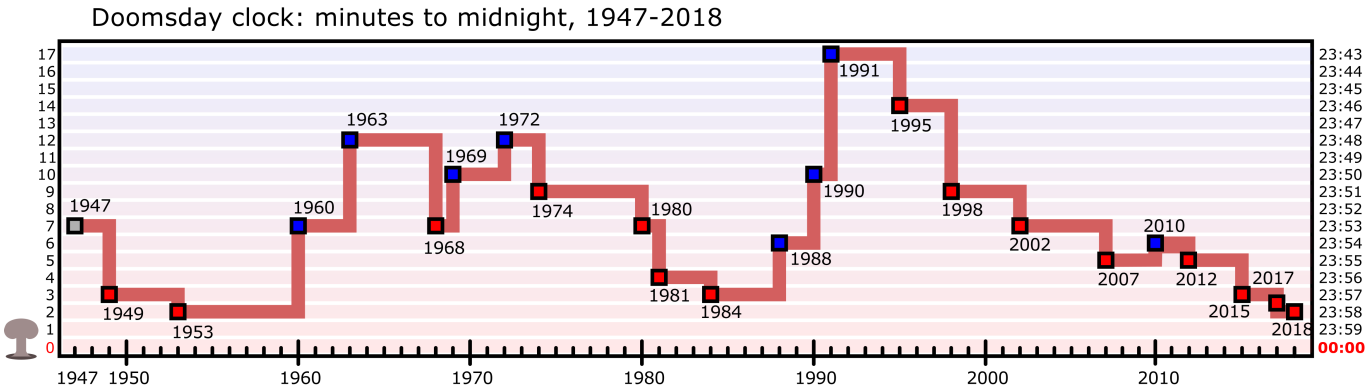
Another reason for my solitude is that while our inner knowing of the Divine is the foundation of all our lives, in practice we humans are social animals. So, *who* we know is generally more important than *what* we know. Most are thus conditioned early in life to live with cultural schizophrenia, not questioning the delusions that thus arise. There is little consensus that this pandemic has led our entire species to be psychologically disturbed, as Erich Fromm pointed out in 1956 in *The Sane Society*. Twenty years later, in *To Have or To Be?*, he outlined a remedy based on a proposed science of humanity transcending materialism and mechanism, much inspired by the mystics Shakyamuni Buddha and Meister Eckhart.

As I continue to express and promulgate such a comprehensive science of humanity, all I can do in life is get up each morning and follow my inner guru as well as I can, inspired by the final chapter in the *Bhagavad Gita*, which teaches that while it is natural to engage in challenging work, it is also essential to be free of egoic attachment to what might result from these activities.[838] This does not mean indifference to the consequences of these endeavours. For as Mohandas Gandhi said, "He who … is without desire for the result and is yet wholly engrossed in the fulfilment of the task before him is said to have renounced the fruits of his action."[839]

So, while I would much prefer to be free of my reputation as a recluse—working in solitude as an autodidact for forty years—if I don't find a group of influential specialists willing and able to engage actively in unifying mysticism and mathematics, I plan to enjoy myself completing Chapters 4 and 5 of this book during 2019, further integrating a view of mathematics as an emergent, generative science of beautiful patterns and relationships.

But what happens after this, I really don't know. On 24th January 2019, the Science and Security Board

of the *Bulletin of the Atomic Scientists* announced that we are now living at a 'new abnormal', leaving their famous Doomsday Clock at two minutes to midnight, having moved it there, as close as it has ever been, the year before, as this Wikipedia diagram illustrates.[840]



The reason why they say that the psychodynamics of society has become a new abnormal during the past two years is that "cyber-enabled information warfare aggravates other major global dangers—including those posed by nuclear weapons and climate change—as it undermines civilization generally." As they say, "to thrive, prosper, and advance, people … need reliable information about their world—factual information, in abundance. Today, however, chaos reigns in much of the information ecosystem on which modern civilization depends."[841]

One principal reason for this critical situation is that the computer is a machine quite unlike any other that the *Homo genus* has invented during the past two thousand millennia, as I mention on page viii in the Prologue. In the 1970s, this 1940s' invention led to what Daniel Bell called the 'Information Society', requiring a quite new work ethic and form of economics, as mentioned on pages 15 and 21 in 'Business Modelling', with the threat of machines with artificial intelligence taking over the workplace. Initially, this situation gave me great hope for our children's and grandchildren's future, for, as Tom Stonier said in *The Wealth of Information*, "Whereas material transactions can lead to competition, information transactions are much more likely to lead to cooperation—information is a resource which can be truly shared," repeating a quotation from page 22.

Sadly, however, neither scientists nor business executives adapted to this rapidly changing social environment, continuing to teach our children and run our business affairs in the false belief that we humans are separate from each other and that we must fight our fellows for a slice of the finite monetary cake. Even though the word *synergy* is being used widely today, it is not generally recognized that the *relationships* between data elements provide even more meaning than these elements looked at in isolation. So, rather than helping us to manage our business affairs with love, peace, and harmony, the transcultural, transdisciplinary Internet has become a war zone, amplifying unresolved psychological disturbances.

Nevertheless, not understanding the root cause of what Fromm, Krishnamurti, and others have called our 'sick society', "The *Bulletin* resolutely believes that human beings can manage the dangers posed by the technology that humans create."

Similarly, on the same day that the *Bulletin of the Atomic Scientists* published their latest report on the existential threats to humanity, 16-year old Greta Thunberg, a climate activist from Stockholm, told an audience of world leaders at Davos that while 'our house is on fire', "*Homo sapiens* has not yet failed. … there is still time to turn everything around. We can still fix this. We still have everything in our own hands."[842]

Would it be so. Sadly, however, in recent years, I have come to realize that most people are so deluded

and out of touch with Reality that there is little chance of making radical changes to the way we live our lives before events catch up with us. As Greta said in Davos, "unless we recognise the overall failures of our current systems, we most probably don't stand a chance." For, as Shakyamuni Buddha said, unless we acknowledge the symptoms of our suffering—the first step in medical diagnosis—there is little chance of finding the cause and hence the remedy.

So whether I have a role to play even in helping to bring spirituality to science, as many are striving for today, often focused on parapsychological phenomena, I really don't know. Would the public announcement that the ultimate problem in human learning has been solved help to resolve the existential crisis humanity faces today?

I ask this question because we can only awaken in the Age of Light by knocking humankind off the pedestal on which we have arrogantly placed ourselves, completing the task that Copernicus and Darwin began with *On the Revolutions of the Celestial Spheres* and *On the Origin of Species* in 1543 and 1859, respectively. Furthermore, the Theory of Everything, which I call Panosophy or the Unified Relationships Theory, is a form of insight, as David Bohm pointed out, mentioned on page 41. So what I write is not actually the megasynthesis of all knowledge that humans have been seeking for millennia; it is merely an expression of Wholeness, as an undivided Continuum, unifying all opposites.

This fundamental law of the Universe lies at the heart of my meditation practice, as I have mentioned several times in this book in various mathematical and psychological contexts, drawing particularly on Indra's Net of Jewels in Huayan Buddhism and graph theory in mathematics. Most significantly, the opposites of birth and death are particular instances of the general principle, not special or significant in any way.

So, when I rest in Stillness with Self-reflective Intelligence, I watch all these opposites, including self and other, dance in consciousness, as *Līlā*, the play of the Divine. In particular, all events in my life, from conception to death, lose their energy, revealing the Authentic Self we all share. Mindfulness, as attention, has allowed all forms to dissolve in continuity, unifying the mystical and mathematical, in what Charles Sanders Peirce called mergence, the opposite of emergence, which leads to mathematics and all other disciplines of learning appearing in consciousness.

This is how I deal equanimously with the turmoil of the global situation, with the inherently unstable economic system, which provides us with the basic necessities in life, once again teetering on the brink because it is not adapting to our rapidly changing psychosocial environment. This is an urgent situation that affects us all, for we now have fewer than twelve months to prepare for the 2020s, when we humans will need to make the biggest change to the way we live our lives together for 5,000 years, since before the invention of money, if we survive that long.

As I trust in the innate kindness and compassion of humans, despite increasing existential fear and cruelty in the world, I still feel that it is possible to meet my fellows as an ordinary human being. For while it might appear that I have some anomalous experiences, what I sense in indivisible, ineffable Wholeness is actually the same for all of us, once we are free of the emotional baggage most of us are burdened with, intuitively feeling into our innate sensuality and thence the True Nature that we all share. So, while remaining in the Presence of the Eternal Now, let us see what miracles might emerge from the hidden depths, as we merge into the Continuum.

Thank you so much for your attention. Do please get in touch if you feel moved to do so.

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This References section, formatted by Microsoft Word, will be split into Notes and Bibliography if and when this book is converted into Adobe InDesign or FrameMaker for professional book layout. At that point, subject and name indexes will also be added.

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[31] David Bohm, *Wholeness and the Implicate Order*, London: Routledge, pp. 3–4.
[32] Although Euclid's mathematical-proof system, described in *The Elements*, is not explicitly based on Aristotle's syllogism, described in *Prior Analytics* in *Organon*, both ways of reasoning are essentially linear, starting with some premises or assumptions and proceeding from there. These two approaches to deductive reasoning merged in 1854 with the publication of George Boole's *Laws of Thought*, which led to the invention of the electronic stored-program computer nearly one hundred years later.
[33] Bacon intended to replace Aristotelian logic in 1620 with his book *Instauratio Magna*. In his Plan for the *Great Renewal*, Bacon sought to place the foundations deeper and further back than ever done before, saying, "What the sciences need is a form of induction which takes experience apart and analyses it, and forms necessary conclusions on the basis of appropriate exclusions and rejections." Bacon, *The New Organon*, ed. Lisa Jardine and Michael Silverthorne, original edition *Novum Organum*, 1620, Cambridge University Press, 2000, p. 17.
[34] I first came across Peirce's notion of abductive reasoning (not in the second edition of the *Oxford English Dictionary*) in Peter Jackson, *Introduction to Expert Systems,* Wokingham, England: Addison Wesley, 1986, p. 115. As he said on p. 1, "An expert system is a computing system capable of representing and reasoning about some knowledge-rich domain, such as internal medicine or geology, with a view to solving problems and giving advice."
Peirce's original paper on the three principal aspects of scientific reasoning was 'Deduction, Induction, and Hypothesis', *Popular Science Monthly*, Vol. 13, August 1878, pp. 470–482, reprinted in *Essential Peirce*, Vol. 1, pp. 186–189.
[35] Albert Einstein, *Einstein on Politics: His Private Thoughts and Public Stands on Nationalism, Zionism, War, Peace, and the Bomb*, eds., David E. Rowe and Robert J. Schulmann, Princeton University Press, 2007, p. 383. Originally published as 'The Real Problem Is in the Hearts of Men', in the *New York Times Magazine* on 23rd June 1946.
[36] Albert Einstein, *The Ultimate Quotable Einstein*, Ed., Alice Calaprice, Foreword, Freeman Dyson, Princeton University Press, 2011, p. 273.
[37] Eckhart Tolle, *The Power of Now: A Guide to Spiritual Enlightenment*, Hodder & Stoughton, 2001, p. 40.
[38] John 8:31–32. "Then said Jesus to those Jews which believed on him, If ye continue in my word, then are ye my disciples indeed; And ye shall know the truth, and the truth shall make you free."
In the Greek, *word* is *logos*, *know* is *gnōsesthe*, from *gnosis*, 'knowledge, wisdom, understanding', and *truth* is *alētheia*, the root of *alethic* 'modalities of truth in logic' and *alethiology* 'study of truth in logic', rarely used. Despite the reference to *Gnosis* and Heraclitus' *Logos*, Jesus' words are normally interpreted at the cognitive, intellectual level, rather than mystical, experiential one, grounded in the Truth.
[39] *Weltanschauung* has a deeper meaning than *worldview*, indicating both scientific observation and spiritual meditation, derived from *Welt* 'world', from Middle High German *wërlt*, from Old High German *weralt*, cognate with *world*, and *Anschauung* 'view', from Middle High German *anschouwunge* 'observation, mystical contemplation'.
[40] F. C. Happold, *Mysticism: A Study and an Anthology*, revised edition, original edition, 1963, Harmondsworth, England: Penguin, 1970, p. 72.
[41] Ilya Prigogine and Isabelle Stengers, *Order Out of Chaos: Man's New Dialogue with Nature*, London,

Flamingo, 1985, p. 233, quoted from A. S. Eddington, *The Nature of the Physical World*, New York: Macmillan, 1948, p. 74.
[42] Calvert Watkins, editor, *The American Heritage Dictionary of Indo-European Roots*, revised second edition, first edition, 1985, Boston: Houghton Mifflin, 2000, language and culture note for *dhghem-*, p. 20.
[43] Carl Haub, Demographer Emeritus, and Toshiko Keneda, Senior Research Associate, at Population Reference Bureau (PRB): https://www.prb.org/howmanypeoplehaveeverlivedonearth/.
[44] E. F. Schumacher, *A Guide for the Perplexed*, original edition, Jonathan Cape, 1977, London: Abacus, 1978, p. 15.
[45] John Templeton, *Possibilities for Over One Hundredfold More Spiritual Information: The Humble Approach in Theology and Science*, Radnor, Pennsylvania: Templeton Foundation Press, 2000, p. 51.
[46] Karl Marx thought he had found the 'laws of motion of society' in Georg Hegel's dialectic (David McLellan, *Karl Marx: The Legacy*, London: British Broadcasting Company, 1983, p. 7). However, Karl Popper was critical of such a claim, writing, "The hope … that we may some day find the 'laws of motion of society', just as Newton found the laws of motion of physical bodies, is nothing but the result of these [holistic] misunderstandings. Since there is no motion of society in any sense similar or analogous to the motion of physical bodies, there can be no such laws." (Karl R. Popper, *The Poverty of Historicism*, second edition, original edition, 1957, London: Routledge & Kegan Paul, 1961, p. 115.)
[47] Alfred North Whitehead and Bertrand Russell, *Principia Mathematica to *56*, shortened edition, 1962, original edition in three volumes, 1910, 1912, and 1913, Cambridge University Press, 1997, p. 360.
[48] en.wikipedia.org/wiki/Systema_Naturae.
[49] Richard Tarnas, *The Passion of the Western Mind: Understanding the Ideas That Have Shaped Our World View*, New York: Harmony Books, 1991, p. 45.
[50] G. H. Hardy, *A Mathematician's Apology*, forward C. P. Snow, Cambridge University Press, 1940, p. 150.
[51] Ibid., p. 88–91.
[52] Ibid., p. 84.
[53] Ibid., p. 85.
[54] Ibid., p. 115.
[55] Alfred North Whitehead, *Science and the Modern World*, Cambridge University Press, 1926, 25.
[56] Hardy, *Apology*, p. 103.
[57] Ibid., p. 104.
[58] Whitehead, *Science*, p. 33.
[59] Ibid.
[60] Ibid., p. 39.
[61] Hardy, *Apology*, p. 109.
[62] Ibid.
[63] Ibid., p. 112.
[64] Aristotle, tr. Hugh Tredennick, *Metaphysics, Books I-IX*, 1003a21–23, p. 147.
[65] Aristotle, *Metaphysics*, Books I-IX, tr. Hugh Tredennick, Cambridge: Harvard University Press; London: William Heinemann, 1933, 1005b20, pp. 161 & 163.
[66] Stanislav Grof with Hal Zina Bennett, *The Holotropic Mind: The Three Levels of Human Consciousness and How They Shape Our Lives*, New York: HarperSanFrancisco, 1990, pp. 38–39.
[67] Ibid., p. 41.
[68] Stanislav Grof, 'The Revision and Re-Enchantment of Psychology: The Legacy of Half a Century of Consciousness Research', pdf file on web, undated.
[69] W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays,* 1st ed. *Mathematical Recreations and Problems*, 1892, 13th ed., Mineola, NY: Dover, 1987, pp. 241–254.
[70] https://en.wikipedia.org/wiki/Seven_Bridges_of_Königsberg.
[71] Bertrand Russell, *History of Western Philosophy*, second edition, 1961, original edition, 1946, London: George Allen & Unwin, 1979, p. 13.
[72] Robert H. Dicke and James P. Wittke, *Introduction to Quantum Mechanics*, first edition, 1960, Addison-

Wesley, 1972.
[73] Albert Einstein, 'Maxwell's Influence on the Evolution of the Idea of Physical Reality', in *James Clerk Maxwell: A Commemorative Volume*, Cambridge University Press, 1931, reprinted in Albert Einstein, *Ideas and Opinions*, Based on *Mein Weltbild*, edited by Carl Seelig, translated by Sonja Barmann, original edition 1954, London: Condor, Souvenir Press, 2005, p. 266.
[74] Alfred Korzybski, *Science and Society: An Introduction to Non-Aristotelian Systems and General Semantics*, fifth edition, original edition, 1933, Englewood, NJ: Institute of General Semantics, 1994, p. 58.
[75] Morris Kline, *Mathematics: The Loss of Certainty*, Oxford University Press, 1980, pp. 216–257.
[76] Gilbert Ryle, *The Concept of Mind*, original edition, Hutchinson, 1949, Harmondsworth, England: Penguin, 1963, pp. 17–24.
[77] Ibid, pp. 17–18.
[78] http://nsidc.org/arcticseaicenews/.
[79] Daniel Bell, *The Coming of Post-Industrial Society*, Basic Books, 1973.
[80] Dominic Walliman, 'The Map of Mathematics', https://youtu.be/OmJ-4B-mS-Y.
[81] Dominic Walliman, 'The Map of Physics', https://youtu.be/ZihywtixUYo.
[82] In the 1980s, David Bohm and I did not talk about the possibility of developing an algebra of algebras in order to establish his theory of the implicate order as sound science, although we did talk about Integral Relational Logic, as it was then emerging in consciousness. And I've not seen any explicit mention of it in any of his books and articles. This idea arises from a review that Danah Zohar wrote of *Wholeness and the Implicate Order*, which I thought was published on 27th July 1980 in the *Observer* newspaper. However, on a brief visit to the British Library on 10th September 2018, I could not find this review on the microfiche copy of this issue. So I will need to return to London at some time to search through other issues of this newspaper and those of the *Sunday Times*.
[83] Dominic Walliman, 'Map of Science (and everything else)', https://youtu.be/ohyai6GIRZg.
[84] Matthew Fox, foreword to Andrew Harvey and Carolyn Baker, *Savage Grace: Living Resiliently in the Dark Night of the Globe*, Bloomington, IN: iUniverse, 2017, p. 4.
[85] Harvey and Baker, *Savage Grace*, location 212 of 2391, Kindle edition.
[86] Arnold Toynbee delineated about twenty civilizations in his monumental *A Study of History*, outlined in Paul Hague, *Wholeness*, *Part III, Our Evolutionary Story*, Chapter
[87] Florian Coulmas, *The Blackwell Encyclopedia of Writing Systems*, WileyBlackwell, 1999, 'Sumerian writing', p. 481.The caption states, *Archaic Sumerian clay tablet of the Uruk III stratum detailing the allotment of malt to a number of people and with stock accounts of barley on the reverse (photography courtesy Margret Nissen)*.
[88] *IBM: A Special Company*, diamond anniversary edition of *Think*, Vol. 55, No. 5, September 1989, p. 44.
[89] Charles W Bachman, 'Software for Random Access Processing', *Datamation*, April 1965, pp. 36-41.
[90] Clive Finkelstein, *Information Engineering: Strategic Systems Development*, Addison-Wesley Longman, 1992, p. 17.
[91] Ted Codd, 'A Relational Model of Data for Large Shared Data Banks' in *Communications of the ACM*, Vol. 13, No. 6, June 1970, pp. 377–387.
[92] Unicode is also known as Universal Coded Character Set (UCS), numbered (ISO/IEC 10646), a way of numbering characters in most of the writing systems in the world. It theoretically consists of a 65,536 (2^{16}) planes, each consisting of 65,536 code points (up to 2^{32} in total), although only seventeen planes are defined, and only five are used, two for private use. To save time in transmission and space in storage, UTF-8 encodes the original 7-bit ASCII characters in one byte. The characters in non-English Latin, Greek, Cyrillic, Hebrew, and Arabic writing systems are encoded in two bytes. Most other modern languages, which are in the Basic Multilingual Plane, require three bytes. Historical scripts, like Phoenician—the first alphabetic script—and Linear-B syllabary are encoded in four bytes, which are needed for many other characters. If additional planes above the ones defined were ever needed, these could be encoded in five or six bytes in UTF-8, the maximum possible in the encoding. So there is plenty of room for expansion!
[93] For instance, in describing the basics of EDSAC, the first productive stored-program computer, David Brailsford tells us that John von Neumann could then directly in machine language, eschewing the need

for an assembler. Hello World (Assemblers, Considered Harmful!?) – Computerphile (https://youtu.be/rf60FSEdbKI).

[94] George Boole, *An Investigation of The Laws of Thought on Which Are Founded the Mathematical Theories of Logic and Probabilities*, original edition, 1854, reprint, New York: Dover, 1958, p. 1.

[95] In 1865, in the first of a series of lectures 'On the Logic of Science', Peirce, a great admirer of Boole's pioneering work, said that all the definitions of logic that had evolved during the previous two millennia could be divided into two classes: "those which do not and those which do give to logic a psychological or human character", favouring the former: "we ought to adopt a thoroughly unpsychological view of logic" (Charles Sanders Peirce, 'On the Logic of Science', First Harvard Lecture, 1865, *Writings of Charles S. Peirce: A Chronological Edition*, Volume 1, 1857–1866, edited by Max H. Fisch, Indiana University Press, 1982, p. 163–167.

Peirce reiterated his determination to keep logic separate from psychology in 1898, when he gave a series of lectures on *Reasoning and the Logic of Things* in Cambridge, Massachusetts. In the exordium for the third lecture titled 'The Logic of Relatives', he said, "My proposition is that logic, in the strict sense of the term, has nothing to do with how you think." (Peirce, *Reasoning and the Logic of Things*, p. 141.)

[96] In 1903, Bertrand Russell, co-author of *Principia Mathematica*, and Gottlob Frege, generally regarded as the founder of first-order predicate logic, agreed with Peirce. For when being troubled by the paradoxes that had been found in the foundations of mathematics, Russell wrote a famous letter to Frege, notably agreeing with his rejection of any psychological element in logic. (Jean van Heijenoort, ed., *From Frege to Gödel: A Source Book in Mathematical Logic*, 1879–1931, Cambridge, MA: Harvard University Press, 1967, pp. 124–125.)

[97] Joseph Weizenbaum, *Computer Power and Human Reason: From Judgement to Calculation*, with new preface, original edition, W. H. Freeman, 1976, Harmondsworth, England: Penguin, 1984, p. 80. This diagram was derived from figures 3.2–3.5 in D. C. Evans, 'Computer Logic and Memory'.

[98] Here is how the OR operator can be expressed in terms of AND and NOT: $P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$.

[99] Clive W. Kilmister, *Language, Logic and Mathematics*, English University Press, 1967, p. 55.

[100] How NOT and AND are expressed in terms of the Sheffer stroke: $\neg P \Leftrightarrow P \uparrow P$ and $P \wedge Q \Leftrightarrow (P \uparrow P) \uparrow (P \uparrow P)$, using the NAND arrow glyph (↑) from Peirce, rather than the stroke, proposed by Jean Nicod in 1917. More information on http://en.wikipedia.org/wiki/Sheffer_stroke.

[101] Peirce, 'A Boolian Algebra with One Constant', in *Collected Papers of Charles Sanders Peirce: Volume IV, The Simplest Mathematics*, edited by Charles Hartstone and Paul Weiss, Cambridge, MA: Harvard University Press, 1933, paras. 12–20, pp. 13–18 and para. 264, pp. 215–216.

[102] Doron Swade, *The Difference Engine: Charles Babbage and the Quest to Build the First Computer*, Penguin Books, 2002, p. 105.

[103] L. F. Menabrea, 'Sketch of the Analytical Engine Invented by Charles Babbage' with notes on memoir by translator, Ada Augusta Lovelace, *Taylor's Scientific Memoirs*, London, Vol. III, 1843, pp. 666–731, reprinted in Philip Morrison and Emily Morrison, editors, *Charles Babbage and His Calculating Engines: Selected Writings* by Charles Babbage and Others, New York: Dover, 1961, p. 252.

[104] Andrew S. Tanenbaum, *Structured Computer Organization*, Englewood Cliffs, NJ: Prentice Hall, 1979, p. 10.

[105] Naomi S. Baron, *Computer Languages: A Guide for the Perplexed*, original edition, Anchor Books, 1986, Harmondsworth, England: Penguin, 1988, pp. 170–189 and 203–212.

[106] Corrado Böhm and Giuseppe Jacopini, 'Flow Diagrams, Turing Machines and Languages with Only Two Formation Rules', *Communications of the ACM* **9** (5), May 1966.

[107] Edward Yourdon, *Techniques of Program Structure and Design*, Englewood Cliffs, NJ: Prentice Hall, 1976, pp. 146–147.

[108] Edsger W. Dijkstra, 'Letters to the editor: Go To Statement Considered Harmful', *Communications of the ACM* March 1968, Vol. 11, No 3, pp. 147–148.

[109] Baron, *Computer Languages*, pp. 346–355.

[110] George Boole, 'On a General Method in Analysis', *Philosophical Transactions of the Royal Society of London*, Vol. 134 (1844), pp. 225–282.

[111] Desmond MacHale, *George Boole: His Life and Work*, Dublin, Ireland: Boole Press, 1985, pp. 61–62.
[112] Ibid., pp. 51, 57, and 64–66.
[113] Rupert Sheldrake, *A New Science of Life: The Hypothesis of Formative Causation*, London: Blond & Briggs, 1981.
[114] *Aristotle's Lagoon*, http://www.bbc.co.uk/programmes/b00q0hh2.
[115] Richard Courant and Herbert Robbins, *What Is Mathematics?: An Elementary Approach to Ideas and Methods*, 2nd rev. ed., ed. Ian Stewart, 1st ed. 1941, Oxford University Press, 1996, p. 113.
[116] Half of Michael A. Nielsen and Isaac L. Chuang's scholarly book *Quantum Computation and Quantum Information* is devoted to the misnamed quantum information.
[117] Claude E. Shannon, 'Information Theory', *Encyclopaedia Britannica*, Vol. 12, fourteenth edition, p. 246B.
[118] Sherman C. Blumenthal, *Management Information Systems: A Framework for Planning and Development*, Englewood Cliffs, NJ: Prentice-Hall, 1969, p. 30.
[119] Norman Lindop, *Report of the Committee on Data Protection*, London: Her Majesty's Stationery Office, 1978, para. 18.28, p. 154.
[120] These definitions were taken from the Web at http://www.ncits.org/tc_home/k5htm/Ansdit.htm. I originally saw them, or something similar, in IBM, *Dictionary of Computing*.
[121] P. M. Heathcote, *'A' Level Information Technology*, Ipswich, England: Payne-Galloway, 1998, pp. 36–37.
[122] John Edward Sadler, *J. A. Comenius and the Concept of Universal Education*, London: George Allen & Unwin, 1966, p. 155.
[123] Jan Amos Comenius, *Continuatio admonitionis fraternæ*, Amsterdam, 1669, tr. in Robert Fitzgibbon Young, *Comenius in England: The Visit of Jan Amos Komenský (Comenius), Czech Philosopher and Educationalist, to London in 1641–1642; Its Bearing on the Origin of the Royal Society, on the Development of the Encyclopaedia, and on the Plans for the Higher Education of the Indians of New England and Virginia*, Oxford University Press, 1932, para. 44, pp. 30–31.
[124] Algirdas Julien Greimas, *The Social Sciences: A Semiotic View*, translated from French by Paul Perron and Frank H. Collins, Minneaplis: University of Minnesota Press, 1990, p. 3.
[125] Lao Tzu, *Tao Teh Ching*, translated by John C. H. Wu, original edition, 1961, Boston: Shambhala, 2005, ch. 1, p. 3.
[126] Thich Nhat Hanh, *Old Path White Clouds: Walking in the Footsteps of the Buddha*, tr. Mobi Ho, Berkeley, CA: Parallax Press, 1991, p. 465.
[127] Dionysius the Areopagite, *Mystical Theology* in *Pseudo-Dionysius: The Complete Works*, Paulist Press International, 1993, p. 141.
[128] Aristotle, *Physics*, 254b7–255b40, pp. 195–200, and 256a4–258b8, pp. 200–207.
[129] Aristotle, *Metaphysics*, 1073a27, p. 153.
[130] Thomas Aquinas, *Summa Theologiæ*, a concise translation by Timothy S. McDermott, originally published, 1265–1273, Allen, TX: Christian Classics, 1989, pp. 12–14.
[131] William Kent, *Data and Reality*, Authorhouse, 2000, p. 10, quoted from Bejamin Lee Whorf, *Language, Thought, and Reality: Selected Writings of B. L. Whorf*, ed. John B. Carroll, foreword by Stuart Chase, Cambridge, MA: MIT Press, 1956.
[132] Albert Einstein, 'Maxwell's Influence on the Evolution of the Idea of Physical Reality', in *James Clerk Maxwell: A Commemorative Volume*, Cambridge University Press, 1931, reprinted in Albert Einstein, *Ideas and Opinions*, Based on *Mein Weltbild*, edited by Carl Seelig, translated by Sonja Barmann, original edition 1954, London: Condor, Souvenir Press, 2005, p. 266.
[133] Alfred Korzybski, *Science and Society: An Introduction to Non-Aristotelian Systems and General Semantics*, fifth edition, original edition, 1933, Englewood, NJ: Institute of General Semantics, 1994, p. 58.
[134] Ferdinand de Saussure, *Course in General Linguistics*, translated by Wade Baskin from *Cours de linguistique générale*, edited by Charles Bally and Albert Sechehaye in collaboration with Albert Reidlinger, 1916, New York: Philosophical Library, 1959, pp. 66–67.
[135] J. F. Sowa, *Conceptual Structures: Information Processing in Mind and Machine*, Reading, MA: Addison-

Wesley, 1984, p. 11.
[136] C. K. Ogden and I. A. Richards, *The Meaning of Meaning: A Study of the Influence of Language upon Thought and the Science of Symbolism*, first published 1923, London: Routledge & Kegan Paul, Ark, 1985, pp. 11 and 279.
[137] Richard L. Gregory assisted by O. L. Zangwill, eds., *The Oxford Companion to the Mind*, Oxford University Press, 1998, article on 'concept', p. 158.
[138] M. C. Escher, *The World of M.C. Escher*, edited by J. C. Locher, Netherlands: Meulenhoff International, 1973, p. 102.
[139] For me, the most enlightened book on education is J. Krishnamurti, *Education and the Significance of Life*, originally published 1953, HarperSanFranscisco, 1981. *The Awakening of Intelligence* also had a profound effect on my own awakening.
[140] Eckhart Tolle, *A New Earth: Awakening to Your Life's Purpose*, London: Penguin, 2006, pp. 138 and 309.
[141] Paul Hague, 'The Evolution of Universals: Being a Universal Human', unpublished.
[142] Osho, *Philosophia Perennis, Volume 2*, Chapter 2, 'Zorba The Buddha', 1st January 1979.
[143] Osho, *The Golden Future*, Session 32, 'The New Man: The Very Salt of the Earth', Cologne: Germany: Rebel Publishing House, 1987, p. 297.
[144] Alan Turing, 'Computing Machinery and Intelligence', *Mind*, LIX, No. 236, 1950, reprinted in Hofstadter & Dennett, *The Mind's I*, pp. 53–67.
[145] Benjamin Woolley, *The Bride of Science: Romance, Reason and Byron's Daughter*, Pan Books, 2000.
[146] Lovelace, 'Memoir on Analytical Engine', p. 284.
[147] Victor Vinge, 'The Technological Singularity', available at http://mindstalk.net/vinge/vinge-sing.html.
[148] Eugene Charniak and Drew McDermott, *Introduction to Artificial Intelligence*, Reading, Massachusetts: Addison-Wesley, 1985, pp. 10–11.
[149] Herbert A. Simon, *The New Science of Management Decision*, Prentice Hall, 1977, p. 6.
[150] Stuart Armstrong, *Smarter Than Us: The Rise of Machine Intelligence*, Machine Intelligence Research Institute, 2014, p. 42.
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[153] Nick Bostrom, *Superintelligence: Paths, Dangers, Strategies*, Oxford University Press, 2014, pp. 12–13.
[154] In October 2017, 3Blue1Brown gave a series brilliant YouTube videos on deep learning, beginning with 'But what *is* a Neural Network? | Chapter 1, deep learning', https://youtu.be/aircAruvnKk.
[155] James Martin, *After the Internet: Alien Intelligence*, Washington. DC: Capital Press, 2000, pp. 28, 30, 32–33, and 40–42.
[156] Richard Dawkins, *The Blind Watchmaker: Why the Evidence of Evolution Reveals a Universe without Design*, first edition, 1986, New York, Norton, 1996.
[157] Ray Kurzweil, et al, *Are We Spiritual Machines?: Ray Kurzweil vs. the Critics of Strong A.I.*, p. 11.
[158] http://www.computer50.org/mark1/new.baby.html. *See also* F. C. Williams and T. Kilburn, 'Electronic Digital Computers', reproduced in Randell, *Origins of Digital Computers*, pp. 415–416.
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[161] Daniel Bell, *The Coming of Post-Industrial Society*, Basic Books, 1973.
[162] Daniel Bell, 'The Social Framework of the Information Society' in *The Computer Age: A Twenty-Year View*, edited by Michael L. Dertouzos and Joel Moses, Cambridge, MA: MIT Press, 1979, p. 173.

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[164] https://www.economist.com/briefing/2017/05/06/data-is-giving-rise-to-a-new-economy.
[165] IBM, *Object-Oriented Interface Design: IBM Common User Access™ Guidelines*, Carmel, IN: Que, , p. 11.
[166] From an earlier edition of the previous book.
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[168] Ibid., p. 2.
[169] Jay W. Forrester, *Industrial Dynamics*, Cambridge: MIT Press, 1961; *Urban Dynamics*, 1969, Pegasus Communications, 2002; and *World Dynamics*, 1971, 2nd ed., MIT Press, 1979.
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[171] Forrester, *World Dynamics*, p. 14.
[172] Ibid., p. 123.
[173] Joseph Weizenbaum, 'Once More: The Computer Revolution', in *The Computer Age: A Twenty-Year View*, eds. Michael L. Dertouzos and Joel Moses, Cambridge, MA: MIT Press, 1979, pp. 446–447. The quotation is from United States, Congress, House, Committee on Banking and Currency, Hearings before the Subcommittee on Urban Growth, October 7, 1970, 91st Congress, 2nd session, pp. 205–265.
[174] Ibid., pp. 445 and 447.
[175] IBM, *Systems Development Method, Business Model: Processes*, p. 43, 10th October 1983.
[176] https://en.wikipedia.org/wiki/System_Architect.
[177] Adobe Systems, *PostScript Language Reference Manual*, Addison Wesley, 1991, pp. 45–47.
[178] Robert Smith, 'Code vs Data (Metaprogramming) – Computerphile', 18th May 2018, https://youtu.be/dw-y3vNDRWk.
[179] Raymond Polivka and Sandra Pakin, *APL: The Language and Its Usage*, Englewood Cliffs, NJ: Prentice Hall, 1975, pp. 421–425.
[180] Leonard Gilman and Allen J. Rose, *APL: An Interactive Approach*, Krieger Publishing Company, 1991. This is the edition that I most probably consulted in the 1990s. However, I have not found this exact citation in the third Wiley edition, borrowed from the library in June 2018. Rather, on page 249 in the subsection titled 'Automatic program generation and modification', Gilman and Rose write this: "Since the earliest days of the stored program computer there has been speculation (mostly in jest, but some serious) about when computers would become self-perpetuating. … No, your APL system can't write or modify programs without your active participation. But it can reduce the amount of work you do by taking over virtually all aspects of the job except thinking." So while they can see the mechanism for programs to write and modify programs, they don't ask where the thinking required to write such program-writing functions originally comes from.
[181] Peter Lane, *Industrial Revolution*, *The Industrial Revolution: The Birth of the Modern Age*, London: Weidenfield and Nicolson, 1978, pp. 24–25.
[182] Trevor Jones, 'A New Society?' in Trevor Jones, ed., *Microelectronics and Society*, Open University Press, 1980, pp. 150–151.
[183] Daniel Bell, 'The Social Framework of the Information Society' in *The Computer Age: A Twenty-Year View*, edited by Michael L. Dertouzos and Joel Moses, Cambridge, MA: MIT Press, 1979, pp. 184–185.
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[206] Byran Magee, *Great Philosophers: An Introduction to Western Philosophy*, Oxford Paperbacks, 2000, p. 86.
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[220] Ken Wilber, *Up from Eden; a Transpersonal View of Human Evolution*, originally published 1981, Wheaton, IL: Quest Books, 1996, p. 12.
[221] Molière (J.-B. Poquelin), *Le Bourgeois Gentilhomme*, 1670, II.iv, Oxford Dictionary of Quotations, Oxford University Press, 1979, p. 353.
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[223] E. F. Schumacher, *A Guide for the Perplexed*, original edition, Jonathan Cape, 1977, London: Abacus, 1978, pp. 15 and 24.
[224] Plato, *Protagoras*, 343b, p. 51. Their leader was Thales of Miletes, who Bertrand Russell considered the first Greek philosopher (Bertrand Russell, *Western Philosophy*, p. 44.)
[225] Ibid., p. 28.
[226] Aristotle, *Categories, On Interpretation, and Prior Analytics*, tr. Harold P. Cooke and Hugh Tredennick, Cambridge: Harvard University Press; London: William Heinemann, 1938.
[227] Francis Bacon, *The New Organon*, ed. Lisa Jardine and Michael Silverthorne, original edition *Novum Organum*, 1620, Cambridge University Press, 2000, p. 17.
[228] Charles Sanders Peirce, 'Deduction, Induction, and Hypothesis', *Popular Science Monthly*, Vol. 13, August 1878, pp. 470–482, reprinted in *The Essential Peirce: Selected Philosophical Writings*, Volume 1 (1867–1893), eds. Nathan Houser and Christian Kloesel, Indiana University Press, 1992, pp. 186–189.
[229] A more detailed description of the history of scientific method is contained within my books *The Theory of Everything: Unifying Polarizing Opposites in Nondual Wholeness* and *The Four Spheres: Healing the Split between Mysticism and Science*.
[230] Karl R. Popper, *Objective Knowledge: An Evolutionary Approach*, revised edition, original edition, 1972, Oxford University Press, 1979, p. 191.
[231] Richard Tarnas, *The Passion of the Western Mind: Understanding the Ideas That Have Shaped Our World View*, New York: Harmony Books, 1991, p. 45.
[232] Chalmers, *Science*, pp. 2–5.
[233] Ibid., p. 89.
[234] Thomas S. Kuhn, *The Structure of Scientific Revolutions*, second edition, enlarged, original edition, 1962, Chicago: University of Chicago Press, 1970, Chapter V, p. 10.
[235] Kuhn, *Scientific Revolutions*, fourth edition, introduced by Ian Hacking, Kindle edition, 2012.
[236] Kuhn, *Scientific Revolutions*, p. 212.
[237] For instance, the Alister Hardy Religious Experience Research Centre in the UK has discovered that 43% of Americans and 48% of British people have had such experiences, which Robert K. C. Forman reported on in 1994, when writing an editorial in the very first issue of the *Journal of Consciousness Studies*, exploring 'controversies in science & the humanities'. (Robert K. C. Forman, 'Mysticism, Language, & the Via Negativa', *Journal of Consciousness Studies*, Vol. 1, No. 1, 1994, 38, referring to David Hay, *Religious Experience Today*, London: Mowbray, 1990. The results of Hardy's researches are recorded in Alister Hardy, *The Spiritual Nature of Man: Study of Contemporary Religious Experience*, 1979, Oxford, England: Religious Experience Research Centre.)
People who have had a near-death experience, when they seem to be out of their bodies, are similarly exposed to Wholeness, describing such experiences as 'coming home' or a 'vision of paradise', as Peter and Elizabeth Fenwick tell us in *The Truth in the Light*. (Peter and Elizabeth Fenwick, *The Truth in the Light: An Investigation of over 300 Near-Death Experiences*, London: Headline Book Publishing, 1996, pp. 97–138.)
Altered States of Consciousness, edited by Charles T. Tart, contains many reports of so-called non-ordinary states of consciousness arising in many different ways, when the brilliant light of Consciousness is revealed, normally hidden from view and experience by clouds of unknowing, by mental structures, which cannot explain what is happening. And Tart, a scientist like Alister Hardy, collects descriptions of

scientists' spiritual experiences, which he publishes in TASTE—The Archives of Scientists' Transcendent Experiences. (http://www.issc-taste.org/)
[238] J. Krishnamurti, *Education and the Significance of Life*, originally published 1953, HarperSan-Franscisco, 1981, p. 18.
[239] Bohm, *Wholeness and the Implicate Order*, p. 1.
[240] Erich Jantsch, 'Towards lnterdisciplinarity and Transdisciplinarity in Education and Innovation', in Léo Apostel, Guy Berger, Asa Briggs, and Guy Michaud, *Interdisciplinarity Problems of Teaching and Research in Universities*, Paris: Organisation for Economic Co-operation and Development, 1972, p. 105.
[241] Basarab Nicolescu, *Manifesto of Transdisciplinarity*, tr. from French by Karen-Claire Voss, State University of New York Press, 2002, pp. 26–31.
[242] Evelyne Blau, *Krishnamurti*: *100 Years*, New York: Stewart, Tabori and Chang, 1995, p. 159.
[243] Bohm, *Wholeness and the Implicate Order*, p. 18.
[244] http://www.noetic.org/about/history.cfm.
[245] Jean Gebser, *The Ever-Present Origin: Foundations and Manifestations of the Aperspectival World*, tr. Noel Barstad and Algis Mickunas, Ohio University Press, 1986, p. 6.
[246] The El Castillo cave, located in Puente Viesgo, Cantabria, Northern Spain, still open to the public, contains palaeolithic art at least 40,800 years, albeit somewhat more primitive (http://cuevas.culturadecantabria.com/el-castillo-2/).
An even earlier example of human drawing is this fragment of rock, estimated to be 73,000 years old, found in Blombos cave in South Africa (https://www.theguardian.com/science/2018/sep/12/earliest-known-drawing-found-on-rock-in-south-african-cave):

So we are constantly needing to revise our timescales of early human development.
[247] Ken Wilber, *Up from Eden: A Transpersonal View of Human Evolution*, originally published 1981, Wheaton, IL: Quest Books, 1996, Chapters 6 and 7 'Great Mother' and 'Great Goddess', pp. 119–156.
[248] Anne Baring and Jules Cashford, *The Myth of the Goddess: Evolution of an Image*, Penguin Books, 1993, pp. 3–5 and 10.
[249] Joseph Campbell, *Historical Atlas of World Mythology - Vol. I: The Way of the Animal Powers, Part 1: Mythologies of the Great Hunt*, New York: Harper & Row, pp. 64 and 71.
[250] Eckhart Tolle, *A New Earth: Awakening to Your Life's Purpose*, London: Penguin, 2006, pp. 138 and 309.
[251] https://en.wikipedia.org/wiki/Recapitulation_theory.
[252] Koestler, *Ghost in the Machine*, pp. 163–165.
[253] Arthur Koestler, *The Sleepwalkers: A History of Man's Changing Vision of the Universe*, original edition, Hutchinson, 1959, Harmondsworth, England: Penguin, Pelican, 1968, pp. 50–52.
[254] *Zeigeist: Addendum*, The Zeitgeist Movement, 2008, movie freely available from http://www.zeitgeistaddendum.com/.
[255] Paulo Coelho, *Veronica Decides to Die*, Harper Collins, 2005, Kindle edition, location 2604 of 2688. A movie based on the book was released in 2009 (https://www.imdb.com/title/tt1068678/).
[256] Edited by Peirce Edition Project (http://peirce.iupui.edu).
[257] Albert Einstein, 'Does the Inertia of a Body Depend on Its Energy Content?' in *Einstein's Miraculous Year: Five Papers That Changed the Face of Physics*, foreword Roger Penrose, Princeton University Press, 2005, pp. 161–164, tr. by John Stachel of '*Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?*', *Annalen der Physik*, 1905, 18 (13): 639–641.

[258] I began to look inwards to discover why we humans behave as we do when I was promoted to a first-line manager in an IBM sales office in January 1974. For the man in charge of managerial education in head office knew that we could not be effective leaders of our staff without some understanding of ourselves and what motivates us all, drawing on Abraham Maslow's hierarchy of needs (Abraham H. Maslow, *Motivation and Personality*, third edition, original edition, 1954, New York: Harper & Row, 1970, pp. 15–23.), Frederick Herzberg's 'hygiene factors' for job satisfaction and dissatisfaction (https://en.wikipedia.org/wiki/Two-factor_theory), and Douglas McGregor's theories X and Y of human motivation (https://en.wikipedia.org/wiki/Theory_X_and_Theory_Y).
At about the same time, some friends introduced me to Eric Berne and Thomas A. Harris's transactional analysis, with its simple parent-adult-child (PAC) model of interpersonal communication (Eric Berne, *Games People Play: The Psychology of Human Relationships*, Penguin Books, 1973; Thomas A. Harris, *I'm OK, You're OK*, Pan Books, 1973).
[259] Letter from Oxford English Dictionary Word and Language Service (OWLS) in 1993.
[260] When I participated in a stirring holotropic breathwork session with Christina and Stanislav Grof in 1992 at a conference in Prague titled 'Science, Spirituality, and the Global Crisis', organized by the International Transpersonal Association, we were asked to draw a mandala at the end of our breathing exercise to depict our experiences, for *mandala* is a Sanskrit word meaning 'disk, circle', a circular figure representing Wholeness or the Universe in Hindu and Buddhist symbolism.
[261] Brian Cox, presenter, *Wonders of the Universe,* DVD, BBC, 2011.
[262] John D. Barrow and Frank J. Tipler, *The Anthropic Cosmological Principle*, Oxford University Press, 1986, p. 166, referencing Stephen G. Brush, *The Temperature of History: Phases of Science and Culture in the Nineteenth Century*, New York: Franklin, 1978.
[263] Heijenoort, *Frege to Gödel*, p. 1.
[264] Mircea Eliade, *Myths, Dreams, and Mysteries: The Encounter Between Contemporary Faiths and Archaic Realities*, tr. Philip Maret, New York: Harper Torchbooks, [1957] 1967, p. 174.
[265] Mircea Eliade, *Patterns in Comparative Religion*, tr. Rosemary Sheed, intro. John Clifford Holt, University of Nebraska Press, [1958] 1996, p. 419.
[266] "The scientific principle that things are usually connected or behave in the simplest or most economical way, especially with reference to alternative evolutionary pathways," *Oxford Dictionary of English*, on MacOS, from Latin from *parcere* 'be sparing'.
[267] https://en.wikipedia.org/wiki/Occam%27s_razor.
[268] http://en.wikipedia.org/wiki/Andromeda–Milky_Way_collision.
[269] Brian Cox, *Wonders of the Universe*, BBC.
[270] National Geographic, *The Death of the Universe*, National Geographic Channel, 2008.
[271] Rees, *Our Final Century*, pp. 146–147.
[272] William James, 'Is Life Worth Living', originally published in the International Journal of Ethics in 1895, in *The Will to Believe, Human Immortality, And Other Essays in Popular Philosophy*, Dover Publications, 1956, pp. 43–44.
[273] Antony Flew, *Philosophy: An Introduction*, London: Teach Yourself Books, Hodder and Stoughton, 1979, p. 62.
[274] Philip J. Davis and Reuben Hersh, *Descartes' Dream: The World According to Mathematics*, Penguin, 1988, pp. 3–4.
[275] F. E. Sutcliffe, introduction to René Descartes, *Discourse on Method Discourse on the Method of Properly Conducting One's Reason and of Seeking the Truth in the Sciences* and *Meditations on the First Philosophy in which the Existence of God and the Real Distinction between the Soul and the Body of Man are Demonstrated*, tr. F. E. Sutcliffe, 1637, Harmondsworth, England: Penguin, 1968, p. 12.
[276] Descartes, *Discourse on Method*, p. 53.
[277] Ibid., pp. 53–54.
[278] Ibid., pp. 12 and 45.
[279] Russell, *Western Philosophy*, p. 542.
[280] Bryan Magee, *The Great Philosophers: An Introduction to Western Philosophy*, Oxford Paperbacks, 2000,

p. 80.
[281] Madan Sarup, *An Introductory Guide to Post-Structuralism and Post-Modernism*, pp. 131–132.
[282] Ken Wilber, *Theory of Everything*, p. xii.
[283] Ibid.
[284] Christian de Quincey, 'A Theory of Everything? A Critical Appreciation of Ken Wilber's *Collected Works*', *Noetic Sciences Review*, March-May 2001, No. 55, p. 15.
[285] F. C. Happold, *Mysticism: A Study and an Anthology*, revised edition, original edition, 1963, Harmondsworth, England: Penguin, 1970, p. 72.
[286] Eknath Easwaran, translator, *The Upanishads*, London: Routledge, Arkana, 1987, pp. 61 and 184.
[287] Stated in one of his satsangs soon after the turn of the millennium.
[288] Ian Stewart, *Concepts of Modern Mathematics*, Harmondsworth: Penguin Books, 1975, p. 286.
[289] Brooks, *Mythical Man-Month*, p. 42.
[290] Jan Christiaan Smuts, *Holism and Evolution*, originally published, 1926, reprint, Highland, NY: Gestalt Journal Press, 1996, p. v.
[291] Ibid., p. 99.
[292] Azriel Levy, *Basic Set Theory*, Berlin: Springer-Verlag, 1979, p. 3.
[293] Thomas Jech, *Set Theory*, 3rd ed., Berlin: Springer, 2006, pp. 3–4.
[294] Bohm, *Wholeness*, pp. 115–116 and 216n.
[295] Charles M. Barker, Helen Curran, and Mary Metcalf, *The 'New' Maths for Teachers and Parents of Primary School Children*, London: Arlington Books, 1964, p. v.
[296] Aristotle, *Categories, On Interpretation, and Prior Analytics*, tr. Harold P. Cooke and Hugh Tredennick, Cambridge: Harvard University Press; London: William Heinemann, 1938, 24b17, p. 201.
[297] Barker, et al., *'New' Maths*, pp. 4–6.
[298] J. Venn, 'On the Diagrammatic and Mechanical Representation of Propositions and Reasonings', *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, Series 5, Vol. 9, No. 59, July 1880, pp. 1–18.
[299] The letters were addressed to Friederike Charlotte of Brandenburg-Schwedt and her younger sister Louise (https://en.wikipedia.org/wiki/Letters_to_a_German_Princess).
[300] Peirce, *Collected Papers: Volume IV*, p. 291.
[301] Peirce, *Collected Papers: Volume IV*, paras. 350–371.
[302] Peter Pin-Shan Chen, 'The Entity-Relationship Model—Toward a Unified View of Data' in *ACM Transactions on Database Systems*, Vol. 1, No. 1, 1976, pp. 9–36.
[303] John Sowa, 'Semantic Networks', *Encyclopedia of Artificial Intelligence*, p. 1496.
[304] Augustus De Morgan, *On the Syllogism, and Other Logical Writings* (from 1846–1868), edited with an Introduction by Peter Heath, London: Routledge & Kegan Paul, 1966, p. 119.
[305] Ibid., p. 208.
[306] Morris Kline, *Mathematics: The Loss of Certainty*, Oxford University Press, 1980, p. 186.
[307] Ketner, et al., *Bibliography of the Published Works of Charles Sanders Peirce*, p. 7.
[308] Peirce, 'Description of a Notation for the Logic of Relatives, Resulting from an Amplification of the Conceptions of Boole's Calculus of Logic', *Memoirs of the American Academy of Arts and Sciences*, Vol. 9, pp. 317–378, 1870. Reprinted by: Welch, Bigelow, and Company, Cambridge, MA, 1870. Also in *Collected Papers: Exact Logic*, ¶¶ 3.45–149, and *Writings*, 1867–1871, pp. 359–429.
[309] Peirce, *Writings*, 1867–1871, p. xlii.
[310] 'On the Algebra of Logic', *American Journal of Mathematics*, Vol. 3, 1880, pp. 15–57. Also in *Collected Papers: Exact Logic*, ¶¶ 3.154–251 and Writings, 1879–1884, pp. 163–209.
[311] Members of Johns Hopkins University, *Studies in Logic*, edited by C. S. Peirce, with an Introduction by Max H. Fisch and a Preface by Achim Eschbach, first published 1883 by Little, Brown, and Company, Boston, MA, Amsterdam/Philadelphia: John Benjamins, 1983.
[312] 'On the Algebra of Logic: A Contribution to the Philosophy of Notation', *American Journal of Mathematics*, Vol. 7, January 1885, pp. 180–202. Also in *Collected Papers: Exact Logic*, ¶¶ 3.359–403 and Writings, 1884–1886, pp. 162–190.

[313] Brady, *From Peirce to Skolem*, p. 113.
[314] Peirce, 'The Logic of Relatives', *The Monist*, Vol. 7, January 1897, pp. 161–217. Also in *Collected Papers: Exact Logic*, paras. 3.456–552.
[315] Peirce, *Reasoning and the Logic of Things: The Cambridge Conferences Lectures of 1898*, edited by Kenneth Laine Ketner, with an Introduction by Kenneth Laine Ketner and Hilary Putnam, Harvard University Press, 1992.
[316] http://www.jfsowa.com/peirce/ms514.htm.
[317] Plato, *The Republic*, second edition, translated by Desmond Lee, original edition, 1955, Harmondsworth, England: Penguin, 1974, 472c, p. 261.
[318] James R. Rumbaugh, Michael R. Blaha, William Lorensen, Frederick Eddy, and William Premerlani, *Object-oriented Modeling and Design*, Englewood Cliffs, NJ: Prentice-Hall, 1991, pp. 36–43.
[319] en.wikipedia.org/wiki/Systema_Naturae.
[320] Melvil Dewey, *Dewey Decimal Classification and Relative Index*, twenty-third edition, first edition, OCLC Online Computer Library Center, 2011.
[321] Hofstadter, *Gödel*, p. 134.
[322] Fritjof Capra, *The Web of Life: A New Synthesis of Mind and Matter*.
[323] Ken Wilber, *Sex, Ecology, Spirituality: The Spirit of Evolution*, Boston: Shambhala, 1995, pp. 21–23.
[324] https://en.wikipedia.org/wiki/Entity–relationship_model.
[325] https://opentextbc.ca/dbdesign01/chapter/chapter-8-entity-relationship-model/.
[326] Barker, *CASE*METHOD™: Entity Relationship Modelling*, Wokingham, England: Addison-Wesley, 1990, p. H-1.
[327] E. F. Codd, 'Derivability, Redundancy and Consistency of Relations Stored in Large Data Banks', IBM Research, RJ 599, 19th August 1969, p. 4.
[328] Eric M. Hammer, 'Semantics for Existential Graphs', *Journal of Philosophical Logic*, 27: 489–503, 1998 and Irving H. Anellis, 'How Peircean was the " 'Fregean' Revolution" in Logic?', Peirce Edition, Institute for American Thought, Indiana University – Purdue University at Indianapolis, Indianapolis, IN, USA.
[329] Fritjof Capra, *The Turning Point: Science, Society and the Rising Culture*, p. 68.
[330] In 'Peirce's Truth-Functional Analysis and the Origin of Truth Tables', Irving H. Anellis tells us that Charles Sanders Peirce presented a truth table of what is today called 'material implication' in an unpublished piece he wrote in 1893 titled 'An Outline Sketch of Synechistic Philosophy'. This was nearly twenty years before similar constructs were found on the verso page of the typed transcript of Bertrand Russell's 1912 lecture on 'The Philosophy of Logical Atomism'. There the matrix for negation is Russell's, alongside of which is the matrix for material implication in the hand of Ludwig Wittgenstein. However, as I have not seen Peirce's sketch, I'm not sure how he made the connection between this truth table and synechism, denoting continuity.
[331] Clement V. Durell, *Projective Geometry*, originally published 1926, London: Macmillan, 1952, p. 108.
[332] Ibid., pp. 90–92.
[333] Ibid., pp. 92–93.
[334] Wikipedia's article on the Stellated octahedron credits the term *stella octangula* to Kepler in 1609, the publication date of *New Astronomy*, and Wolfram Mathworld's article on Stella Octangula credits this term to Kepler's *De Nive Sexangula* 'Six-sided snowflake', published in 1611. However, I cannot find this term in the English translation of either of these books. Moreover, even though Kepler showed two stellations of the dodecahedron in *The Harmony of the World* (pp. 111, 116, and 397–398) he did not include a diagram of the stellated octahedron, as the congruence of two tetrahedra.
[335] Magnus J. Wenninger, *Polyhedron Models*, Cambridge University Press, 1971, p. 35. Figures are from https://polyhedra.mathmos.net/.
[336] Jech, *Set Theory*, p. 3.
[337] Kline, *Mathematics*, pp. 209–212.
[338] Peirce, 'The Architecture of Theories', *The Monist*, Vol. 1, January 1892, pp. 321–337, in *Essential Peirce*, Vol. 1, pp. 296.
[339] Euclid, *The Thirteen Books of Euclid's Elements*, Vol. II, tr. Thomas L. Heath, New York: Dover

Publications, Book IX, Proposition 20, pp. 412–413.
[340] https://en.wikipedia.org/wiki/Prime_number_theorem.
[341] https://primes.utm.edu/howmany.html.
[342] Aristotle, *Metaphysics*, 1005b20–25, 1011b24, p. 199.
[343] Alexis de Tocqueville, *Democracy in America*, translated by George Lawrence in 1966 and edited by J. P. Mayer in 1969, originally published as *De la démocratie en Amérique* in two volumes in 1835 and 1840, London: Fontana Press, 1994, section 'Tyranny of the Majority', pp. 250–253.
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[345] http://www.reaganlibrary.com/reagan/speeches/first.asp.
[346] Aristotle, *Metaphysics*, Books I-IX, translated by Hugh Tredennick, Cambridge: Harvard University Press; London: William Heinemann, 1933, 1005b20, pp. 161 & 163.
[347] Osho, *The Hidden Harmony: Discourses on the Fragments of Heraclitus*, second edition, Cologne: The Rebel Publishing House, 1992, p. 1.
[348] Ken Wilber, *Integral Spirituality: A Startling New Role for Religion in the Modern and Postmodern World*, Boston, MA: Shambhala Publications, 2006, pp. 20–21.
[349] Ibid., pp. 30–32.
[350] Monier Monier-Williams, *A Sanskrit-English Dictionary: Etymologically and Philologically Arranged with Special Reference to Cognate Indo-European Languages*, first published 1872, new edition revised by E. Leumann and C. Cappeller, et al, 1899, Ashford, Kent, England: Bay Foreign Language Books, 2003, p. 1284.
[351] Bertrand Russell, 'Reflections on My Eightieth Birthday' in *Portraits from Memory and Other Essays*, first edition, 1956, Nottingham, England: Spokesman, 1995, p. 53.
[352] Alfred North Whitehead and Bertrand Russell, *Principia Mathematica to *56*, shortened edition, 1962, original edition in three volumes, 1910, 1912, and 1913, Cambridge University Press, 1997, p. 360.
[353] Russell, *Principles of Mathematics*, p. 523.
[354] Kline, *Mathematics*, p. 221.
[355] Whitehead and Russell, *Principia Mathematica*, p. 37.
[356] Ernest Nagel and James R. Newman, *Gödel's Proof*, original edition, 1959, London: Routledge & Kegan Paul, 1971, pp. 50–51.
[357] Ian Stewart, *Concepts of Modern Mathematics*, Harmondsworth, England: Penguin, 1975, p. 116.
[358] http://www.bbc.co.uk/sn/tvradio/programmes/horizon/einstein_symphony_prog_summary.shtml.
[359] Aurobindo, *The Life Divine*, first published as a serial, 1914–1919, original edition, 1939–1940, Sri Aurobindo Ashram, 2001, p. 141.
[360] Nathan Houser and Christian Kloesel, note to 'One, Two, Three: Kantian Categories' in *The Essential Peirce: Selected Philosophical Writings*, Volume 1 (1867–1893), edited by Nathan Houser and Christian Kloesel, Indiana University Press, 1992, p. 242.
[361] Joseph Brent, *Charles Sanders Peirce: A Life*, revised and expanded edition, originally published 1993, Indiana University Press, 1998, p. 173.
[362] Peirce's 1904 letter to Francis C. Russell, published in Brent, *Charles Sanders Peirce*, p. 323.
[363] Paul Carus, *The Gospel of Buddha According to Old Records*, originally published as *The Gospel of Buddha: Compiled from Ancient Records*, Open Court, 1894, Open Court, 2004.
[364] Ibid.
[365] Ibid., p. 316.
[366] Morris Kline, *Mathematics: The Loss of Certainty*, Oxford University Press, 1980, pp. 199–204.
[367] Peirce, 'Immortality in the Light of Synechism', in *Essential Peirce*, Vol. 2, p. 2. First published in *Collected Papers of Charles Sanders Peirce: Volume VII, Science and Philosophy*, edited by Arthur W. Burks, Cambridge, MA: Harvard University Press, 1958, paragraphs 565–578.
[368] Ibid., p. 3.
[369] Baring, *Dream of the Cosmos*, pp. 489–520.
[370] Peirce, 'Immortality', p. 3.

[371] Ibid.
[372] Brent, *Charles Sanders Peirce*, p. 213.
[373] Bohm, *Wholeness and the Implicate Order*, p. 48.
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[431] https://en.wikipedia.org/wiki/Comparison_of_programming_languages_(array)#Array_system_cross-reference_list.

[432] Polivka and Pakin, *APL*, pp. 375–376. The system variable `⎕IO` stores the base index, which can also be changed with the `)ORIGIN` system command.
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[434] Alfred North Whitehead and Bertrand Russell, *Principia Mathematica to *56*, shortened edition, 1962, original edition in three volumes, 1910, 1912, and 1913, Cambridge University Press, 1997, p. 360.
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[437] A000040, The prime numbers in OEIS.
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[439] Ibid., Book IX. Proposition 20 (pp. 412–413) states "Prime numbers are more than any assigned multitude of prime numbers" and Proposition 14 (pp. 402–403) states, "If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it."
[440] David M. Burton, *The History of Mathematics: An Introduction*, seventh edition, McGraw-Hill, [1998] 2011, p. 177.
[441] Ibid., pp. 180–181.
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[443] https://primes.utm.edu/notes/proofs/infinite/kummers.html.
[444] https://en.wikipedia.org/wiki/Primorial_prime.
[445] https://en.wikipedia.org/wiki/5040_(number).
[446] A000396, Perfect numbers *n*: *n* is equal to the sum of the proper divisors of *n* in OEIS.
[447] Euclid, *The Elements*, Vol. II, Book VII, Definition 22: "A **perfect number** is that which is equal to its own parts," p. 278.
[448] Euclid, *The Elements*, Vol. II, Book IX. Proposition 36 (pp. 421–426) states, "If as many numbers as we please beginning from an unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product with be perfect."
[449] Burton, *History of Mathematics*, pp. 504–507.
[450] Ibid., p. 506.
[451] Ibid.
[452] J. J. O'Connor and E. F. Robertson, Pietro Antonio Cataldi, *MacTutor History of Mathematics archive*, University of St Andrews, http://www-history.mcs.st-andrews.ac.uk/Biographies/Cataldi.html.
[453] https://en.wikipedia.org/wiki/Mersenne_prime.
[454] In 1876, Lucas proved that M_{67} is composite, but he could not produce the actual factors. Then, at the October 1903 meeting of the American Mathematical Society, Frank Nelson Cole had a paper on the programme with the modest title *On the Factorization of Large Numbers*. When called on to speak, Cole walked to the blackboard, and, saying nothing, raised the number two to the 67th power and subtracted one, giving 147,573,952,589,676,412,927. Silently, he moved to a clean part of the board and multiplied longhand this product:

$$193{,}707{,}721 \times 761{,}838{,}257{,}287 = 147{,}573{,}952{,}589{,}676{,}412{,}927$$

Cole took his seat without uttering a word to a standing ovation, the first and only time on record at such a meeting. He later confided to a friend that it had taken 'three years of Sundays' to find the prime factors of M_{67} (Burton, *History of Mathematics*, p. 508).
[455] Burton, *History of Mathematics*, p. 508.
[456] Ibid. Burton also credits Paul Seelhoff with independently finding M_{61} in 1886 and E. Fauquembergue for independently finding M_{107} between 1914 and 1917.
[457] Ibid., p. 509.
[458] https://en.wikipedia.org/wiki/Raphael_M._Robinson.

[459] https://en.wikipedia.org/wiki/SWAC_(computer).
[460] https://www.mersenne.org.
[461] http://primes.utm.edu/largest.html.
[462] https://oeis.org/A000043. The Mersenne primes themselves are A001348, while the Mersenne numbers are A000225, where $a(n) = 2^n - 1$.
[463] https://www.mersenne.org/primes/press/M77232917.html.
[464] https://en.wikipedia.org/wiki/Amicable_numbers.
[465] James Grime, 'Is Zero Even?', Numberphile, ed. Brady Haran, https://youtu.be/8t1TC-5OLdM.
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[467] https://youtu.be/ij-EK-MZv2Q.
[468] https://youtu.be/U2zLoTG6VFY.
[469] **modular arithmetic**. (2008). Encyclopædia Britannica. *Encyclopaedia Britannica 2008 Ultimate Reference Suite*. Chicago: Encyclopædia Britannica.
[470] OED, **rational**, adj. A. 5. a. Heath gives this definition: "A **ratio** is a sort of relation in respect of size between two magnitudes of the same size," Euclid, *The Elements*, Vol. II, Book V, p. 114.
[471] https://en.wikipedia.org/wiki/Rational_number.
[472] http://mathworld.wolfram.com/EgyptianFraction.html.
[473] http://www.midicode.com/tunings/greek.shtml.
[474] http://en.wikipedia.org/wiki/Pythagorean_tuning.
[475] Kepler, *Harmony of the World*, p. 138.
[476] Stanley Sadie, editor, *The Grove Concise Dictionary of Music*, Grove Publications, 1988, p. 181.
[477] http://en.wikipedia.org/wiki/Just_intonation.
[478] http://en.wikipedia.org/wiki/Cent_(music).
[479] John H. Conway and Richard Guy, *The Book of Numbers*, Copernicus Books, 1998, p. 169–171.
[480] http://mathworld.wolfram.com/RepeatingDecimal.html and http://mathworld.wolfram.com/DecimalExpansion.html.
[481] Euclid, *Elements*, Book VII, Proposition 2, pp. 298–300.
[482] https://en.wikipedia.org/wiki/Euclidean_algorithm.
[483] Ibid., Proposition 1, pp. 296–297.
[484] An idea I got from http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html.
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[486] Shakespeare, *All's Well That Ends Well*, v. iii. 307.
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[495] Ibid.
[496] Boyer and Merzbach, *History of Mathematics*, p. 68.
[497] Ibid., p. 82.
[498] Ibid., p. 297 and Boyer and Merzbach, *History of Mathematics*, p. 356.
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[500] Burton, *History of Mathematics*, p. 690.
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[502] Burton, *History of Mathematics*, p. 691.
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[504] Boyer and Merzbach, *History of Mathematics*, p. 540.
[505] https://en.wikipedia.org/wiki/Cantor's_diagonal_argument.
[506] Burton, *History of Mathematics*, p. 694.
[507] Boyer and Merzbach, *History of Mathematics*, p. 541.
[508] Ibid., p. 543.
[509] Burton, *History of Mathematics*, p. 690.
[510] Conway and Guy, *Book of Numbers*, pp. 239–241.
[511] Mathologer, https://youtu.be/c9nUAXUSuII.
[512] Boyer and Merzbach, *History of Mathematics*, p. 546.
[513] Paul Hague, *Through Evolution's Accumulation Point: Towards Its Glorious Culmination*, 2016, unpublished.
[514] Conway and Guy, *Book of Numbers*, p. 186.
[515] blackpenredpen, 'Infinite Continued Fractions, simple or not?', https://youtu.be/vw4eYXMWYEw.
[516] http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html#section6.1.
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[518] ei pi, 'Radicals and Continued Fractions', https://youtu.be/IKdLhkqJk9g.
[519] Conway and Guy, *Book of Numbers*, p. 186.
[520] https://en.wikipedia.org/wiki/Continued_fraction#Generalized_continued_fraction.
[521] https://quod.lib.umich.edu/c/cme/AFZ9170.0001.001/1:170?rgn=div1;view=fulltext.
[522] William Jones was a self-taught mathematician from Wales, one of Anglesey's most famous sons. He was the father of William, a philologer and jurist, who was the first to notice the similarities in Sanskrit, Persian, Greek, Latin, and Celtic, leading to the discovery of the putative Proto-Indo-European language. Gareth Ffowc Roberts, 'Pi Day 2015: meet the man who invented π', https://www.theguardian.com/science/alexs-adventures-in-numberland/2015/mar/14/pi-day-2015-william-jones-the-welshman-who-invented-pi.
[523] William Jones, *Synopsis Palmariorum Matheseos* '*A Summary of Achievements in Mathematics*', p. 127. The first part of the extensive subtitle is *A New Introduction to the Mathematics: Containing the Principles of Arithmetic & Geometry, Demonstrated in a Short and Easy Method*.
[524] Ibid., pp. 263 and 243.
[525] Burton, *History of Mathematics*, p. 321.
[526] Boyer and Merzbach, *History of Mathematics*, p. 255.
[527] Burton, *History of Mathematics*, pp. 328–329.
[528] Ibid., pp. 321 and 324–326.
[529] Weisstein, Eric W. 'Cubic Formula'. From *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/CubicFormula.html . Weisstein, Eric W. 'Quartic Equation'. From *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/QuarticEquation.html.
[530] Enrique A. González-Velasco, *Journey through Mathematics: Creative Episodes in Its History*, Springer, 2011, p. 191. Writing in Latin, the quote is *Tales numeros vocabimus numeros integros complexos* in *Commentationes societatis regiae scientiarum Gottingensis recentiones*, vol. VII.
[531] George Gamow, *One Two Three … Infinity: Facts and Speculations of Science*, 2nd ed., 1st ed., 1947, New York: Viking Press, 1961, p. 33.
[532] Burton, *History of Mathematics*, pp. 631–632. Wessel's paper was written in Danish for the Royal Academy of Sciences in Denmark, not noticed by his contemporaries. It lay buried for a century before his contribution was acknowledged.
[533] Ibid., p. 632. Argand's contribution was privately published in 1806 as '*Essai sur une manière de*

représenter les quantités imaginaires dans les constructions géométriques' ('Essay on a method of representing imaginary quantities'), republished in 1813 in the French journal *Annales de Mathématiques*. So his contribution fared somewhat better than that of Wessel.
[534] https://en.wikipedia.org/wiki/Gaussian_integer.
[535] Hamilton wrote to his son Archibald Henry in 1865, "Every morning in the early part of [October 1843], on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me: 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head: 'No, I can only add and subtract them'." The full context of the quote is given in Michael J. Crowe, *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System*, 1st ed., 1967, Dover Publications, 1994, pp. 29–30. Also see a history of the idea in John C. Baez and John Huerta, 'The Octonions', 7th April 2010, quoted in John H. Conway and Derek A. Smith, *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*, Wellesley, MA: A. K. Peters, 2003, p. 7.
[536] William Rowan Hamilton, letter dated 20th November 1843 to *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science,* XXV, pp. 489–495 and William Rowan Hamilton, 'Letter to Graves [dated 17th October 1883] on Quaternions; or on a New System of Imaginaries in Algebra, in *The Mathematical Papers of Sir William Rowan Hamilton, Volume III: Algebra*, eds. H. Halberstam and R. E. Ingram, Cambridge University Press, 1967, pp. 106–110.
[537] Burton, *History of Mathematics*, pp. 633–636.
[538] William Rowan Hamilton, 'Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time', *Transactions of the Royal Irish Academy*, vol. 17 (1837), pp. 293–422, read 4th November 1833 and 1st June 1835. David R. Wilkins published a slightly edited version of this important paper in 2000, now available on the Web. The references are on pages 107 and 111. The original is also available in *Mathematical Papers, Volume III: Algebra*, pp. 3–96.
[539] Peter Guthrie Tait, 'Quaternions', *Encyclopædia Britannica*, 9th ed., Vol. XX, pp. 160–164.
[540] http://mathworld.wolfram.com/EulerFour-SquareIdentity.html.
[541] Kline, *Mathematics*, pp. 90–91.
[542] Michael J. Crowe, *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System*, 1967, New York: Dover Publications, 1994.
[543] http://worrydream.com/refs/Crowe-HistoryOfVectorAnalysis.pdf.
[544] James Clerk Maxwell, *A Treatise on Electricity and Magnetism*, Oxford: Clarendon Press, 1873, p. 9.
[545] Hamilton, 'On Quaternions', *Philosophical Magazine*, vol. xxix, July 1846, p. 27, reproduced in Wilkins transcript, p. 13. The full quotation is: "The algebraically real part may receive … all values contained on the one *scale* of progression of number from negative to positive infinity; we shall call it therefore the *scalar part*, or simply the *scalar* of the quaternion. … On the other hand, the algebraically *imaginary* part, being geometrically constructed by a straight line, or radius vector, which has, in general, for each determined quaternion, a determined length and determined direction in space, may be called the *vector part*, or simply the *vector* of the quaternion … *a quaternion is in general the sum of its own scalar and vector parts*."
[546] Crowe, *History of Vector Analysis*, p. 54-55.
[547] Hermann Grassmann, *Extension Theory*, tr. Lloyd C. Kannenberg from *Die Ausdehnungslehre: Vollständig und in strenger Form*, 1862, Providence, RI: American Mathematical Society, p. ix.
[548] Hermann Grassmann, *A New Branch of Mathematics: The* Ausdenungslehre *of 1844 and Other Works*, tr. Lloyd C. Kannenberg from *Die Lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, 1st ed., 1844, 2nd ed., 1878, Chicago: Open Court, 1995, pp. 33 and 11–12.
[549] Ibid., p. 9.
[550] Crowe, *History of Vector Analysis*, p. 69.
[551] Ibid., p. 86.
[552] Ibid., p. 120, as quoted in Silvanus P. Thompson, *The Life of William Thomson*, Vol. II, London 1910, p. 1138, and p. 119, as quoted in G. Crystal, 'Professor Tait' in *Nature*, Vol. 64, 1901, p. 306.
[553] Oliver Heaviside, *Electromagnetic Theory* Vol. I, London: The Electrician Printing and Publishing Company, 1893, pp. 134–135.

[554] P. G. Tait, *An Elementary Treatise on Quaternions*, 3rd ed., 1st & 2nd eds., 1867 & 1873, Cambridge University Press, 1890, p. vi.
[555] Edwin Bidwell Wilson, *Vector Analysis: A Text-book for the Use of Students of Mathematics and Physics: Founded Upon the Lectures of J. Willard Gibbs*, New York: Charles Scribner's Sons, 1901, p. xi.
[556] https://en.wikipedia.org/wiki/Quaternion_Society.
[557] 3Blue1Brown, 'What are quaternions, and how do you visualize them? A story of four dimensions', https://youtu.be/d4EgbgTm0Bg and 'Quaternions and 3d rotation, explained interactively' https://youtu.be/zjMuIxRvygQ. The interactive videos are at https://eater.net/quaternions or http://www.3imaginary1real.com, for "owning that domain just felt like the right thing to do", as Grant said.
[558] 'Extract from a Letter to William R. Hamilton to John T. Graves, Esq.', Note A to 'Researches Respecting Quaternions, First Series', in Hamilton, *Mathematical Papers, Volume III: Algebra*, pp. 217–226.
[559] John C. Baez, 'The Octonions', *Bulletin of the American Mathematical Society*, Vol. 39, No. 2, 21st December 2001, pp. 145–205, p. 146.
[560] In a letter to Graves dated 8th July 1844, Hamilton refers to these letters dated 26th December 1843, 4th January 1844, and 18th January 1844, in Hamilton, *Mathematical Papers, Volume III: Algebra*, p. 650.
[561] http://mathworld.wolfram.com/DegensEight-SquareIdentity.html.
[562] Arthur Cayley, Postscript to 'On Jacobi's Elliptic Functions, in reply to the Rev. Brice Bronwin; and on Quaternions', *Philosophical Magazine*, Third Series, Vol. XXVI, No. CLXXII, March 1845, pp. 210-211.
[563] John T. Graves, Postscript to 'On a Connection between the General Theory of Normal Couples and the Theory of Complete Quadratic Functions of Two Variables', *Philosophical Magazine*, Third Series, Vol. XXVI, No. CLXXIII, April 1845, p. 320.
[564] Note from William Rowan Hamilton respecting the researches of John T. Graves to John Radford Young, 'On an Extension of a Theorem of Euler, with a Determination of the Limit beyond which it fails', *The Transactions of the Royal Irish Academy*, Vol. XXI, 1848, pp. 338–341. This letter to Young, dated 7th March 1848, about Graves' discovery of octaves is also reproduced in Hamilton, *Mathematical Papers, Volume III: Algebra*, pp. 653–655.
[565] L. E. Dickson, 'On Quaternions and Their Generalization and the History of the Eight Square Theorem', *Annals of Mathematics*, Second Series, Vol. 20, No. 3, March 1919, pp. 155-171.
[566] https://en.wikipedia.org/wiki/Cayley–Dickson_construction.
[567] Baez, 'The Octonions', pp. 150–152.
[568] https://en.wikipedia.org/wiki/James_Cockle.
[569] James Cockle in *Philosophical Magazine*, 'On Certain Functions Resembling Quaternions and on a New Imaginary in Algebra', 1848, Vol. 33, pp. 435–439, 'On a New Imaginary in Algebra', 1849, Vol. 34, pp. 37–47, 'On the Symbols of Algebra and on the Theory of Tessarines', 1849, Vol. 34, pp. 406–410, 'On the True Amplitude of a Tessarine', 1850, Vol. 36, pp. 290–292, and 'On Impossible Equations, on Impossible Quantities and on Tessarines' 1850, Vol. 37, pp. 281–283.
[570] Mathoma, 'Split Complex Numbers', 2016 playlist: https://www.youtube.com/playlist?list=PLpzmRsG7u_gqc-EKA_lul8s0pvGEls211.
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with 99 volumes of a planned 132 available, as of December 2018 (https://www.frommann-holzboog.de/editionen/20?lang=en-gb).
When Peirce died in 1914, his widow donated all his writings to the department of philosophy at Harvard, which number some 80,000 manuscript-pages. As the Peirce Edition Project says on its home page, "much of what Peirce wrote remains in manuscript form, unpublished and in significant disarray." If all the writings that exist at Harvard and elsewhere were to be published, they would constitute around a hundred volumes. As it is, the Peirce Edition Project plans to publish thirty volumes chronologically, having reached 1892 with Volume 8, the seventh to be published (http://www.iupui.edu/~peirce/).
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[683] Joseph Warren Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite*, Princeton University Press, 1990, pp. 53 and 118–119.
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[691] Galina I. Sinkevich, 'Georg Cantor from St. Petersburg. Childhood and History of the Family. Archival Research', *Kwartalnik historii nauki i techniki*, Warsaw, September 2018.
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[700] Georg Cantor, '*Mitteilungen zur Lehre vom Transfiniten*', in *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, ed. Ernst Zermelo, Berlin: J. Springer, 1932, §VII, pp. 409–411, originally published in *Zeitschrift für Philosophie und philosophische Kritik 91*, 1887.
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[715] http://www.vatican.va/holy_father/benedict_xvi/encyclicals/documents/hf_ben-xvi_enc_20051225_deus-caritas-est_en.html.
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Carolyn Eisler explains in 'Introductions to the New Elements of Mathematics' in Carolyn Eisler, *Studies in the Scientific and Mathematical Philosophy of Charles S. Peirce: Essays*, The Hague, The Netherlands: Mouton, 1979, pp. 308–376.
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[727] Charles S. Peirce, 'Multitude and Quantity', c. 1897, in *The New Elements of Mathematics, Volume 3*, ed. Carolyn Eisele, pp. 87 and 100.
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[743] Georg Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers*, tr. P. E. B. Jourdain from *Beiträge zur Begründung der transfiniten Mengenlehre*, New York: Dover, 1915, pp. 85 and v.
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[745] Burton, *History of Mathematics*, p. 678.
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[748] https://en.wikipedia.org/wiki/Galileo%27s_paradox.
[749] Burton, *History of Mathematics*, p. 678.
[750] Dauben, *Georg Cantor*, p. 99.

[751] Ibid., pp. 125–126. However, Dauben (p. 328) points out that the distinction between *reellen* and *realen* is lost in Uwe Parpart's first English translation of *Grundlagen*.
[752] https://en.wikipedia.org/wiki/Ordinal_arithmetic.
[753] Dauben, *Georg Cantor*, p. 214.
[754] Cantor, *Contributions*, §20, 'The ε-Numbers of the Second Number-Class', pp. 195–201.
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